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Existence of Solutions of Higher Order Sturm-Liouville Boundary Value Problems

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Abstract. We consider a higher order Sturm-Liouville boundary value problem (BVP) where the nonlinear term involves derivatives – such cases are seldom investigated in the literature due to the complexity introduced by the derivatives. A *new* technique is used to obtain existence criteria for at least one positive solution of the BVP. We demonstrate the usefulness of the results obtained by an example with *known* positive solutions.

INTRODUCTION

We consider the higher order Sturm-Liouville boundary value problem (BVP)

$$\begin{cases} y^{(m)}(t) + F(t, y(t), y'(t), \dots, y^{(q)}(t)) = 0, & t \in [0, 1] \\ y^{(k)}(0) = 0, & 0 \leq k \leq m-3; \quad \zeta y^{(m-2)}(0) - \theta y^{(m-1)}(0) = 0, \quad \rho y^{(m-2)}(1) + \delta y^{(m-1)}(1) = 0 \end{cases} \quad (1)$$

where $m \geq 3$, $1 \leq q \leq m-2$ and F is continuous at least in the domain of interest. The constants ζ , θ , ρ and δ are such that $\theta \geq 0$, $\delta \geq 0$, $\theta + \zeta > 0$, $\delta + \rho > 0$, and $\kappa \equiv \zeta\rho + \zeta\delta + \theta\rho > 0$. These assumptions allow ζ and ρ to be negative.

There is a vast amount of research done on the existence of positive solutions of Sturm-Liouville BVPs, possibly due to its application in the modelling of many physical phenomena. For recent developments in (1) as well as in other types of BVPs, the reader is referred to the monographs [2, 3] and the hundreds of references cited therein. We note that in most of these investigations the nonlinear terms considered do *not* involve derivatives of the dependent variable, only a relatively small number of papers tackle nonlinear terms that involve derivatives.

For example, Fink [6] has studied the radial symmetric form of the semilinear elliptic equation $\Delta y + \lambda q(|x|)f(y) = 0$ in $\mathbb{R}^N$, which turns out to be a particular second order Sturm Liouville eigenvalue problem that has y' in the nonlinear term, viz.,

$$y'' + \frac{N-1}{t} y' + \lambda q(t)f(y) = 0, \quad t \in (0, 1); \quad y'(0) = y(1) = 0.$$

Later, Wong [9] has considered (1) when $q = m-2$, the existence of a solution (not necessarily positive) is obtained by assuming that (1) has lower and upper solutions v and w such that $v^{(m-2)}(t) \leq w^{(m-2)}(t)$ on $[0, 1]$, and F is monotonically increasing with respect to v and w in some sense. Thereafter, Grossinho and Minhós [8] established the existence of a solution to a related problem of (1) when $q = m-1$, their method requires again the assumption of lower and upper solutions, in addition F must satisfy *Nagumo-type condition* on some set $A \subset [0, 1] \times \mathbb{R}^m$, viz., there exists a continuous function $h : [0, \infty) \rightarrow (0, \infty)$ such that $|F(t, u_1, \dots, u_m)| \leq h(|u_m|)$ for $(t, u_1, \dots, u_m) \in A$; and $\int_0^\infty \frac{s}{h(s)} ds = \infty$. The relatively few number of papers on problems involving derivative-dependent nonlinearities indicates that problems of this type are more difficult to tackle analytically.

Motivated by the above research, in the current work we shall develop a different and new technique to tackle BVP (1). Our technique requires *neither* the existence of lower and upper solutions *nor* the assumption of Nagumo-type condition, both of these conditions are not easy to check in practical applications. The focus of this paper is on the existence of at least one positive solution of (1). By a *positive solution* y of (1), we mean $y \in C^{(m)}[0, 1]$ satisfying (1) and $y(t) \geq 0$ for $t \in [0, 1]$. Due to the presence of derivatives in the nonlinear term, our present work

naturally generalizes and extends the known results for Sturm Liouville BVPs [1, 4, 5, 6, 7, 10, 11, 12, 13] as well as complements the work on other types of BVPs in the literature. We remark that our conditions/assumptions, which do not involve lower and upper solutions and Nagumo-type condition, are comparatively easy to check – this practical usefulness will be illustrated by an example with *known* positive solutions.

EXISTENCE CRITERIA

We shall first state some properties of certain Green's function which are important in deriving the main results.

Lemma 1. [10, 12, 13] Let $G(t, s)$ be the Green's function of the *second order* Sturm-Liouville BVP

$$-w''(t) = 0, \quad t \in (0, 1); \quad \zeta w(0) - \theta w'(0) = 0, \quad \rho w(1) + \delta w'(1) = 0. \quad (2)$$

- (a) $G(t, s) \geq 0$ for $(t, s) \in [0, 1] \times [0, 1]$ and $G(t, s) > 0$ for $(t, s) \in (0, 1) \times (0, 1)$.
- (b) $G(t, s) \leq LG(s, s)$ for $(t, s) \in [0, 1] \times [0, 1]$ where $L = \max \{1, \theta/(\theta + \zeta), \delta/(\delta + \rho)\}$.
- (c) $G(t, s) \geq K_\eta G(s, s)$ for $(t, s) \in [\eta, 1 - \eta] \times [0, 1]$, where $\eta \in (0, \frac{1}{2})$ is fixed and $K_\eta = \min\{(\delta + \rho\eta)/(\delta + \rho), [\delta + \rho(1 - \eta)]/(\delta + \rho\eta), (\theta + \zeta\eta)/(\theta + \zeta), [\theta + \zeta(1 - \eta)]/(\theta + \zeta\eta)\}$.
- (d) The Green's function of the following *n-th order* Sturm-Liouville BVP is $g_n(t, s)$, where $D_t^{n-2}g_n(t, s) = G(t, s)$,

$$-w^{(n)}(t) = 0, \quad t \in (0, 1); \quad w^{(k)}(0) = 0, \quad 0 \leq k \leq n-3; \quad \zeta w^{(n-2)}(0) - \theta w^{(n-1)}(0) = 0, \quad \rho w^{(n-2)}(1) + \delta w^{(n-1)}(1) = 0.$$

To begin the investigation, consider the initial value problem

$$y^{(q)}(t) = x(t), \quad t \in [0, 1]; \quad y(0) = y'(0) = y''(0) = \dots = y^{(q-1)}(0) = 0. \quad (3)$$

which leads to the representation

$$y^{(k)}(t) = \int_0^t \int_0^{s_1} \int_0^{s_2} \dots \int_0^{s_{q-k-1}} x(s_{q-k}) ds_{q-k} \dots ds_1 = J^{q-k}x(t), \quad 0 \leq k \leq q \quad (4)$$

where $J^{q-k}x$ denotes the $(q-k)$ -time integral of x , with $J^0x(t) \equiv x(t)$.

In view of (3) and (4), we rewrite (1) to get the following $(m-q)$ -th order Sturm-Liouville BVP

$$\begin{cases} x^{(m-q)}(t) + F(t, J^q x(t), J^{q-1}x(t), \dots, Jx(t), x(t)) = 0, & t \in [0, 1] \\ x^{(k)}(0) = 0, & 0 \leq k \leq m-q-3; \quad \zeta x^{(m-q-2)}(0) - \theta x^{(m-q-1)}(0) = 0, \quad \rho x^{(m-q-2)}(1) + \delta x^{(m-q-1)}(1) = 0. \end{cases} \quad (5)$$

If (5) has a solution x^* , then the BVP (1) has a solution y^* given by

$$y^*(t) = J^q x^*(t) = \int_0^t \int_0^{s_1} \int_0^{s_2} \dots \int_0^{s_{q-1}} x^*(s_q) ds_q \dots ds_1. \quad (6)$$

Hence, the existence of a solution of (1) follows from the existence of a solution of (5). If x^* is nontrivial/positive, so is $y^{*(k)}$, $0 \leq k \leq q$. We shall study (1) via (5).

Let the Banach space $B = \{x \in C^{(m-q)}[0, 1] \mid x^{(k)}(0) = 0, \quad 0 \leq k \leq m-q-3\}$ be equipped with the norm $\|x\| = \sup_{t \in [0, 1]} |x^{(m-q-2)}(t)|$. Throughout, let $\eta \in (0, \frac{1}{2})$ be fixed. Define the cone C in B by $C = \{x \in B \mid x^{(m-q-2)}(t) \geq 0, \quad t \in [0, 1]; \min_{t \in [\eta, 1-\eta]} x^{(m-q-2)}(t) \geq \gamma \|x\|\}$ where $\gamma = K_\eta/L$.

Our main tool is *Krasnosel'skii's fixed point theorem in a cone*. So let the operator $S : B \rightarrow B$ be defined by

$$Sx(t) = \int_0^1 g_{m-q}(t, s) F(s, J^q x(s), J^{q-1}x(s), \dots, Jx(s), x(s)) ds, \quad t \in [0, 1]. \quad (7)$$

Noting Lemma 1(d), it is clear that a fixed point of S is a solution of (5). To obtain a positive solution of (1), we shall seek a fixed point of the operator S in the cone C .

The conditions that are used in the main results are listed below. In these conditions, $\tilde{u} = (u_1, \dots, u_{q+1})$, $\tilde{K} = \{u \in C[0, 1] \mid u(t) \geq 0, \quad t \in [0, 1]\}$ and $K = \{u \in \tilde{K} \mid u(t) > 0 \text{ on some subset of } [0, 1] \text{ of positive measure}\}$.

- (A1) F is continuous on $[0, 1] \times \tilde{K}^{q+1}$, with $F(t, \tilde{u}) \geq 0$ for $(t, \tilde{u}) \in [0, 1] \times \tilde{K}^{q+1}$; and $F(t, \tilde{u}) > 0$ for $(t, \tilde{u}) \in [0, 1] \times K^{q+1}$.
(A2) There exist continuous functions β and f with $\beta : [0, 1] \rightarrow [0, \infty)$, $f : [0, \infty)^{q+1} \rightarrow [0, \infty)$ and f is nondecreasing in each of its arguments, such that $F(t, \tilde{u}) \leq \beta(t)f(\tilde{u})$ for $(t, \tilde{u}) \in [0, 1] \times \tilde{K}^{q+1}$.
(A3) There exists $a > 0$ such that $a > Mf\left(\frac{a}{(m-2)!}, \frac{a}{(m-3)!}, \dots, \frac{a}{(m-q-2)!}\right)$ where $M = \sup_{t \in [0, 1]} \int_0^1 G(t, s)\beta(s)ds$.
(A4) Let $z \in (\eta, 1 - \eta)$ be fixed. There exists a continuous function α with $\alpha : [z, 1 - \eta] \rightarrow (0, \infty)$, such that $F(t, \tilde{u}) \geq \alpha(t)f(\tilde{u})$ for $(t, \tilde{u}) \in [z, 1 - \eta] \times K^{q+1}$.
(A5) Let $z \in (\eta, 1 - \eta)$ be fixed. There exists $b > 0$ such that $b \leq Nf\left(\frac{(z-\eta)^{m-2}\gamma b}{(m-2)!}, \frac{(z-\eta)^{m-3}\gamma b}{(m-3)!}, \dots, \frac{(z-\eta)^{m-q-2}\gamma b}{(m-q-2)!}\right)$ where $N = \sup_{t \in [0, 1]} \int_z^{1-\eta} G(t, s)\alpha(s)ds$ and $\gamma = K_\eta/L$.

Using Krasnosel'skii's fixed point theorem in a cone, we obtain the first result that gives the existence of a *nontrivial positive* solution.

Theorem 2. Let (A1)–(A5) hold. Then, (1) has a *nontrivial positive* solution $y^* \in C^{(m)}[0, 1]$ such that for $0 \leq k \leq m-2$,

$$0 \leq y^{*(k)}(t) \begin{cases} < \frac{t^{m-k-2}}{(m-k-2)!} a \leq \frac{a}{(m-k-2)!}, & t \in [0, 1], \text{ if } a > b \\ \leq \frac{t^{m-k-2}}{(m-k-2)!} b \leq \frac{b}{(m-k-2)!}, & t \in [0, 1], \text{ if } a < b \end{cases} \quad (8)$$

and

$$y^{*(k)}(t) \begin{cases} \geq \frac{(t-\eta)^{m-k-2}}{(m-k-2)!} \gamma b, & t \in [\eta, 1 - \eta], \text{ if } a > b \\ > \frac{(t-\eta)^{m-k-2}}{(m-k-2)!} \gamma a, & t \in [\eta, 1 - \eta], \text{ if } a < b. \end{cases} \quad (9)$$

Our next result guarantees the existence of *two nontrivial positive* solutions.

Theorem 3. Let (A1)–(A5) and (A5) $_{|b=b'}$ hold, where $0 < b' < a < b$. Then, (1) has (at least) two *nontrivial positive* solutions $y_1, y_2 \in C^{(m)}[0, 1]$ such that for $0 \leq k \leq m-2$,

$$\begin{cases} 0 \leq y_1^{(k)}(t) < \frac{t^{m-k-2}}{(m-k-2)!} a \leq \frac{a}{(m-k-2)!}, & t \in [0, 1] \\ y_1^{(k)}(t) \geq \frac{(t-\eta)^{m-k-2}}{(m-k-2)!} \gamma b', & t \in [\eta, 1 - \eta] \\ 0 \leq y_2^{(k)}(t) \leq \frac{t^{m-k-2}}{(m-k-2)!} b \leq \frac{b}{(m-k-2)!}, & t \in [0, 1] \\ y_2^{(k)}(t) > \frac{(t-\eta)^{m-k-2}}{(m-k-2)!} \gamma a, & t \in [\eta, 1 - \eta]. \end{cases} \quad (10)$$

EXAMPLE

Consider the Sturm-Liouville BVP

$$\begin{cases} y^{(5)}(t) + F(t, y(t), y'(t), y''(t), y'''(t)) = 0, & t \in [0, 1] \\ y(0) = y'(0) = y''(0) = 0, & 2y^{(3)}(0) - y^{(4)}(0) = 0, & -y^{(3)}(1) + 3y^{(4)}(1) = 0 \end{cases} \quad (11)$$

where

$$F(t, y, y', y'', y''') = \frac{9}{625} (530 + 90t + 27t^2 + 9t^3 + t^4 - 3t^5) (y + y' + y'' + y''' + 100). \quad (12)$$

Here, $m = 5$, $q = 3$, $\zeta = 2$, $\theta = 1$, $\rho = -1$ and $\delta = 3$. Let $\eta = \frac{1}{4}$ and $z = \frac{1}{2}$. A direct computation gives $L = \frac{3}{2}$, $K_{\frac{1}{4}} = \frac{1}{2}$ and $\gamma = \frac{1}{3}$. Clearly, (A1), (A2) and (A4) are satisfied with

$$\alpha(t) = \beta(t) = \frac{18}{125} (530 + 90t + 27t^2 + 9t^3 + t^4 - 3t^5), \quad f(u_1, u_2, u_3, u_4) = \frac{1}{10} (u_1 + u_2 + u_3 + u_4 + 100).$$

To check (A3) and (A5), in view of Lemma 1(b) and 1(c), we solve the stricter but easier inequalities

$$a > \left[\int_0^1 LG(s, s)\beta(s)ds \right] f\left(\frac{a}{3!}, \frac{a}{2!}, a, a\right) \quad \text{and} \quad b \leq \left[\int_{\frac{1}{2}}^{\frac{3}{4}} K_{\frac{1}{4}}G(s, s)\alpha(s)ds \right] f\left(\frac{\gamma b}{4^3 3!}, \frac{\gamma b}{4^2 2!}, \frac{\gamma b}{4}, \gamma b\right)$$

to get $a \in [26.577, \infty)$ and $b \in (0, 1.4883]$. Hence, (A3) and (A5) are satisfied if $a \in [26.577, \infty)$ and $b \in (0, 1.4883]$.

Now, an application of Theorem 2 gives a *nontrivial positive* solution $\bar{y} \in C^{(5)}[0, 1]$ of (11), (12) satisfying (8) and (9) for the case $a > b$. Since $a \in [26.577, \infty)$ and $b \in (0, 1.4883]$, we further obtain for $0 \leq k \leq 3$,

$$\begin{cases} 0 \leq \bar{y}^{(k)}(t) < \frac{t^{3-k}}{(3-k)!}(26.577) \leq \frac{26.577}{(3-k)!}, & t \in [0, 1] \\ \bar{y}^{(k)}(t) \geq \frac{1}{(3-k)!} \left(t - \frac{1}{4}\right)^{3-k} \gamma(1.4883), & t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \end{cases} \quad (13)$$

In fact, a positive solution of (11), (12) is known to be

$$y^*(t) = \frac{1}{50} (50t^3 + 25t^4 - 3t^5). \quad (14)$$

By direct computation, we find for $0 \leq k \leq 3$,

$$y^{*(k)}(t) \leq c_k, \quad t \in [0, 1] \quad \text{and} \quad y^{*(k)}(t) \geq d_k \left(t - \frac{1}{4}\right)^{3-k}, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right] \quad (15)$$

where c_k and d_k are respectively the smallest and the largest constants for the inequalities to hold, and they are given as follows

$$c_0 = 1.44, \quad c_1 = 4.7, \quad c_2 = 10.8, \quad c_3 = 14.4, \quad d_0 = 4.5267, \quad d_1 = 9.7453, \quad d_2 = 14.7375, \quad d_3 = 8.775. \quad (16)$$

Since $c_k < (26.577)/(3-k)!$ and $d_k > \gamma(1.4883)/(3-k)!$, this y^* could just be $\bar{y}$ in (13). Hence, the conclusion (13) is somewhat validated.

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