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A simplified model of catenary action in reinforced concrete frames under axially restrained conditions

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From previous experimental studies on missing column scenarios, catenary action could provide beneficial effects on structures during load redistribution after a target column is suddenly removed under an alternative load path procedure to assess structural resistance to progressive collapse. However, full-scale structural tests are costly and time consuming. Equally, finite-element models require in-depth expertise and large computing resources. Practising engineers are not amenable to this approach for preliminary sizing of members or for a quick assessment of structural resilience. To address this need, this paper provides a simplified semi-analytical model for predicting structural response under a column loss scenario. The model is constructed based on finite-element simulations and is verified by test results. The simplified model idealises the structural response of reinforced concrete beam–column sub-assemblages under column loss scenarios as a piecewise multi-linear curve. It shows good agreement with published test results and validated finite-element simulations.

Notation

A	area of beam section, $A = A_c + A_s$	f_u	ultimate strength of longitudinal rebars
A_c	sectional area of concrete	f_y	yield strength of longitudinal rebars
A_s	sectional area of total longitudinal rebars at the end joint	$H, H_{\max}$	horizontal reaction and maximum horizontal reaction of the sub-assemblage
A_s^{top}	sectional area of top longitudinal rebars in the beam near the end joint	I_g	moment of inertia of the gross section area
c_1, c_2	locations of curtailment region	K_A	horizontal spring of the end-joint restraint
d	distance from extreme compression fibre of concrete to tension reinforcement	K_R	rotational spring of the end-joint restraint
$d_A, d_B, d_{C_1}, d_C, d_{D_1}, d_D, d_E, d_F, d_G$	displacements at points A, B, C ₁ , C, D ₁ , D, E, F, G	k_{flex}	relative bending stiffness
d_{beam}	concrete beam depth	k_{hor}	normalised horizontal stiffness
$d_{\text{mid}}, d_{\text{end}}$	diameters of tensile rebars at the middle and end joints	k_{rot}	normalised rotational stiffness
d'	distance from extreme compression fibre of concrete to compression reinforcement	k_{span}	span–depth ratio
E_c	elastic modulus of concrete, $E_c = 4.7(f_{ck})^{1/2}$ GPa	k_{topbar}	steel-to-concrete stiffness ratio
E_{eq}	equivalent elastic modulus of beam section, $E_{\text{eq}} = (E_c A_c + E_s A_s)/(A_c + A_s)$	L_n	clear span of the single beam
EI	the flexural stiffness of the beam of the removed column when the sub-assemblage is non-symmetrically designed	$L_n^{\text{mid}}, L_n^{\text{end}}$	equivalent cantilever spans
E_s	elastic modulus of longitudinal rebars	L_{sp}	plastic hinge length at the end joint (for calculation of d_G)
F_{top}^u	ultimate tensile force of the top reinforcement, $F_{\text{top}}^u = f_u A_s^{\text{top}}$	$L_{\text{sp}}^{\text{mid}}, L_{\text{sp}}^{\text{end}}$	plastic hinge lengths at the middle and end joints (for calculation of d_D)
F_{top}^y	yield force of the top reinforcement, $F_{\text{top}}^y = f_y A_s^{\text{top}}$	$M_{\text{mid}}, M_{\text{end}}$	sagging moment capacities of beam section near the middle joint and hogging moment capacity of beam section near the end joint
f_{ck}	concrete compressive strength	$P_A, P_B, P_{C_1}, P_C, P_{D_1}, P_D, P_E, P_F, P_G, P_B^d, P_D^d, P_G^d$	applied loads at points A, B, C ₁ , C, D ₁ , D, E, F, G
		$P_{\text{left}}, P_{\text{right}}, P_{\text{total}}$	equivalent dynamic loads at displacements d_B, d_D, d_G
		R_A	simplified responses of the ‘left’, ‘right’ and total systems
		$\alpha_1, \alpha_2, \beta_1, \beta_2$	capacity of the horizontal restraint variables of the proposed model for sub-assemblages

Δ_c	beam deformation caused by constant yield curvature at curtailment region
Δ_u	beam deformation caused by ultimate curvature at plastic hinge
Δ_u^{mid}	beam deformation caused by ultimate curvature at plastic hinge (segment XY)
Δ_{XY}, Δ_{YZ}	contributions of beam deflection within segments XY and YZ
Δ_y	beam deformation caused by linear distribution of curvature along the beam
Δ_y^{mid}	beam deformation caused by linear distribution of curvature along segment XY
δ	translational deformation at the end joint
ε_u	ultimate strain of longitudinal reinforcement
ε_y	yield strain of longitudinal reinforcement
θ	rotational deformation at the end joint
θ_D	rotational angle
ϕ_y^c	yield curvature of concrete beam section at the curtailment region under hogging moment
ϕ_y, ϕ_u	yield and ultimate curvatures of concrete beam section at the end joint under hogging moment
ϕ_y', ϕ_u'	yield and ultimate curvatures of concrete beam section at the middle joint under sagging moment

Introduction

Although progressive collapse of buildings has been identified as a very low-probability event, its catastrophic consequence through the loss of human lives has raised significant research interest during the past two decades. In terms of design guides for progressive collapse mitigation (DOD, 2013; GSA, 2003), the assumption of a single column being suddenly removed is generally accepted in the engineering community. The surviving structure will be analysed for double-span bridging effect and alternative paths for possible load redistribution. Recent experimental work (Lim *et al.*, 2015; Lu *et al.*, 2016; Ren *et al.*, 2016; Sadek *et al.*, 2011; Sasani and Kropelnicki, 2008; Su *et al.*, 2009; Valipour *et al.*, 2013; Yi *et al.*, 2008; Yu and Tan, 2013a, 2013b) and numerical studies (Bao *et al.*, 2014; Pham *et al.*, 2016; Qian *et al.*, 2015; Valipour and Foster, 2010) on reinforced concrete (RC) beam–column structures under the single column removal scenario have confirmed the contribution of upper-bound mechanisms, such as compressive arch action and subsequent catenary action, besides flexural mechanisms. While compressive arch action (at small deformations) is an efficient means to enhance the maximum flexural capacity of the beam section, catenary action requires large deformations and is considered as the last line of defence mechanism to prevent the structure from collapse.

In the previous quasi-static tests on beam–column sub-assemblages, catenary action was observed to start kicking in

when the middle-joint displacement of a two-span beam had reached around one beam depth. Subsequently, as the deflection kept increasing, beam bottom rebars near the middle joint started fracturing, leading to a sudden drop in load capacity. The displacement corresponding to this failure was about 1/8 to 1/11 of the single clear span in the tests of Yu and Tan (2013a, 2013b) and Lim *et al.* (2015); whereas in Yi *et al.* (2008) and Sadek *et al.* (2011), that ratio was 1/6 and 1/5, respectively. Suffice to say, the fracture of bottom rebars is significantly influenced by span–depth ratio, rebar arrangement and ultimate strain of longitudinal reinforcement. If the downward movement were to be continued after the fracture of bottom rebars (Lim *et al.*, 2015; Sasani and Kropelnicki, 2008; Yu and Tan, 2013a, 2013b), the load capacity would increase again and the final load resistance might be greater than both the peak strength provided by compressive arch action and the catenary action stage before the fracture of bottom rebars.

The behaviour of RC beam–column structures against the column loss scenario involves various non-linear phenomena such as changing of geometry under large displacements, crushing of concrete at large strain, yielding and post-yielding behaviour of reinforcement. The use of sophisticated numerical simulations, either with physics-based solid elements or with simplified fibre elements incorporating a component-based joint model (Bao *et al.*, 2014; Pham *et al.*, 2016; Yu and Tan, 2013a), can provide reasonable predictions compared to actual response. Nonetheless, users are required to possess a high level of modelling skill, knowledge of the finite-element method (FEM) and vast computing resources when resorting to numerical simulations. In the conceptual design stage where engineers need to explore various options to arrive at reasonable and economical solutions, a well-validated and simple approach is preferred instead of detailed finite-element simulations. In this regard, Li *et al.* (2014) proposed equations for a straight type of catenary mechanism under point load application at the middle joint, defining the relationship between applied loads and corresponding deformations. However, Li's model assumes perfectly rigid horizontal restraints at both sides of the two-span beam, which is difficult to achieve in practice. Other analytical methods have been proposed (Park and Gamble, 2000; Valipour *et al.*, 2013; Yu and Tan, 2014) to predict the maximum structural capacity of beams or one-way slabs under axial compression condition, but they are not applicable for the catenary action phase. In addition, as progressive collapse is a dynamic phenomenon, strain energy from structural response, rather than the peak static resistance, is more important in assessing the maximum dynamic load and corresponding deformation.

To study the behaviour of RC substructures under the single column removal scenario, a physics-based model applying the FEM from a related study (Pham *et al.*, 2016) is employed to investigate important variables affecting catenary action. Based on actual tests and numerical predictions, a semi-analytical

model is proposed to provide a practical and yet reliable procedure to predict the overall response of two-dimensional RC beam–column structures. This simplified approach takes into account the effect of partial restraint, as well as structural response before and after the fracture of bottom rebars in the double-span beam. Equivalent dynamic capacity is also considered in the proposed model by using a simplified energy-based method (Izzuddin and Nethercot, 2009; Izzuddin *et al.*, 2008). Published test results (Lim *et al.*, 2015; Yu and Tan, 2013a, 2013b) are used to validate the proposed model and to assess its conservatism for design purposes.

Research significance

The paper presents a semi-analytical approach for RC beam–column substructures under progressive collapse, taking into account the effect of catenary action generated from horizontal restraint conditions. The purpose of this method is to provide a quick and reliable tool to practising engineers for preliminary sizing of structural members in a RC building subjected to the threat of single column removal.

Structural investigations on catenary action

Proposed simplified structural response under supporting column removal

From the test results (Lim *et al.*, 2015; Sasani and Kropelnicki, 2008; Yu and Tan, 2013a, 2013b), a typical structural response with adequate horizontal restraints can be presented as in Figure 1(a) by the curve OABCDEFG. It should be noted that curve OA represents the initial stage in which steel reinforcement behaves elastically and there is little concrete damage. At point A, longitudinal rebars in the beam–column joints start yielding and cracks start developing within the concrete tensile regions if the section is under-reinforced. The first peak at point B denotes the maximum flexural capacity considering compressive arch action enhancement. Thereafter, vertical load gradually decreases due to crushing of concrete at the compressive zones, as well as a reduction of

compressive arch action. Point C in Figure 1(a) marks the onset of catenary action and usually coincides with a vertical deflection proportional to one beam depth. Within the ascending phase CD, mobilisation of catenary action is identified by a change of axial force in beams from compression to tension. The sudden drop DE marks the fracture of the bottom longitudinal rebars in the beams at the middle joint region. For symmetrical structures, it is conservatively assumed that the bottom rebars from both sides of the middle joint fracture simultaneously. In actual tests, bottom rebars from two sides of the joint could fracture at different times due to imperfections in the symmetric condition. After which, if the middle-joint displacement continues to increase then the top reinforcing bars will contribute to the residual capacity of catenary mechanism until they also fracture.

To provide a simplified response representing the structural capacity of beam–column components, the typical response OABCDEFG is represented by a polyline with controlling points from A to G (Figure 1(b)). For parts BC and CD, additional intermediate points C_1 and D_1 are proposed to make these curves fit better with the actual response in Figure 1(a). For horizontal lines C_1D_1 and EF, it is assumed that the structural capacity remains unchanged while the displacement keeps increasing. Point C_1 is defined so that C is the mid-point of C_1D_1 . A, B, C, D, E and G are considered as the primary points of the proposed curve, which can be directly obtained from structural analysis. However, C_1 , D_1 and F are secondary points, which are dependent on the locations of the primary points.

Proposed normalised factors for sub-assembly behaviour

A series of normalised parameters is used to investigate the effects of important geometrical, material and external restraint factors on the behaviour of beam–column structures. To make the problem tractable, it is assumed that there is symmetry in loading, material and boundary conditions; thus,

Figure 1. Simplified response of beam–column structures: (a) actual response; (b) simplified piecewise response

only one half of the double-span beam is considered (Figure 2). The middle joint has vertical deflection as the only degree of freedom, while its horizontal and rotational degrees of freedom are restrained owing to symmetry at the centre. For the end joint, a pin-on-roller support with an axial spring (K_A) and a rotational spring (K_R) are considered as boundary conditions. The horizontal force from the axial spring is denoted as H . Such an arrangement allows the consideration of boundary effects (or imperfections) incorporating both horizontal and rotational restraints at the end joint. In Figure 2, E_{eq} is the equivalent elastic modulus of the beam section; δ and θ are the translational and rotational deformations at the end joint, respectively. A semi-analytical model based on the simplified response shown in Figure 1(b) is developed employing the normalised factors, as presented in the following.

The first normalised parameter is the span–depth ratio k_{span} defined by Equation 1. This factor describes the governing behaviour of the structure, which is either pure flexure or flexure with shear.

$$1. \quad k_{span} = \frac{L_n}{d_{beam}}$$

where L_n is the single clear span before the column is removed and d_{beam} is the overall beam depth.

The normalised horizontal stiffness, k_{hor} , which describes the axial stiffness of the beam relative to adjacent lateral restraints, is defined by Equation 2. The value of k_{hor} depends on the axial stiffness of adjacent members to withstand a horizontal

force, which is sensitive to the number of bays next to the removed column. A small value of k_{hor} denotes a weak boundary restraint compared to the beam axial stiffness. A previous study by Pham *et al.* (2016) shows that if k_{hor} is less than 0.011, catenary action in beams is unlikely to be mobilised.

$$2. \quad k_{hor} = \frac{K_A L_n}{E_{eq}(A_c + A_s)}$$

$$3. \quad E_{eq} = (E_c A_c + E_s A_s) / (A_c + A_s)$$

where E_c , A_c are, respectively, the elastic modulus and the total area of uncracked concrete; E_s , A_s are, respectively, the elastic modulus and the total area of steel reinforcement at the end joint; and E_{eq} is the equivalent elastic modulus of the beam section.

The normalised bending stiffness k_{flex} , which represents the flexural capacity of the beam compared to its flexural stiffness, is given by Equation 4. This factor relates the bending moment resistance of the section to the beam elastic stiffness. For example, a high value of k_{flex} denotes that the beam is highly reinforced.

$$4. \quad k_{flex} = \frac{M_{mid} + M_{end}}{E_{eq} I_g / L_n}$$

where I_g is the moment of inertia of the uncracked concrete cross-section; M_{mid} and M_{end} are, respectively, the sagging and

Figure 2. Proposed half-type model: (a) initial stage; (b) large-deformation stage; (c) bending moment

the hogging plastic moment capacities (yielding state) of the beam section at the middle joint and the end joint. M_{mid} and M_{end} can be obtained from the simplified equation proposed by Paulay and Priestley (1992)

$$5. \quad M = (d - d')A_s^{\text{ts}}f_y$$

where d and d' are the distances from the extreme compression fibre of concrete to the centre of tension and compression reinforcement, respectively; A_s^{ts} and f_y are the cross-sectional area and yield strength of tension longitudinal rebars, respectively. Equation 5 is applicable for under-reinforced sections in the absence of beam axial forces.

The steel-to-concrete stiffness ratio, k_{topbar} relates the axial stiffness of the top reinforcement to that of the concrete material in the beam section at the end joint (Equation 6). A small reinforcement ratio yields a small k_{topbar} or vice versa.

$$6. \quad k_{\text{topbar}} = \frac{E_s A_s^{\text{top}}}{E_c A_c}$$

where A_s^{top} is the area of top reinforcement in the beam near the end joint.

Finally, Equation 7 specifies the normalised rotational stiffness k_{rot} of the end joint. It is a ratio of the rotational stiffness K_R of the boundary condition to the beam elastic stiffness. If the end restraint has low rotational stiffness, it will limit the capacity of compressive arch action.

$$7. \quad k_{\text{rot}} = \frac{K_R}{E_{\text{eq}} I_g / L_n}$$

Investigations on structural behaviour

To investigate the effects of the five normalised variables on the overall response of the sub-assembly structures, a physics-based FEM model developed in a related study

(Pham *et al.*, 2016), which has been validated by quasi-static tests (Yu and Tan, 2013a, 2013b), was employed. This model was constructed using an explicit finite-element software LS-Dyna (Hallquist, 2007). It employed eight-node solid elements with reduced integration scheme to simulate concrete materials, and two-node Hughes–Liu beam elements with 2×2 Gauss quadrature integration to simulate longitudinal reinforcement. A continuous surface cap model Mat_159 was employed to simulate the behaviour of the concrete material. It was shown from previous studies (Bao *et al.*, 2014) that this material model can simulate actual responses and failure modes of RC structures against progressive collapse under quasi-static condition. An isotropic elasto-plastic material model ‘mat piecewise linear plasticity’ (Mat_024) was used for the steel reinforcement. Mesh sizes of $12.5 \times 12.5 \times 12.5 \text{ mm}^3$ (joint regions) and $12.5 \times 12.5 \times 25 \text{ mm}^3$ (other regions) were used for solid elements, whereas a mesh size of 25 mm was used for the beam elements. Although element erosion is not a physical phenomenon for concrete material, this attribute allows modelling of spalling and separation of concrete under extremely high tensile force. In the FEM model, the criterion for elements to be eroded was based on the maximum principal strain value. A value of 0.1 was chosen for the maximum principal strain as the erosion criterion. Element erosion was also applied for reinforcement material as it reaches the maximum tensile strain to represent fracture of rebars. Bond-slip behaviour between concrete and reinforcement was modelled applying a one-dimensional contact function (Contact_1D) between concrete solid elements and rebar beam elements (Bao *et al.*, 2014; Shi *et al.*, 2009). Figure 3 illustrates the one-half FEM model; Figure 4 shows some validation results of the model with experimental data from Yu and Tan (2013b).

Based on the validated model, parametric studies were conducted for the five factors ranging from k_{span} to k_{rot} and concrete strength; the results are shown in Figure 5. The purpose of the comparison is to find critical parameters that should be incorporated into the simplified model for catenary action. While direct comparisons are made for k_{span} , k_{hor} and k_{rot} , the

Figure 3. FEM model for sub-assembly structure (Pham *et al.*, 2016)

Figure 4. Validation results of FEM model (adapted from Pham *et al.* (2016)): (a) specimen S3; (b) specimen S4; (c) specimen S5

Figure 5. Investigation of catenary action mobilisation: (a) span–depth ratio, k_{span} ; (b) horizontal stiffness, k_{hor} ; (c) top rebars (%); (d) bottom rebars (%); (e) rotational stiffness, k_{rot} ; (f) concrete strength, f_{ck}

influences of k_{flex} and k_{topbar} are indirectly illustrated by comparing different ratios of top and bottom rebars at the end joint. From both numerical simulations (Pham *et al.*, 2016) presented in Figure 5(a) and test results from Yu and Tan (2013b), it is shown that, if k_{span} is smaller than 7, then after the first peak, the structural response will decrease quickly and there is no ascending phase; that is, flexural and compressive arch mechanisms will govern the response and shear dominates the failure instead of catenary action.

In terms of axial restraint, five values of k_{hor} are compared, ranging from 0.011 to 2.271 (Pham *et al.*, 2016). The results presented in Figure 5(b) clearly show that k_{hor} has a significant effect on structural response, as well as on the development of the catenary mechanism in the beams. In this comparison, the value of $k_{hor} \geq 2$ can be considered as the perfectly rigid condition. However, if the horizontal restraint is too weak, say $k_{hor} < 0.01$ (for $k_{span} = 11$), the catenary mechanism cannot be efficiently mobilised. This observation agrees with Yu and Tan

(2013a) regarding the sensitivity of catenary action on horizontal restraint stiffness.

Regarding the effect of longitudinal reinforcement on catenary action, Figures 5(c) and 5(d) show that, although the top rebars significantly affect the structural response before and after the fracture of the bottom rebars, the bottom rebars only affect the behaviour prior to their fracture at the middle joint (from point A to D). After the fracture, the load-carrying capacity drops to point E and the bottom rebars can no longer contribute to catenary action.

The normalised rotational stiffness, k_{rot} , is studied in the range of 0.04 to the perfectly rigid condition. Results show that rotational restraint stiffness has less effect on structural response after the first peak (Figure 5(e)). To gain a thorough understanding of the effect of k_{rot} on catenary action, a more extensive investigation is made in which the span–depth ratio k_{span} is also considered. Five values of k_{span} (7, 9, 11, 12.8 and 15) and five values of k_{rot} (0.04, 0.41, 0.85, 4.36 and 8.9) are taken into account simultaneously. It is found that the effect of rotational stiffness on the curve from C₁ to G (Figure 1(b)) is almost constant if both k_{span} and k_{rot} satisfy Equation 8. Moreover, the case of beam–column sub-assemblages with very little rotational restraint at the end supports ($k_{rot} < 0.2$) is not realistic for concrete structures. Hence, it is concluded that the development of catenary action does not depend on k_{rot} .

$$8. \quad \frac{5}{k_{span}} + k_{rot} \geq 1$$

Considering concrete strength, it is observed that f_{ck} greatly affects the peak compressive arch action (point B) but has limited effect on the remaining points from C to G (Figure 5(f)). In this phase, catenary action governs the response and is only dependent on k_{span} , k_{hor} and k_{topbar} . Connection gaps, which exist in actual tests, can remarkably affect the mobilisation of compressive arch action (Yu and Tan, 2014). Nevertheless, it has very little influence on residual response after all the bottom rebars have fractured (part EG). Moreover, this boundary imperfection only occurs in the

laboratory tests and has little physical meaning for cast in situ concrete structures. In summary, all the four important factors (k_{span} , k_{hor} , k_{flex} , k_{topbar}) have been incorporated into the proposed model, while the three other less important ones (k_{rot} , concrete strength and connection gaps) can be ignored.

Simplified semi-analytical model on catenary action

Based on the comparison in the previous section, a comprehensive numerical parametric study on catenary action is employed to obtain a semi-analytical approach for predicting structural response. All the investigated parameters are summarised in Table 1. A total number of 120 case studies are considered, including four types of rebar arrangement, five different spans and six values of horizontal stiffness. The reinforcement contents are chosen within the practical range of beam design. Constant parameters include the beam cross-sectional dimensions, concrete grade and reinforcement strength. Perfect rotational restraint and zero connection gap are assumed, as they have little effect on catenary action. From numerical predictions, the values of controlling points (from A to G of Figure 1(b)) can be defined and semi-analytical equations are proposed as follows.

Point A

Assuming yielding of both the top rebars near the end joint and the bottom rebars near the middle joint, plastic hinge theory is employed to predict the capacity at point A (Figure 1(b)) as shown in Equation 9. Based on numerical results of displacement when structural capacity reaches the value of P_A , the displacement at point A (d_A) is determined by Equation 10. In this equation, α_1 is a coefficient that considers the effect of horizontal restraint stiffness (k_{hor}) defined by Equation 11. Nonetheless, since the proposed model assumes rigid condition for rotational restraint, it tends to overestimate the stiffness at point A and yields a smaller value of ϕ_y compared to test results.

$$9. \quad P_A = \frac{2(M_{mid} + M_{end})}{L_n}$$

Table 1. Numerical parametric studies

Parameters	Value					
Concrete	f_{ck} = 30 MPa					
Beam dimension	150 mm wide by 250 mm depth					
Reinforcement	f_y = 520 MPa, f_u = 620 MPa, ε_y = 0.27%, ε_u = 12%					
Top rebars	3H8	3H10	3H13	3H16		
mm ² /%	151 (0.45%)	236 (0.7%)	398 (1.2%)	603 (1.8%)		
Bottom rebars	2T8	2T10	2T10	2T10		
mm ² /%	101 (0.3%)	157 (0.47%)	157 (0.47%)	157 (0.47%)		
Span: mm	1750/2250/2750/3200/3750					
Horizontal stiffness: kN/m	5×10^3	10^4	2×10^4	4×10^4	10^5	10^6

where M_{mid} and M_{end} are plastic moment capacities of beams at the middle and the end joints, respectively, computed following Equation 5.

$$10. \quad d_A = \frac{\alpha_1(M_{\text{mid}} + M_{\text{end}})L_n^2}{3E_C I_g}$$

$$11. \quad \alpha_1 = \frac{1}{(1.25 + k_{\text{hor}})^4} + 0.56$$

Point B

In this investigation, the analytical model proposed by Yu and Tan (2014) is employed to define the load-carrying capacity and the corresponding deformation at B. This approach considers imperfections in boundary conditions, such as connection gaps at restraints, and stiffness of both horizontal and rotational restraints.

Point C

At point C (Figure 1(b)), the effect of compressive arch action almost ceases and catenary action is about to be mobilised. The applied load at C, P_C , is assumed to be equal to P_A (flexural strength). The vertical deflection at C, d_C , is assumed to be proportional to d_{beam} , and depends on k_{span} and k_{hor} . A large value of k_{span} and a small value of k_{hor} lead to a large value of d_C . On the other hand, a small d_C denotes the dominant influence of arching action. Consequently, Equations 12 and 13 are used together to define d_C .

$$12. \quad d_C = \alpha_2 d_{\text{beam}}$$

where α_2 is defined by Equation 13

$$13. \quad \alpha_2 = (0.4 + 0.065k_{\text{span}} - 12k_{\text{flex}}) \left[\frac{1}{(1.07 + k_{\text{hor}})^{15}} + 1 \right]$$

Line FG

From the results of parametric studies, it is shown that the gradient of line FG greatly depends on the yield force of the top reinforcement F_{top}^y at the end joint and the span–depth ratio k_{span} , while it is less influenced by axial restraint ratio k_{hor} . Nonetheless, the position of FG is dominated by the stiffness of axial restraint. Therefore, a linear function for line FG is proposed herein

$$14. \quad P_{\text{FG}}(d) = \beta_1 \frac{F_{\text{top}}^y}{L_n} (d - \beta_2 d_{\text{beam}})$$

β_1 and β_2 are specified by Equations 15 and 16, respectively.

$$15. \quad \beta_1 = (1.2 + 0.24k_{\text{span}} - 9k_{\text{topbar}})$$

$$16. \quad \beta_2 = (-0.45 + 0.125k_{\text{span}} - 3.6k_{\text{topbar}}) \times \left[\frac{1}{(1.05 + k_{\text{hor}})^{10}} + 1 \right]$$

Line D₁D

Observations from both the test and numerical results show that line D₁D tends to be parallel to FG (Figure 1(b)). Hence, for simplicity it is assumed that they both share the same gradient, which mostly depends on the yield strength F_{top}^y of the top reinforcement (Equation 14). The reduction in terms of load-carrying capacity between D₁D and FG is due to positive moment at the middle joint M_{mid} , which terminates after all the bottom rebars have fractured at D. Equation 17 shows the load capacity of the structure within D₁D.

$$17. \quad \begin{aligned} P_{\text{D}_1\text{D}}(d) &= P_{\text{FG}}(d) + \frac{2M_{\text{mid}}}{L_n} \\ &= \beta_1 \frac{F_{\text{top}}^y}{L_n} (d - \beta_2 d_{\text{beam}}) + \frac{2M_{\text{mid}}}{L_n} \end{aligned}$$

After obtaining the separate functions of FG and D₁D, the load-carrying capacities at points D and G can be defined by specifying corresponding displacements of these points. In the next section, an analytical approach based on plastic hinge length will be used to determine the corresponding deflections d_D and d_G .

Displacements at G and D

The deformation at point D (d_D) is calculated based on the deflection caused by segments XY (Δ_{XY}) and YZ (Δ_{YZ}) shown in Figure 6(a), in which Y is the contra-flexural point. In segment XY, a plastic hinge is assumed to have fully developed in the beam section near the middle joint with a plastic hinge length $L_{\text{sp}}^{\text{mid}}$ (Figure 6(b)). Beam curvature within the plastic hinge length is assumed to reach the ultimate value ϕ_u' , whereas the beam curvature along XY is uniformly distributed from ϕ_y' (yield value) to zero. Hence, $\Delta_{XY} = \Delta_u^{\text{mid}} + \Delta_y^{\text{mid}}$ in which Δ_u^{mid} is due to ultimate curvature ϕ_u' within $L_{\text{sp}}^{\text{mid}}$, and Δ_y^{mid} comes from linear distribution of curvature along XY.

In segment YZ, the plastic hinge is partially developed within a plastic hinge length $L_{\text{sp}}^{\text{end}}$ (Figure 6(b)). Along YZ, beam curvature is assumed to be linear from ϕ_y (yield value) to zero. Since the plastic hinge is not yet fully developed, not all of the beam curvature within $L_{\text{sp}}^{\text{end}}$ reaches ultimate value ϕ_u (Figure 6(b)). Therefore, Δ_{YZ} is defined based on Δ_{XY} , noting that both segments have the same rotational angle θ_D (Figure 6(a)).

When the structure reaches the maximum capacity at G and finally collapses at the two ends, a plastic hinge is assumed to have fully developed at the end joint. Near the middle joint,

Figure 6. Proposed response and curvature distribution at point D: (a) assumption of moment distribution; (b) assumption of curvature distribution

Figure 7. Proposed response and curvature distribution at point G: (a) structural response; (b) curvature distribution

however, concrete spalling penetrates through most of the beam sections, except at the region surrounding the top rebars. Thus, the middle joint can be considered as a pin (Figure 7(a)). Within the region of top rebar cut-off points (curtailment regions), several flexural cracks are observed owing to negative

bending moment reaching the yield capacity of the beam section. Total displacement d_G is divided into three components, that is $d_G = \Delta_y + \Delta_u + \Delta_c$. The first component Δ_y is due to linear distribution of curvature along the beam: from zero at the middle joint to ϕ_y at the end joint. The second component Δ_u is

the displacement caused by ultimate curvature ϕ_u within the ideal plastic hinge length L_{sp} near the end joint. The last term Δ_c comes from curvature distribution along the curtailment region L_c where some top rebars are cut off. Based on test results and numerical studies, it is assumed that the remaining top rebars along the curtailment region attain the yield strength. Also the curvature reaches the yield value ϕ_y^c and is constant in this part (Figure 7(b)).

To calculate the displacements d_D and d_G , a method proposed by Paulay and Priestley (1992) for cantilever columns under horizontal displacements is applied. The detailed derivations of d_D and d_G are presented in Appendix 1. Afterwards, the load capacities at G and D are obtained from Equations 14 and 17, respectively.

Point E

After the fracture of bottom rebars at both sides of the middle joint, the vertical load capacity has a sudden drop from D to E. Results obtained from tests and FEM analyses show that the applied loads at D and E are proportional to the ratio of plastic moment resistance at the end joint and total plastic moment resistance from the middle and the end joints. Knowing the value of P_D , it is possible to compute P_E from Equation 18.

$$18. \quad \frac{P_E}{P_D} = \frac{M_{end}}{M_{mid} + M_{end}}$$

Points C_1 , D_1 , F

After all the primary points A, B, C, D, E, G and the equations of straight lines D_1D and FG have been determined, the secondary points C_1 , D_1 , F can then be defined based on these data. Point D_1 is the intersection of D_1D and C_1C , that is $P_{D_1} = P_C$. Similarly, point F belongs to the line FG and $P_F = P_E$. Lastly, C_1 is determined from C_1D_1 in which C is the mid-point. However, if the results show that P_D is smaller than P_C then point D_1 does not exist. In this case, P_D will be equal

to P_C while d_D remains unchanged and C_1 is chosen so that $C_1C = CD$ (Figure 8(a)). This scenario represents the case when bottom rebars fracture early before catenary action is efficiently mobilised. On the other hand, if E lies below line FG then F is determined as the intersection of lines DE and FG and point E is no longer needed (Figure 8(b)). It demonstrates the response of a sub-assembly with relatively large horizontal restraint stiffness.

Minimum value for horizontal stiffness ratio k_{hor}

Observations from different span–depth ratios show that, after the fracture of bottom rebars at the middle joint, catenary action can be mobilised efficiently at the latter part (FG) only if k_{hor} is not less than a certain value defined by Equation 19. Therefore, in the proposed response, polyline EFG will be considered as a horizontal line EG if k_{hor} does not satisfy the stiffness requirement in Equation 19. In this case, P_G is equal to P_E and point F is no longer needed (Figure 8(c)).

$$19. \quad \begin{aligned} &0.011 \quad \text{if } k_{span} \geq 11 \\ &k_{hor} \geq 0.019 \quad \text{if } 11 \geq k_{span} \geq 9 \\ &0.029 \quad \text{if } 9 \geq k_{span} \geq 7 \end{aligned}$$

Minimum strength of axial restraint, R_A

In the numerical simulations, the horizontal restraint stiffness K_A is assumed to behave elastically. Nonetheless, in actual buildings, boundary restraints may respond beyond the elastic range and may be subject to large inward deflections if ductile failure governs, or may suddenly fail in a brittle manner. In this approach, the horizontal restraint is only allowed to behave elastically with the maximum elastic strength, R_A . If the horizontal reaction force exceeds R_A , catenary action cannot be developed in the FG part. To fully mobilise catenary action in the beam–column structure, the capacity of horizontal restraint R_A must be greater than the maximum horizontal reaction H_{max} , which can be fully developed in the sub-assembly before the top rebars fracture. At point G, it is

Figure 8. Special cases of the simplified model: (a) $P_D < P_C$; (b) point E lies below FG; (c) k_{hor} too small. D_{new} is the new location of point D, in the case when P_D is smaller than P_C

assumed that the beam top rebar near the end joint section reaches its ultimate strength, f_u . Through numerical predictions and neglecting the tensile force from the bottom rebars as well as the tensile strength of concrete, $H_{\max}$ is related to the ultimate tensile force of the top reinforcement, F_{top}^u , by Equation 20.

$$20. \quad R_A \geq H_{\max} = F_{\text{top}}^u \left[1 - \frac{1}{(k_{\text{span}} - 4.5)^{1.6}} \right] \left[1 - \frac{1}{(k_{\text{hor}} + 1.05)^{20}} \right] \quad \text{for } k_{\text{span}} \geq 7$$

When both k_{span} and k_{hor} are large ($k_{\text{span}} \geq 11$ and $k_{\text{hor}} \geq 0.12$), $H_{\max}$ is equal to F_{top}^u . But when k_{span} is small or k_{hor} is weak, $H_{\max}$ can be reduced to 50% of F_{top}^u .

For the single column loss scenario, K_A and R_A will be calculated based on structural properties of adjacent column members. If $R_A \geq H_{\max}$, then catenary action can be fully developed following all the equations of the proposed model. But if $R_A < H_{\max}$, catenary action is neglected after all the bottom rebars have fractured and EFG will be replaced by a horizontal line EG, similar to the case of relatively small k_{hor} presented in Figure 8(c).

Parameters of the primary controlling points (load and corresponding displacement, etc.) derived from the proposed model are compared to values obtained from finite-element parametric studies (Table 1). Results are presented in Appendix 2. Suffice to say, acceptable agreement is obtained between semi-analytical and finite-element predictions.

Dynamic assessment of sub-assemblages under missing column scenario

Once the static non-linear behaviour of the sub-assemblage is already defined, the dynamic effect can be easily achieved by employing the energy-based approach by Izzuddin *et al.* (2008). Based on energy conservation, this method transforms

the non-linear static curve into a pseudo-static response which represents the maximum dynamic displacement for a certain value of dynamic load. The maximum value from the pseudo-static curve represents the ultimate progressive collapse load that the structure can sustain. Employing this approach to the proposed simplified static polyline, there are three possible peaks of dynamic response at displacements d_B , d_D and d_G (Figure 9). The equivalent dynamic loads at these deflections, which are P_B^d , P_D^d and P_G^d , are derived by Equations 21–23. The largest value from these three dynamic loads is the ultimate dynamic capacity of the sub-assemblage.

$$21. \quad P_B^d = [P_A d_A + (P_B + P_A)(d_B - d_A)]/2d_B$$

$$22. \quad P_D^d = [P_A d_A + (P_B + P_A)(d_B - d_A) + (P_C + P_B)(d_{C1} - d_B) + 2P_C(d_{D1} - d_{C1}) + (P_D + P_{D1})(d_D - d_{D1})]/2d_D$$

$$23. \quad P_G^d = [P_A d_A + (P_B + P_A)(d_B - d_A) + (P_C + P_B)(d_{C1} - d_B) + 2P_C(d_{D1} - d_{C1}) + (P_D + P_{D1})(d_D - d_{D1}) + P_E(d_F - d_E) + (P_G + P_F)(d_G - d_F)]/2d_G$$

In summary, the solution procedure to obtain the simplified semi-analytical method is presented in Figure 10.

Test validations using the semi-analytical model

To verify the application of the proposed model for catenary action and structural response, experimental results include seven tests from Yu and Tan (2013a, 2013b), and two tests from Lim *et al.* (2015) are also used to construct the structural response using the equations and solution procedure described in the previous section. Other tests on two-dimensional

Figure 9. Dynamic assessment for the simplified response: (a) dynamic load P_B^d at d_B ; (b) dynamic load P_D^d at d_D ; (c) dynamic load P_G^d at d_G

Figure 10. Procedure of the simplified model

beam–column structures (Sadek *et al.*, 2011; Su *et al.*, 2009; Yi *et al.*, 2008) are not used for the validation, as the stiffness of horizontal restraints was not directly measured in these tests. Input information as well as all necessary variables for deriving the model are presented in Tables 2 and 3. Comparisons between semi-analytical and test results are presented in Figure 11. The proposed model provides reasonably good predictions of structural response, and yet requires much less effort compared to detailed FEM simulations. For specimen PR from Lim *et al.* (2015), because the test was terminated at a deflection of 396 mm before any fracture of top longitudinal rebars occurred, the response is defined up to this displacement only. In most of the tests, lateral boundary conditions for both sides were well restrained by either the strong wall or the A-frame. Hence, the ultimate strength R_A of these restraints is considered to be relatively large. However, for specimen PR, one side of

the frame was partially restrained while the other was fully restrained. The actual test results showed no enhancement from catenary action after the fracture of rebars at the middle joint. During the test, horizontal reaction of the partially restrained side gradually increased to 54 kN before the adjacent column was pulled inwards, indicating the yielding state of the restraint. Thus, in the simplified model, the strength of horizontal restraint R_A is limited to 54 kN, which is smaller than $H_{\max}$ calculated from Equation 20 (107.9 kN). As a result, the simplified response of specimen PR after the fracture of the bottom rebars is a horizontal line EG (Figure 8(c)), which shows good agreement with the actual test result.

By assuming perfectly rigid rotational restraint, the proposed model neglects rotational deformation at the end joint, and therefore tends to overestimate the stiffness of the substructure

Table 2. Test data and proposed model parameters

Test		$F_{a,y}$: kN	K_A : KN/m	k_{span}	k_{hor}	k_{flex}	k_{topbar}	α_1	α_2	β_1	β_2	Middle joint		End joint		Mid-span ϕ_y^c : m ⁻¹
												ϕ_u' : m ⁻¹	ϕ_y' : m ⁻¹	ϕ_u : m ⁻¹	ϕ_y : m ⁻¹	
Yu and Tan (2013a, 2013b)	S1	148	90 000	11.0	0.237	0.0117	0.0600	0.76	0.99	3.30	0.77	0.64	0.0325	0.546	0.0325	0.0325
	S2	120	140 000	11.0	0.372	0.0104	0.0488	0.70	0.99	3.40	0.77	0.64	0.0338	0.6	0.0338	0.0338
	S3	199	145 000	11.0	0.375	0.0140	0.0825	0.70	0.95	3.10	0.65	0.64	0.0357	0.549	0.0357	0.0357
	S4	199	145 000	11.0	0.369	0.0164	0.0825	0.71	0.92	3.10	0.65	0.64	0.0325	0.6	0.0325	0.0325
	S5	199	145 000	11.0	0.361	0.0193	0.0825	0.71	0.89	3.10	0.65	0.655	0.0325	0.655	0.0325	0.0325
	S6	309	145 000	11.0	0.357	0.0213	0.1249	0.74	0.86	2.72	0.49	0.64	0.0325	0.549	0.0325	0.0325
	S7	199	145 000	8.6	0.288	0.0128	0.0825	0.88	0.81	2.52	0.35	0.6	0.0325	0.6	0.0325	0.0325
Lim <i>et al.</i> (2015)	FR	118	20 000	12.3	0.083	0.0221	0.1016	0.88	1.05	3.25	0.93	0.774	0.0487	0.774	0.0487	0.0487
	PR	118	5000	12.3	0.021	0.0221	0.1016	0.94	1.19	3.25	1.09	0.774	0.0487	0.774	0.0487	0.0487

Table 3. Controlling points from the proposed model

Test		Point A		Point B		Point C ₁		Point C		Point D ₁		Point D		Point E		Point F		Point G	
		d_A	P_A	d_B	P_B	d_{C1}	P_{C1}	d_C	P_C	d_{D1}	P_{D1}	d_D	P_D	d_E	P_E	d_F	P_F	d_G	P_G
Yu and Tan (2013a, 2013b)	S1	17.7	33.6	60.0	43.7	181.9	33.6	314.2	33.6	314.2	33.6	314.7	33.7	314.7	21.9	214.7	21.9	578.8	68.8
	S2	14.3	29.5	70.0	38.4	185.4	29.5	312.2	29.5	312.0	29.5	337.8	33.4	337.8	21.6	337.8	21.6	629.6	65.0
	S3	19.8	40.9	57.5	53.7	195.6	40.9	280.0	40.9	280.0	40.9	280.0	40.9	280.0	29.3	283.5	29.3	629.9	105.0
	S4	23.8	48.9	55.0	61.1	168.5	48.9	292.5	48.9	292.5	48.9	378.2	68.1	292.5	48.5	378.2	48.5	666.9	113.3
	S5	28.6	58.6	55.0	69.0	151.1	58.6	292.8	58.6	292.8	58.6	434.2	90.3	292.8	61.0	434.2	61.0	718.2	124.7
	S6	32.1	65.6	55.0	70.9	159.8	65.6	271.9	65.6	271.9	65.6	337.0	85.5	271.9	65.5	337.0	65.5	665.3	165.7
	S7	15.2	62.5	52.5	78.6	160.0	62.5	246.6	62.5	246.6	62.5	246.6	62.5	246.6	37.5	246.8	37.5	508.5	98.5
Lim <i>et al.</i> (2015)	FR	32.9	25.8	52.6	28.6	118.8	25.8	257.8	25.8	257.8	25.8	293.8	32.0	257.8	21.7	293.8	21.7	552.0	66.1
	PR	35.4	25.8	54.0	27.0	142.1	25.8	286.4	25.8	286.4	25.8	293.8	27.1	286.4	16.7	293.8	16.7	400.0	16.7

Note: unit of displacement (d_A to d_G) – mm; unit of applied load (P_A to P_G) – kN

Figure 11. Test validation using the proposed model

at the initial stage (part OA). As a result, the values of d_A calculated from the model are smaller than those from actual tests. Nonetheless, this assumption has little bearing on the total strain energy of the static response, which is important for dynamic assessment of progressive collapse. In terms of maximum displacement and corresponding ultimate load capacity (point G), the simplified model provides relatively good agreement with all the test results (Table 4). A comparison study between the model predictions and test results on maximum reaction force $H_{\max}$ at point G and maximum equivalent dynamic load using the Izzuddin method show very good agreement (Table 4).

Discussion on the semi-analytical model for sub-assemblages

Limits of application

Since the simplified model is developed based on FEM analyses, there are some assumptions that should be clearly stated when applying this approach. First of all, the model can only be used for structures with concrete cylinder strength ranging

from 20 to 50 MPa. Structures using higher strength or fibre concrete are outside the scope of this research. Second, the longitudinal reinforcement consists of deformed bars with yield strength up to 600 MPa. It is assumed that, at the beam sections near the end joint and the middle joint, the ratio of top bars is larger than bottom bars, which is generally satisfied in normal RC gravity design. The anchorage length of reinforcement into the joint regions follows code provisions, for example Eurocode 2 (BS EN 1992-1-1:2004; BSI, 2004), to ensure good bonding between concrete and reinforcement. In terms of structural geometry, the model is only valid when $k_{\text{span}} \geq 7$ where flexure dominates behaviour instead of shear. With regard to catenary action developed in the double-span beam after the fracture of the bottom reinforcement, it is shown that catenary action is only significant when the requirements for horizontal stiffness (Equation 19) and strength (Equation 20) are satisfied. Concerning rotational stiffness, k_{rot} has to satisfy Equation 8 to ensure rotational resistance at the end joint will not affect catenary action. This requirement was easily fulfilled in the tests for both sub-assemblages (Yu and Tan, 2013a, 2013b) and beam-column frames (Lim *et al.*, 2015). The proposed model

Table 4. Test validation results

Test	Maximum displacement: mm			Maximum static applied load: kN			F_u^a : kN	Maximum horizontal reaction H_{max} : kN			Maximum dynamic load: kN		
	Test			Test				Test			Test		
	Analytical	Analytical/test	Analytical/test	Analytical	Analytical/test	Analytical/test		Analytical	Analytical/test	Analytical/test	Analytical	Analytical/test	Analytical/test
Yu and Tan (2013a, 2013b)	S1	573	578.8	1.01	68.9	68.8	200	170	188	1.11	34.2	39.9	1.17
	S2	612	629.6	1.03	67.6	65.0	172	156	163	1.05	35.3	36.9	1.05
	S3	729.3	626.9	0.86	124.4	105.0	236	228	224	0.98	62.5	56.5	0.90
	S4	614.3	666.9	1.09	103.7	113.3	236	217	224	1.03	65.1	64.7	0.99
	S5	655.9	718.2	1.09	105.1	124.7	236	225	224	1.00	70.8	75.1	1.06
	S6	675.3	665.3	0.99	139.9	165.7	369	346	350	1.01	88.5	90.8	1.03
Lim et al. (2015)	S7	628.5	508.5	0.81	106.0	98.5	236	219	211	0.96	74.6	68.6	0.92
	FR	530	552.0	1.04	64.4	66.1	141	152	125	0.82	34.5	34.1	0.99
	PR	N/A			N/A			N/A			26.3	24.7	0.94
			Mean			Mean			Mean			Mean	1.00
			COV			COV			COV			COV	0.08
			0.99			1.03			0.99				0.08
			0.10			0.11			0.08				0.08

N/A, Specimen PR is not taken into account due to early testing termination; COV, coefficient of variation

neglects connection gaps in the restraint boundaries. Such gaps have little effect on the final catenary action part (FG); they only affect the first peak of compressive arch action (point B).

Discussion on symmetric conditions

The simplified model is based on half-specimen simulations, assuming symmetry for geometry, material and boundary conditions. In practice, it is likely that the restraint stiffness values (K_A and R_A) from two sides of the two-span beam structure are not the same (e.g. the number of adjacent bays on each side is different). In this case, the smaller horizontal stiffness value should be applied.

In circumstances where geometry and material properties from two sides of the double-span beam are not identical, or the joint may not be located at the middle of the double span, then the total behaviour of the sub-assembly P_{total} may be approximated as the average of the simplified responses from two symmetric systems which are similar to the single-span structures on the left and the right sides of the removed column (Figure 12(a)), the so-called ‘left’ and the ‘right’ systems, corresponding to the same middle-joint displacement, d , as shown in Equation 24.

$$24. \quad P_{total}(d) = \frac{P_{left}(d) + P_{right}(d)}{2}$$

However, $P_{left}(d)$ and $P_{right}(d)$ are not totally independent. They are coupled with each other at specific displacements when rebars fracture. Owing to differences in structural response, the displacements corresponding to top and bottom rebar fractures of the ‘left’ and the ‘right’ systems are not the same. Figure 12(b) shows an example of the two simplified responses in which the bottom rebars of the ‘left’ system fracture earlier compared to those of the ‘right’ system ($d_{D,L} < d_{D,R}$), and vice versa regarding the fracture of the top rebars ($d_{G,L} > d_{G,R}$).

To simply combine the responses of the two systems into the approximate total response, P_{total} , it is assumed that after the fracture of the bottom rebars from the ‘left’ system (at $d = d_{D,L}$), the bottom rebars from the ‘right’ system are also considered to have fractured. Therefore, the behaviour of the ‘right’ system will change from $D_{1,R}D_R E_R F_R$ to $D_{1,R}D_R E_R F_R$, as shown in Figure 12(b). Similarly, after the top rebars from the ‘right’ system have fractured (at $d = d_{G,R}$) causing a total failure to this structure, behaviour of the ‘left’ system is also considered to have terminated and segment $F_L G_L$ is replaced by $F_L G_L'$, as represented in Figure 12(b).

To verify this coupling method for double-span beams with different geometry/material properties from the two sides, FEM analyses are conducted for an example shown in

Figure 12. Combined response from two sides of the sub-assembly: (a) the 'left' and the 'right' systems; (b) simplified responses of the 'left', the 'right' and the total systems

Figure 13(a). Three simulations are performed representing responses from the 'left', the 'right' and the 'total' structures, and are compared with predictions from the simplifying approach using the coupling of rebar fracture from the 'left' and the 'right' systems (Figure 13(b)). It is shown that the simplified method gives a conservative result compared to FEM prediction.

Conclusions

The behaviour of beam-column structures under a single column removal scenario is greatly affected by the restraint stiffness of boundary conditions. In this paper, parametric studies based on numerical analyses are developed to gain a better understanding of the sub-assembly structural resistance, especially the development and contribution of catenary action before and after the fracture of the bottom rebars at the middle joint. It is shown that, while rotational stiffness only affects the initial compressive arch action and has much less influence on the remaining response, horizontal stiffness has a greater effect on both compressive arch action and subsequent catenary action. Other factors such as concrete strength, connection gap or bottom reinforcement in beams has little effect on the subsequent catenary action phase. The assumption of elastic stiffness for horizontal restraint simplifies the analysis but is difficult to achieve in practice. Instead, if the strength of

the horizontal restraint is smaller than the maximum horizontal reaction required by the sub-assembly, then catenary action will not be fully mobilised after the fracture of the bottom rebars.

The proposed semi-analytical model provides a quick means for estimating the overall behaviour of beam-column sub-assemblies under concentrated loading condition. It simplifies the non-linear response into a piecewise multi-linear curve with some major points to be determined through equations. The assumptions of relatively good rotational restraint and bonding condition yield reasonable predictions as the model has been validated by test results. Strain energy as well as the maximum dynamic load can then be easily obtained from the simplified static response. Although the model is developed based on symmetric assumptions of structural and boundary conditions, it can be conservatively applied for non-symmetric sub-assemblies.

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Figure 13. Example of sub-assembly with two different sides: (a) double-span structure; (b) FEM and simplified predictions

Appendix 1: derivations of displacements at points D and G

Point D

At point D, it is assumed that the beam sections at the middle and the end joints remain the maximum plastic moment capacities, M_{mid} and M_{end} , besides the development of catenary action (Figure 6(a)). To calculate the plastic hinge lengths near the middle joint (L_{sp}^{mid}) and near the end joint (L_{sp}^{end}), the Paulay and Priestley (1992) method is applied. However, the equation of plastic hinge length from this method is proposed for a cantilever column subject to horizontal displacement at the free end. To employ this equation, therefore, equivalent cantilever spans, L_n^{mid} and L_n^{end} are derived, which represent

the beam spans of segments XY and YZ (Figure 6(a)). As a result, L_n^{mid} and L_{sp}^{mid} can be computed from Equation 25. Thereafter, plastic hinge lengths L_{sp}^{mid} and L_{sp}^{end} are defined using Equations 26 and 27, respectively. Subsequently, yield and plastic displacements of segment XY can be obtained by Equations 28 and 29, respectively.

$$25. \quad \frac{L_n^{mid}}{L_n^{end}} = \frac{M_{mid}}{M_{end}}$$

$$26. \quad L_{sp}^{mid} = 0.08L_n^{mid} + 0.022d_{mid}f_y$$

$$27. \quad L_{sp}^{end} = 0.08L_n^{mid} + 0.022d_{end}f_y$$

where d_{mid} and d_{end} are the diameters of longitudinal tensile reinforcement at the middle and the end joints.

$$28. \quad \Delta_y^{mid} = \frac{\phi'_y L_n^{mid^2}}{3}$$

$$29. \quad \Delta_u^{mid} = (\phi'_u - \phi'_y) L_{sp}^{mid} (L_n^{mid} - 0.5L_{sp}^{mid})$$

where ϕ'_y and ϕ'_u are the yield and ultimate curvatures of the concrete beam section at the middle joint under sagging moment. These values depend on the cross-sectional geometry and material properties of the concrete and reinforcement stress-strain curve. Park and Paulay (1975) have developed a rational method to calculate these values to construct the moment-curvature response of a flexural section. Nowadays, there are several convenient tools to obtain this response curve. In this study, a cross-sectional design tool integrated in the structural analysis software SAP 2000 (Habibullah and Wilson, 2005) is employed to determine the yield and ultimate values of curvature at different beam sections based on the material and geometry properties. The input data used in SAP 2000 are based on concrete cylinder strength, yield strength and ultimate tensile strength of reinforcement with corresponding strains. The simple parametric stress-strain curves of the reinforcement and of the unconfined concrete model are automatically generated by SAP 2000. After obtaining Δ_{XY} , Δ_{YZ} is defined using Equation 30.

$$30. \quad \frac{\Delta_{XY}}{\Delta_{YZ}} = \frac{L_n^{mid}}{L_n^{end}}$$

Point G

In terms of vertical loading resistance, the beam at this stage behaves as a cantilever with the span equal to L_n . Hence, the method from Paulay and Priesley (Equation 31) can be directly applied to calculate the plastic hinge length at the end joint. All the components of d_G (Δ_y , Δ_u and Δ_c) are defined by Equations 32, 33 and 34, respectively.

$$31. \quad L_{sp} = 0.08L_n + 0.022d_{end}f_y$$

$$32. \quad \Delta_y = \frac{\phi_y L_n^2}{3}$$

$$33. \quad \Delta_u = (\phi_u - \phi_y) L_{sp} (L_n - 0.5L_{sp})$$

$$34. \quad \Delta_c = \int_{c_1}^{c_2} \phi_y^c (L_n - x) dx$$

where ϕ_u and ϕ_y are the yield and the ultimate curvatures of the concrete beam section at the end joint under hogging moment. ϕ_y^c is the yield curvature of the concrete beam section at the curtailment region under hogging moment. c_1 and c_2 denote locations of the curtailment region (Figure 7(b)). The method for deriving ϕ_y , ϕ_u and ϕ_y^c is similar to ϕ'_y and ϕ'_u .

During the validation of the proposed model by the FEM predictions, it is found that the model-predicted displacements at G have good agreement with FEM values if $k_{span} > 9$. If $k_{span} \leq 9$, however, the proposed equation provides smaller values compared to FEM results. The reason is that tensile forces are ignored in the beams. Previous studies have shown that axial tensile force in columns can also help in increasing the ductility of the plastic hinge, especially when the span-depth ratio is small. To overcome this shortcoming, a modification to Equation 31 for determining the plastic hinge length is made when $k_{span} \leq 9$. In such situations, L_{sp} will be derived by Equation 35.

$$35. \quad L_{sp} = \text{maximum}(0.08L_n, d_{beam}) + 0.022d_{end}f_y$$

Appendix 2: comparisons between the simplified model and the FEM results

Parameters derived from the proposed model such as displacements and loads at controlling points are compared to FEM predictions of 120 cases (Table 1). Basically, the values from the proposed model are either close to, or more conservative than, the numerical predictions.

For point A

A comparison of d_A between the proposed model and the corresponding FEM displacement for the same applied load P_A is shown in Figure 14(a).

For point C

Results of P_C and d_C are compared between FEM analyses and the proposed equations. The comparisons show relatively good agreement. Values from the simplified model are more conservative than FEM predictions in most cases (Figures 14(b) and 14(c)).

For line FG

In Equation 14, $(\beta_1 F_{top}^y / L_n)$ represents the gradient of FG while $(\beta_2 d_{beam})$ is the root of the equation $P_{FG}(d) = 0$, which represents the location of FG. Comparisons of the gradient

Figure 14. Comparisons between simplified and FEM models: (a) d_A (mm) (mean = 0.94, COV = 0.11); (b) d_C (mm) (mean = 1.03, COV = 0.09); (c) P_C (kN) (mean = 0.89, COV = 0.07); (d) gradient of line FG (kN/mm) (mean = 1.02, COV = 0.11); (e) root of line FG (mm) (mean = 1.12, COV = 0.18); (f) P_E/P_D (mean = 1.05, COV = 0.10); (g) d_D (mm) (mean = 1.01, COV = 0.12); (h) d_G (mm) (mean = 0.92, COV = 0.17); (i) H_{max} (kN) (mean = 1.01, COV = 0.11)

and the root from the function of line FG between numerical and simplified methods are carried out and illustrated in Figures 14(d) and 14(e). The predicted values for the gradient are relatively close to the FEM results. However, the proposed model gives more conservative values of the root compared to FEM when k_{hor} is relatively small. Thus, the use of this model will provide safe results for the case of weak axial restraints.

For point E

Figure 14(f) describes a comparison of P_E/P_D between the numerical predictions and the results from Equation 18. The errors are within an acceptable range.

Displacement at D and G

Comparisons are made between FEM and analytical results for the displacements d_D and d_G . The results are shown in

Figures 14(g) and 14(h). Basically, the simplified method has good agreement with numerical predictions for d_D and gives more conservative results for d_G .

Maximum tensile reaction, $H_{\max}$

The analytical values of $H_{\max}$ show relatively good agreement with those computed by FEM (Figure 14(i)).

REFERENCES

- Bao Y, Lew HS and Kunnath SK (2014) Modeling of reinforced concrete assemblies under column-removal scenario. *Journal of Structural Engineering* **140**(1): 04013026.
- BSI (2004) BS EN 1992-1-1: 2004: Eurocode 2: Design of concrete structures – general rules and rules for buildings. BSI, London, UK.
- DOD (Department of Defense) (2013) *Unified Facilities Criteria (UFC) 4-023-03. Design of Buildings to Resist Progressive Collapse*. DOD, Washington, DC, USA.
- GSA (General Services Administration) (2003) *Progressive Collapse Analysis and Design Guidelines for New Federal Office Buildings and Major Modernization Projects*. GSA, Washington, DC, USA.
- Habibullah A and Wilson E (2005) *SAP 2000 'Static and Dynamic Finite Element Analysis of Structures'*. Computers and Structures Inc., Berkeley, CA, USA.
- Hallquist JO (2007) *LS-DYNA Keyword User's Manual Version 971*. Livermore Software Technology Corporation, Livermore, CA, USA.
- Izzuddin B and Nethercot D (2009) Design-oriented approaches for progressive collapse assessment: load-factor vs ductility-centred methods. In *Proceedings of Structures Congress 2009: Don't Mess with Structural Engineers: Expanding Our Role*, Austin, TX, USA (Griffis L, Helwig T, Waggoner M and Holt M (eds)). American Society of Civil Engineers (ASCE), Reston, VA, USA, pp. 1–10.
- Izzuddin B, Vlassis A, Elghazouli A and Nethercot D (2008) Progressive collapse of multi-storey buildings due to sudden column loss—part I: simplified assessment framework. *Engineering Structures* **30**(5): 1308–1318.
- Li Y, Lu X, Guan H and Ye L (2014) Progressive collapse resistance demand of reinforced concrete frames under catenary mechanism. *ACI Structural Journal* **111**(5): 1225–1234.
- Lim NS, Lee CK and Tan KH (2015) Experimental studies on 2-D RC frame with middle column removed under progressive collapse. *Proceedings of fib Symposium 2015, Copenhagen, Denmark*.
- Lu X, Lin K, Li Y et al. (2016) Experimental investigation of RC beam–slab substructures against progressive collapse subject to an edge-column-removal scenario. *Engineering Structures*, <http://dx.doi.org/10.1016/j.engstruct.2016.07.039>.
- Park R and Gamble WL (2000) *Reinforced Concrete Slabs*. John Wiley & Sons, Hoboken, NJ, USA.
- Park R and Paulay T (1975) *Reinforced Concrete Structures*. John Wiley & Sons, Hoboken, NJ, USA.
- Paulay T and Priestley M (1992) *Seismic Design of Reinforced Concrete and Masonry Buildings*. Wiley, New York, NY, USA.
- Pham AT, Tan KH and Yu J (2016) Numerical investigations on static and dynamic responses of reinforced concrete sub-assemblages under progressive collapse. *Engineering Structures*, <http://dx.doi.org/10.1016/j.engstruct.2016.07.042>.
- Qian K, Li B and Zhang Z (2015) Testing and simulation of 3D effects on progressive collapse resistance of RC buildings. *Magazine of Concrete Research* **67**(4): 163–178, <http://dx.doi.org/10.1680/mac.14.00178>.
- Ren P, Li Y, Lu X, Guan H and Zhou Y (2016) Experimental investigation of progressive collapse resistance of one-way reinforced concrete beam–slab substructures under a middle-column-removal scenario. *Engineering Structures* **118**: 28–40, <https://doi.org/10.1016/j.engstruct.2016.03.051>.
- Sadek F, Main JA, Lew HS and Bao Y (2011) Testing and analysis of steel and concrete beam–column assemblies under a column removal scenario. *Journal of Structural Engineering* **137**(9): 881–892.
- Sasani M and Kropelnicki J (2008) Progressive collapse analysis of an RC structure. *The Structural Design of Tall and Special Buildings* **17**(4): 757–771.
- Shi Y, Li ZX and Hao H (2009) Bond slip modelling and its effect on numerical analysis of blast-induced responses of RC columns. *Structural Engineering and Mechanics* **32**(2): 251–267.
- Su Y, Tian Y and Song X (2009) Progressive collapse resistance of axially-restrained frame beams. *ACI Structural Journal* **106**(5): 600–607.
- Valipour HR and Foster SJ (2010) Finite element modelling of reinforced concrete framed structures including catenary action. *Computers and Structures* **88**(9): 529–538.
- Valipour H, FarhangVesali N and Foster S (2013) A generic model for investigation of arching action in reinforced concrete members. *Construction and Building Materials* **38**: 742–750, <https://doi.org/10.1016/j.conbuildmat.2012.09.046>.
- Yi WJ, He QF, Xiao Y and Kunnath SK (2008) Experimental study on progressive collapse-resistant behavior of reinforced concrete frame structures. *ACI Structural Journal* **105**(4): 433–439.
- Yu J and Tan KH (2013a) Experimental and numerical investigation on progressive collapse resistance of reinforced concrete beam column sub-assemblages. *Engineering Structures* **55**: 90–106, <https://doi.org/10.1016/j.engstruct.2011.08.040>.
- Yu J and Tan KH (2013b) Structural behavior of RC beam–column subassemblages under a middle column removal scenario. *Journal of Structural Engineering* **139**(2): 233–250.
- Yu J and Tan KH (2014) Analytical model for the capacity of compressive arch action of reinforced concrete sub-assemblages. *Magazine of Concrete Research* **66**(3): 109–126, <http://dx.doi.org/10.1680/mac.13.00217>.

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