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Numerical Solution of Fourth-order Fractional Diffusion Wave Model

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Abstract. In this paper, we shall construct a new numerical scheme for fourth-order fractional diffusion wave model. The solvability, stability and convergence of proposed method are established in l_2 norm and it is shown that the numerical scheme *improves* the earlier work done. Simulation is carried out to verify the accuracy and efficiency of the numerical scheme.

INTRODUCTION

Fourth-order fractional diffusion equations [1, 2] may be used to model groove formation on the metal surface with a grain boundary or beam transverse vibration. Explicit general solutions of fourth-order fractional diffusion wave problems are given in the work of Agrawal [3, 4]. Thereafter a decomposition method is applied successfully in [5]. However, for a wide range of applications, it is hardly practical to apply these analytical techniques. On numerical treatment of such problems, methods of finite difference [6], compact finite difference [7, 8], quintic B-spline [9] and quintic spline [10] have been applied successfully. It is observed that compact finite difference as well as quintic spline methods could achieve $O(h^4)$, where h is the spatial step size, and this has been the *best* result until the present work.

In this paper, we shall propose an efficient fully discrete numerical scheme which improves the spatial accuracy to at least $O(h^{4.5})$ for a fourth-order fractional diffusion wave problem discussed in [6, 7, 8]. The problem is stated as follows:

$$\begin{cases} {}^C D_t^\gamma u + b^2 \frac{\partial^4 u}{\partial x^4} = f(x, t), & x \in (0, L), t \in [0, T] \\ u(x, 0) = \phi(x), \quad \frac{\partial u(x, 0)}{\partial t} = \psi(x), & x \in [0, L] \\ u(0, t) = h_1(t), \quad u(L, t) = h_2(t), \quad \frac{\partial^2 u(0, t)}{\partial x^2} = g_1(t), \quad \frac{\partial^2 u(L, t)}{\partial x^2} = g_2(t), & t \in [0, T] \end{cases} \quad (1)$$

where $1 < \gamma < 2$, L , T , b are some positive constants and $f(x, t)$, $\phi(x)$, $\psi(x)$, $h_1(t)$, $h_2(t)$, $g_1(t)$, $g_2(t)$ are continuous functions in corresponding domains. Applying the fractional integral operator J_t^α , where $\alpha = \gamma - 1 \in (0, 1)$, to (1) gives

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + b^2 J_t^\alpha \frac{\partial^4 u}{\partial x^4} = g(x, t), & x \in (0, L), t \in [0, T] \\ u(x, 0) = \phi(x), & x \in [0, L] \\ u(0, t) = h_1(t), \quad u(L, t) = h_2(t), \quad \frac{\partial^2 u(0, t)}{\partial x^2} = g_1(t), \quad \frac{\partial^2 u(L, t)}{\partial x^2} = g_2(t), & t \in [0, T] \end{cases} \quad (2)$$

where $g(x, t) = J_t^\alpha f(x, t) + \frac{\partial u(x, 0)}{\partial t} = J_t^\alpha f(x, t) + \psi(x)$ and $J_t^\alpha f(x, t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x, s) ds$.

THE CN-PQS NUMERICAL SCHEME

In this section, we shall construct an efficient numerical scheme using Crank-Nicolson technique and parametric quintic spline (CN-PQS) for problem (2). To begin, let $h = \frac{L}{M}$ and $\tau = \frac{T}{N}$ be the spatial and temporal step sizes, and $\Omega_h = \{(x_0, x_1, \dots, x_M) | x_j = jh, 0 \leq j \leq M\}$ and $\Omega_\tau = \{(t_0, t_1, \dots, t_N) | t_n = n\tau, 0 \leq n \leq N\}$ be the uniform partitions of $[0, L]$ and $[0, T]$ respectively. Then, denote $U_h = \{u = (u_0, u_1, \dots, u_M) | u_0 = u_M = 0\}$ and $U_h^* = \{u | u \in U_h, \delta_x^2 u_0 = \delta_x^2 u_M = 0\}$. For any $u(\cdot, t_n) = u^n = (u_0^n, u_1^n, \dots, u_M^n) \in U_h$, define $\delta_t u_j^{n-\frac{1}{2}} = (u_j^n - u_j^{n-1})/\tau$ and

$$\delta_x u_{j+\frac{1}{2}}^n = \frac{1}{h}(u_{j+1}^n - u_j^n), \quad \delta_x^2 u_j^n = \frac{1}{h}(\delta_x u_{j+\frac{1}{2}}^n - \delta_x u_{j-\frac{1}{2}}^n), \quad \delta_x^3 u_{j+\frac{1}{2}}^n = \frac{1}{h}(\delta_x^2 u_{j+1}^n - \delta_x^2 u_j^n), \quad \delta_x^4 u_j^n = \frac{1}{h}(\delta_x^3 u_{j+\frac{1}{2}}^n - \delta_x^3 u_{j-\frac{1}{2}}^n). \quad (3)$$

Next, we shall introduce the weighted shifted Grünwald Letnikov method [11, 12] for fractional integral operator J_t^α ($0 < \alpha < 1$) and parametric quintic spline operator H for spatial discretization.

Lemma 1. [11, 12] If $u(x_j, t) = \bar{u}(t) \in C^2[0, T]$, then it holds that (refer to [11, 12] for λ_k)

$$J_t^\alpha u(x_j, t_n) = \tau^\alpha \sum_{k=0}^n \lambda_k u(x_j, t_{n-k}) + O(\tau^2). \quad (4)$$

Definition 1. A function $Q(x; \xi) \in C^{(4)}[0, L]$ is said to be the *parametric quintic spline with parameter ξ* (with respect to mesh Ω_h) if its restriction $Q_j(x; \xi) \equiv Q_j(x)$ in $[x_{j-1}, x_j]$, $1 \leq j \leq M$ satisfies $Q_j(x_i) = u_i$, $i = j-1, j$ and

$$Q_j^{(4)}(x) + \xi^2 Q_j''(x) = (F_j + \xi^2 W_j) \frac{x - x_{j-1}}{h} + (F_{j-1} + \xi^2 W_{j-1}) \frac{x_j - x}{h},$$

where $Q_j''(x_i) = W_i$, $Q_j^{(4)}(x_i) = F_i$, $i = j-1, j$

The continuity of the first and third derivative of spline $Q(x, \xi)$ at $x = x_j$ yields two spline relations involving u_j , W_j and F_j which, as in [13], lead to $HF_j := pF_{j-2} + qF_{j-1} + sF_j + qF_{j+1} + pF_{j+2} = \delta_x^4 u_j$, $2 \leq j \leq M-2$ (refer to [13] for p, q, s). We further have two more relations $HF_1 := (28F_0 + 245F_1 + 56F_2 + F_3)/360 = \delta_x^4 u_1$ and $HF_{M-1} := (28F_M + 245F_{M-1} + 56F_{M-2} + F_{M-3})/360 = \delta_x^4 u_{M-1}$.

Now, we are in a position to develop the CN-PQS numerical scheme for problem (2). Discretizing (2) at $x = x_j, t = t_{n-\frac{1}{2}} = (n - \frac{1}{2})\tau$, using the Crank-Nicolson method for the first term $\frac{\partial u(x,t)}{\partial t}$ and the fact

$$J_t^\alpha \frac{\partial^4 u(x_j, t_{n-\frac{1}{2}})}{\partial x^4} = \frac{1}{2} \left[J_t^\alpha \frac{\partial^4 u(x_j, t_n)}{\partial x^4} + J_t^\alpha \frac{\partial^4 u(x_j, t_{n-1})}{\partial x^4} \right] + O(\tau^2)$$

and then applying the result (4) to J_t^α as well as parametric quintic spline operator H on both sides of the equation, we get the CN-PQS numerical scheme

$$H \frac{u_j^n - u_j^{n-1}}{\tau} + b^2 \tau^\alpha \frac{1}{2} \left[\sum_{k=0}^n \lambda_k \delta_x^4 u_j^{n-k} + \sum_{k=0}^{n-1} \lambda_k \delta_x^4 u_j^{n-1-k} \right] = H g_j^{n-\frac{1}{2}}, \quad 1 \leq j \leq M-1 \quad (5)$$

with initial condition $u_j^0 = \phi(x_j)$ and boundary conditions $u_0^n = h_1(t_n)$, $u_M^n = h_2(t_n)$, $\delta_x^2 u_0^n = g_1(t_n)$, $\delta_x^2 u_M^n = g_2(t_n)$.

THEORETICAL RESULTS

In this section, we shall establish solvability, stability and convergence of CN-PQS numerical scheme in terms of l_2 norm. To begin, for $u, v \in U_h$, define $(u, v) = h \sum_{j=1}^{M-1} u_j v_j$. Obviously, $\|u\| = \sqrt{(u, u)}$ is a kind of norm.

Next, using Taylor expansion at $(x_j, t_{n-\frac{1}{2}})$ of formula (5), we get the following result for the local truncation error T_j^n .

Lemma 2. If $p = -\frac{1}{720}$, $q = \frac{124}{720}$, $s = \frac{474}{720}$, then $T_j^n = O(h^k + \tau^2)$, where $k = 4$ for $j = 1, M-1$ and $k = 6$ for $2 \leq j \leq M-2$.

For the rest of the paper, we shall take the values of p, q and s as in Lemma 2. Using energy methodology, we establish the following theoretical results.

Theorem 1. (Solvability) The proposed CN-PQS scheme (5) has a unique solution at each time level $t = t_n$.

Theorem 2. (Stability) The solution of CN-PQS scheme depends continuously on data $\phi(x)$ and $g(x, t)$.

Theorem 3. (Convergence) Assume that $u(x, t) \in C^{8,2}([0, L], [0, T])$ is the exact solution of (2) while $\{u_j^n, 0 \leq j \leq M, 0 \leq n \leq N\}$ is the numerical solution obtained from the CN-PQS scheme (5). Let $e_j^n = u(x_j, t_n) - u_j^n$ denote the error at (x_j, t_n) , then we have $\|e^n\| \leq C(\tau^2 + h^{4.5})$ where $C > 0$ is a constant.

Remark 1. From Theorem 3, the spatial accuracy is improved to at least $O(h^{4.5})$. Note that it is a theoretical result. In practice, it is possible to get even higher accuracy which is demonstrated in experiment later.

EXPERIMENT

In this part, we shall present an example to verify the effectiveness and efficiency of the proposed CN-PQS numerical scheme. To this aim, we shall compute l_2 norm error at $t = T$, namely $e_2(\tau, h) = \|e^N\|$, and corresponding temporal and spatial convergence orders.

Example [6] Consider problem (1) with $L = T = b^2 = 1$, $\phi(x) = 0$, $\psi(x) = e^x$ and $h_1(t) = g_1(t) = t^{3+\gamma} + t$, $h_2(t) = g_2(t) = e(t^{3+\gamma} + t)$, $f(x, t) = e^x(\Gamma(\gamma + 4)t^3/6 + t^{3+\gamma} + t)$. The exact solution is $u(x, t) = e^x(t^{3+\gamma} + t)$.

First, we fix $h = \frac{1}{30}$ and vary τ as in Table 1 where we compute l_2 norm error and the temporal convergence order. It is observed that $O(\tau^2)$ is achieved for all cases. Next, we let $\tau = \frac{1}{50000}$ and vary h as in Table 2 where we present l_2 norm error and spatial convergence order. It is observed that at least $O(h^{4.5})$ is obtained.

TABLE 1. l_2 norm error at $t = 1$ and temporal convergence order: $t - \text{order} = \log_2 \left(\frac{e_2(\tau, h)}{e_2(\tau/2, h)} \right)$

τ	$\gamma = 1.2$	$t - \text{order}$	$\gamma = 1.5$	$t - \text{order}$	$\gamma = 1.8$	$t - \text{order}$
$\frac{1}{5}$	2.17E-03	-	3.04E-03	-	4.53E-03	-
$\frac{1}{10}$	5.35E-04	2.0182	7.97E-04	1.9331	1.07E-03	2.0775
$\frac{1}{20}$	1.35E-04	1.9909	2.01E-04	1.9893	2.79E-04	1.9448
$\frac{1}{40}$	3.37E-05	1.9960	5.04E-05	1.9949	7.06E-05	1.9836
$\frac{1}{80}$	8.44E-06	1.9982	1.26E-05	1.9977	1.77E-05	1.9956
$\frac{1}{160}$	2.11E-06	1.9992	3.15E-06	1.9989	4.43E-06	1.9985

Further, the numerical/exact solutions and absolute error when $\gamma = 1.2$, $\tau = \frac{1}{50000}$ and $h = \frac{1}{20}$ are plotted in Figures 1 and 2. From the figures, it is visually seen that our CN-PQS scheme gives a good approximation to exact solution at mesh points.

FIGURE 1. Approximate solution by CN-PQS method and exact solution when $\gamma = 1.2$, $h = \frac{1}{20}$, $\tau = \frac{1}{50000}$.

TABLE 2. l_2 norm error at $t = 1$ and spatial convergence order: $x - \text{order} = \log_2 \left(\frac{e_2(\tau, h)}{e_2(\tau, h/2)} \right)$

h	$\gamma = 1.2$	$x - \text{order}$	$\gamma = 1.5$	$x - \text{order}$	$\gamma = 1.8$	$x - \text{order}$
$\frac{1}{5}$	7.73E-08	-	7.61E-08	-	7.43E-08	-
$\frac{1}{10}$	1.23E-09	5.9791	1.20E-09	5.9919	1.15E-09	6.0088
$\frac{1}{20}$	1.44E-11	6.4139	1.10E-11	6.7660	2.63E-11	5.4542

FIGURE 2. Absolute error of CN-PQS method when $\gamma = 1.2$, $h = \frac{1}{20}$, $\tau = \frac{1}{50000}$.

CONCLUSION

In this work, we propose a new high order numerical scheme for a class of fourth-order fractional diffusion-wave problems. The solvability, stability and convergence of the proposed method are established using discrete energy methodology. Then, an example is presented to demonstrate the efficiency and consistency of our numerical scheme. It is shown that our method gives second order accuracy in temporal and at least of order 4.5 accuracy in spatial which improves previous work.

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