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Parametric Quintic Spline Approach for Two-dimensional Fractional Sub-diffusion Equation

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Abstract. In this paper, we shall tackle the numerical treatment of two-dimensional fractional sub-diffusion equations using parametric quintic spline. It is shown that this numerical scheme is solvable, stable and convergent with high accuracy which improves some earlier work. Finally, we carry out an experiment to demonstrate the efficiency of our numerical scheme.

INTRODUCTION

Fractional sub-diffusion model describes slow diffusion process [1] which describes more aptly the nature of real observations. Many useful techniques have been introduced in [2, 3] to solve particular fractional diffusion models analytically. However, this is extremely difficult if not impossible in most cases. Therefore, from a practical point of view, it is very important to seek approximate solutions instead.

There have been many numerical investigations on fractional sub-diffusion problems, e.g., finite difference [4], compact finite difference [4, 5, 6, 7, 8, 9, 10], polynomial and nonpolynomial spline method [11], orthogonal spline collocation (OSC) [12] and alternating direction implicit (ADI) scheme [5]. Very recently, we have constructed an efficient numerical scheme for one-dimensional fractional sub-diffusion problem [13] using parametric quintic spline approach and demonstrated the efficiency of the scheme both theoretically and numerically.

In this work, we shall study an efficient numerical scheme for a class of two-dimensional fractional sub-diffusion problems [14] using *parametric quintic spline* coupled with $L2 - 1_\sigma$ technique [4]. The problem is stated as follows:

$$\begin{cases} {}^C_0 D_t^\gamma u = d \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u + f(x, y, t), & (x, y) \in \Omega = (0, x_R) \times (0, y_R), \quad t \in [0, T], \\ u(x, y, t) = \phi(x, y, t), & (x, y) \in \partial\Omega, \quad t \in [0, T], \\ u(x, y, 0) = \psi(x, y), & (x, y) \in \bar{\Omega} = \Omega \cup \partial\Omega \end{cases} \quad (1)$$

where ${}^C_0 D_t^\gamma u$ ($0 < \gamma < 1$) is the Caputo fractional derivative, $d > 0$ is the diffusion constant, $\partial\Omega$ is the boundary of Ω and $\phi(x, y, t)$, $\psi(x, y)$, $f(x, y, t)$ are some known continuous functions.

PARAMETRIC QUINTIC SPLINE APPROACH

In this section, we shall derive the parametric quintic spline scheme after some preliminary work. To begin, let us consider the uniform partition: $P : 0 < x_1 < \dots < x_m < \dots < x_M = x_R$, $P' : 0 < y_1 < \dots < y_n < \dots < y_N = y_R$ of $\bar{\Omega}$ and uniform mesh: $P'' : 0 < t_1 < \dots < t_k < \dots < t_K = T$ of $[0, T]$ with $x_m = mh_x$, $y_n = nh_y$, $t_k = k\tau$ where $h_x = \frac{x_R}{M}$, $h_y = \frac{y_R}{N}$, $\tau = \frac{T}{K}$ are step sizes in x , y and t dimensions respectively.

For any function $v(x, y, t)$, denote $v(x_m, y_n, t_k)$ by $v_{m,n}^k$. Let $U_{M,N} = \{u \mid u = \{u_{m,n}\} \ 0 \leq m \leq M, \ 0 \leq n \leq N\}$ and $U_{M,N}^* = \{u \in U_{M,N} \mid u_{m,n} = 0, \text{ if } (x_m, y_n) \in \partial\Omega\}$ be spaces needed later. For any $u \in U_{M,N}$, we define the following

notations in x dimension

$$\delta_x u_{m+\frac{1}{2},n} = \frac{u_{m+1,n} - u_{m,n}}{h_x}, \delta_x^2 u_{m,n} = \frac{\delta_x u_{m+\frac{1}{2},n} - \delta_x u_{m-\frac{1}{2},n}}{h_x}, \delta_x^3 u_{m+\frac{1}{2},n} = \frac{\delta_x^2 u_{m+1,n} - \delta_x^2 u_{m,n}}{h_x}, \delta_x^4 u_{m,n} = \frac{\delta_x^3 u_{m+\frac{1}{2},n} - \delta_x^3 u_{m-\frac{1}{2},n}}{h_x},$$

$\bar{\delta}_x^2 u_{m,n} = [\delta_x^2 + \alpha h_x^2 \delta_x^4] u_{m,n}$, $2 \leq m \leq M-2$ and $\bar{\delta}_x^2 u_{m,n} = \delta_x^2 u_{m,n}$, $m = 1, M-1$, where α is a parameter [13]. It is noted that δ_y^i , $i = 1, 2, 3, 4$ and $\bar{\delta}_y^2$ can be similarly defined in y dimension.

First, we present $L2-1_\sigma$ approximation [4] of Caputo fractional derivative.

Lemma 1. [4] For $u(x_m, y_n, t) = g(t) \in C^3[0, T]$, $0 < \gamma < 1$, we have (refer to [4] for r_j 's and $\mu, \sigma, t_{k-1+\sigma}$)

$${}_0^C D_t^\gamma u(x_m, y_n, t_{k-1+\sigma}) = \mu \left[r_0 u_{m,n}^k - \sum_{j=1}^{k-1} (r_{k-j-1} - r_{k-j}) u_{m,n}^j - r_{k-1} u_{m,n}^0 \right] + O(\tau^2). \quad (2)$$

Next, we propose a parametric quintic spline operator which relates second derivative and function values based on our previous work [13].

Definition 1. For any $u \in U_{M,N}$, the parametric quintic spline operator in x dimension H_x is defined as follows: $H_x u_{m,n} := p(u_{m-2,n} + u_{m+2,n}) + q(u_{m-1,n} + u_{m+1,n}) + s u_{m,n}$, $2 \leq m \leq M-2$ and $H_x u_{m,n} := (u_{m-1,n} + u_{m+1,n})/12 + 10/12 u_{m,n}$, $m = 1, M-1$.

Remark 1. If $p = \frac{1}{12}\alpha + \frac{1}{240}$, $q = \frac{8}{12}\alpha + \frac{24}{240}$ and $s = 1 - p - q$ ($0 < \alpha < \frac{1}{12}$), given $t = t_k$, for sufficiently smooth function u , applying Taylor expansion in x dimension, we obtain

$$H_x \frac{\partial^2 u(x_m, y_n, t_k)}{\partial x^2} = \bar{\delta}_x^2 u_{m,n}^k + O(h_x^r)$$

where $r = 6$, $2 \leq m \leq M-2$ and $r = 4$, $m = 1, M-1$.

Remark 2. Operator H_y in y dimension can be similarly defined and the relation between H_y and $\bar{\delta}_y^2$ can be established as in Remark 1.

Discretizing problem (1) at $(x_m, y_n, t_{k-1+\sigma})$, using Lemma 1 and the fact that $u(x_m, y_n, t_{k-1+\sigma}) = u_{m,n}^{k-1+\sigma} + O(\tau^2)$, $u_{m,n}^{k-1+\sigma} := \sigma u_{m,n}^k + (1 - \sigma) u_{m,n}^{k-1}$ and implementing two-dimensional operator $H = H_x H_y = H_y H_x$, we get

$$\mu H \left[r_0 u_{m,n}^k - \sum_{j=1}^{k-1} (r_{k-j-1} - r_{k-j}) u_{m,n}^j - r_{k-1} u_{m,n}^0 \right] = d \left(H_x \bar{\delta}_y^2 + H_y \bar{\delta}_x^2 \right) u_{m,n}^{k-1+\sigma} + H f_{m,n}^{k-1+\sigma}, \quad (3)$$

with boundary conditions $u_{m,n}^k = \phi(x_m, y_n, t_k)$, $(x_m, y_n) \in \partial\Omega$ and initial conditions $u_{m,n}^0 = \psi(x_m, y_n)$, $(x_m, y_n) \in \bar{\Omega}$.

STABILITY, SOLVABILITY AND CONVERGENCE

In this section, we shall establish stability, solvability and convergence of the proposed numerical scheme (3) using discrete energy methodology. To begin, for $u, v \in U_{M,N}^*$, define an inner product $(\cdot, \cdot)$ as

$$(u, v) = h_x h_y \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} u_{m,n} v_{m,n}.$$

It is easily seen that $\|u\| = (u, u)^{\frac{1}{2}}$ is a l_2 norm.

Using energy method, it is shown that our scheme is stable, solvable and convergent with high accuracy under the condition $0.05 < \alpha < 0.06$.

Theorem 1. (Stability) The numerical scheme (3) is robust to initial data $\psi(x, y)$ and non homogeneous data $f(x, y, t)$.

Theorem 2. (Solvability) The system (3) admits a unique solution at each time level $t = t_k$.

Theorem 3. (Convergence) Assume that the exact solution $u(x, y, t) \in C^{8,8,3}(\bar{\Omega} \times [0, T])$. The error $e_{m,n}^k$ satisfies

$$\|e^k\| \leq C(\tau^2 + h_x^{4.5} + h_y^{4.5}), \quad 1 \leq k \leq K$$

where C is a constant that depend on α only.

Remark 3. The above spatial convergence rate is a theoretical one. In practice, a higher convergence is possible and this is illustrated in simulation part below.

EXPERIMENT

In this section, we shall present an example to demonstrate the efficiency of the numerical scheme (3) and also to compare the performance with compact finite difference method. To have proper comparison, we shall apply (2) to approximate Caputo fractional derivative while employing the parametric quintic spline (PQS) or compact finite difference (CFD) in the spatial dimensions.

In our experiment, we shall compute and compare l_2 norm error at $t = T$ ($\|e^K\|$) and spatial convergence rate (SCR) where

$$\|e^K(h_x, h_y, \tau)\| = \sqrt{h_x h_y \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} (e_{m,n}^K)^2}, \quad \text{SCR} = \log_2 \left(\frac{\|e^K(h, h, \tau)\|}{\|e^K(\frac{h}{2}, \frac{h}{2}, \tau)\|} \right).$$

Example [9] Consider problem (1) with $\Omega = (0, \pi) \times (0, \pi)$, $d = T = 1$, $\phi(x, y, t) = 0$, $(x, y) \in \partial\Omega$, $\psi(x, y) = 0$, $(x, y) \in \bar{\Omega}$ and $f(x, y, t) = t^3 \left(\frac{\Gamma(4+\gamma)}{6} + 2t^\gamma \right) (\sin x)(\sin y)$. The exact solution is $u(x, y, t) = t^{3+\gamma}(\sin x)(\sin y)$.

In this example, let us fix $\gamma = 0.9$, $\tau = \frac{1}{5000}$, $\alpha = \frac{1}{18}$ and vary h ($= h_x = h_y$). We shall calculate the error $\|e^K(h, h, \tau)\|$ and spatial convergence rate for both PQS and CFD methods. The results are shown in Table 1. Obviously, our method provides the smallest error for all cases and high spatial convergence rate which indicates a better performance than CFD.

TABLE 1. l_2 norm error at $t = 1$ and spatial convergence order when $\gamma = 0.9$, $\alpha = \frac{1}{18}$, $\tau = \frac{1}{5000}$

h	Parametric quintic spline	SCR	Compact finite difference	SCR
$\frac{\pi}{5}$	1.11E-04	-	3.49E-04	-
$\frac{\pi}{10}$	1.14E-06	6.6104	2.15E-05	4.0195
$\frac{\pi}{20}$	4.12E-08	4.7839	1.30E-06	4.0539

For visualization, we plot the exact solution, numerical solutions and absolute errors of PQS and CFD methods at $t = 1$ when $\gamma = 0.9$, $\alpha = \frac{1}{18}$, $\tau = \frac{1}{5000}$ and $h = \frac{\pi}{20}$ in Figures 1 and 2. Obviously, our PQS method gives a better approximation than CFD method.

FIGURE 1. Exact solution at $t = 1$ when $\gamma = 0.9$, $h = \frac{\pi}{20}$, $\tau = \frac{1}{5000}$, $\alpha = \frac{1}{18}$.

FIGURE 2. Approximate solutions and absolute errors at $t = 1$ using compact finite difference method (top) and parametric quintic spline (bottom) when $\gamma = 0.9$, $h = \frac{\pi}{20}$, $\tau = \frac{1}{5000}$, $\alpha = \frac{1}{18}$.

CONCLUSION

In this paper, we propose an efficient numerical scheme for two-dimensional fractional sub-diffusion problems using parametric quintic spline. It is shown that the developed numerical scheme is uniquely solvable, stable in l_2 norm sense and convergent with accuracy $O(\tau^2 + h_x^{4.5} + h_y^{4.5})$. The spatial convergence improves that of the compact finite difference method as it only achieves $O(h_x^4 + h_y^4)$. An experiment is carried out to demonstrate the efficiency of the proposed method as well as to compare with the compact finite difference method.

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