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Dynamical quantum phase transitions in non-Hermitian lattices

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In closed quantum systems, dynamical phase transitions are identified by the nonanalytic behavior of the return probability as a function of time. In this work, we study the nonunitary dynamics following quenches across exceptional points in a non-Hermitian lattice realizable by optical resonators. Dynamical quantum phase transitions with topological signatures are found when an isolated exceptional point is crossed during the quench. A winding number defined by a real, noncyclic geometric phase is introduced, whose value features quantized jumps at critical times of these phase transitions and remains constant elsewhere, playing the role of a topological order parameter. This work provides a simple framework to study dynamical and topological quantum phase transitions in non-Hermitian systems.

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I. INTRODUCTION

Dynamical quantum phase transitions (DQPTs) are characterized by the nonanalytic behavior of physical observables as functions of time [1,2]. These transitions happen in general if the system is ramped through a quantum critical point. As a promising framework to classify quantum dynamics of nonequilibrium many-body systems, DQPTs have been studied intensively in recent years [3–22]. The generality and topological feature of DQPTs were demonstrated in both lattice and continuous systems [23], across different spatial dimensions [24–28], and under various dynamical protocols [29–31]. The defining features of DQPTs have also been observed in recent experiments [32–34].

Following the initial proposal, most studies on DQPTs focus on closed quantum systems undergoing unitary time evolution. Efforts have been made to generalize DQPTs to systems prepared in mixed states [35–37]. However, DQPTs in systems with gain and loss, and therefore subject to nonunitary evolution, are largely unexplored. One class of such systems can be described by non-Hermitian Hamiltonians [38–41] (with the so-called $\mathcal{PT}$ -symmetric quantum mechanics being one example [40,41]). These non-Hermitian systems, realizable in various platforms like photonic lattices [42], phononic media [43], LRC circuits [44], and cold atoms [45,46], have attracted great attention in recent years due to their nontrivial dynamical [41,47–57], topological [58–73], and transport properties [74–81]. Many of these features can be traced back to non-Hermitian degeneracy (i.e., exceptional) points mediating gap closing and reopening transitions on the complex plane [82–87]. In this work, we explore DQPTs in non-Hermitian systems, with a focus on topological signatures in nonunitary evolution following quenches across exceptional points (EPs).

II. THEORY

We start by summarizing the theoretical framework of DQPTs for systems described by non-Hermitian lattice Hamiltonians. The nonunitary time evolution of the system is governed by the Schrödinger equation $i \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle$. For concreteness, we present the formalism with a one-dimensional, two-band lattice model in mind, while the generalization to multiple-band systems is straightforward [88].

Consider a system described by Hamiltonian $H = H^f \theta(t) + H^i \theta(-t)$, where $\theta(t)$ is the step function. At time $t = 0$, the system undergoes a sudden quench with its Hamiltonian switched from H^i to H^f . At time $t > 0$, the return amplitude of the system to its initial state $|\Psi^i\rangle$ is given by $G(t) = \langle \Psi^i | e^{-iH^f t} | \Psi^i \rangle$. In our study, the postquench Hamiltonian H^f is non-Hermitian, and the evolution operator $e^{-iH^f t}$ is therefore nonunitary. Nontrivial dynamics is expected if $|\Psi^i\rangle$ is not a right eigenvector of H^f . In particular, when the evolving state $|\Psi^i(t)\rangle = e^{-iH^f t} |\Psi^i\rangle$ becomes orthogonal to the initial state at some critical time t_c , we have $G(t_c) = 0$. Then the rate function of the return amplitude [89,90], $f(t) = \lim_{N \rightarrow \infty} \frac{-1}{N} \ln G(t)$, becomes nonanalytic at $t = t_c$, where N is the number of degrees of freedom of the system. Similar to DQPTs in unitary evolution, we identify the vanishing of $G(t)$ at critical times as signatures of DQPTs in non-Hermitian systems.

In momentum space, the pre- and postquench Hamiltonian $H^{i/f}$ can be expressed as $H^{i/f} = \sum_k \mathcal{H}_k^{i/f} |k\rangle \langle k|$, where $k \in [0, 2\pi)$ is the quasimomentum and $\mathcal{H}_k^{i/f} = d_x^{i/f}(k) \sigma_x + d_y^{i/f}(k) \sigma_y + d_z^{i/f}(k) \sigma_z$ is the Bloch Hamiltonian, with $\sigma_{x,y,z}$ being Pauli matrices in their usual representation. The right eigenvectors $|\psi_k^{i/f\pm}\rangle$ of $\mathcal{H}_k^{i/f}$ satisfy $\mathcal{H}_k^{i/f} |\psi_k^{i/f\pm}\rangle = \pm d_k^{i/f} |\psi_k^{i/f\pm}\rangle$, with $d_k^{i/f} = \sqrt{[d_x^{i/f}(k)]^2 + [d_y^{i/f}(k)]^2 + [d_z^{i/f}(k)]^2}$. Here we focus on the case in which $\mathcal{H}_k^i$ is Hermitian with a gapped spectrum, and the initial state is the ground state of the prequench Hamiltonian H^i , i.e., $|\Psi_i\rangle = \prod_k |\psi_k^{i-}\rangle$.

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The translational symmetry allows us to treat the dynamics of each state $|\psi_k^{i-}\rangle$ separately. The return amplitude is then given by $G^-(t) = \prod_k \mathcal{G}_k^-(t)$, where $\mathcal{G}_k^-(t) = \cos(d_k^f t) - i \sin(d_k^f t) \langle \psi_k^{i-} | \frac{\mathcal{H}_k^f}{d_k^f} | \psi_k^{i-} \rangle$. In the thermodynamic limit, the rate function of return probability reads

$$g^-(t) = -\frac{1}{2\pi} \int_0^{2\pi} dk \ln[|\mathcal{G}_k^-(t)|^2]. \quad (1)$$

A DQPT is expected when a line of Lee-Yang-Fisher (LYF) zeros [91–94], defined as

$$z_{n,k}^- = i \frac{\pi}{d_k^f} \left(n + \frac{1}{2} \right) + \frac{1}{d_k^f} \operatorname{arctanh} \left(\langle \psi_k^{i-} | \frac{\mathcal{H}_k^f}{d_k^f} | \psi_k^{i-} \rangle \right), \quad n \in \mathbb{Z}, \quad (2)$$

crosses the imaginary time axis at a critical momentum k_c , yielding a critical time $t_{c,n}^- = -i z_{n,k_c}^-$ at which $\mathcal{G}_{k_c}^-(t_c) = 0$. Note that these considerations hold for both Hermitian and non-Hermitian systems.

To capture topological signatures of DQPTs in non-Hermitian systems, we study the time dependence of the winding number,

$$v_D(t) = \frac{1}{2\pi} \int_0^{2\pi} dk [\partial_k \phi_k^G(t)], \quad (3)$$

where the geometric phase of the return amplitude is given by $\phi_k^G(t) \equiv \phi_k(t) - \phi_k^{\text{dyn}}(t)$, with the total phase $\phi_k(t) = -i \ln \left[\frac{\mathcal{G}_k^-(t)}{|\mathcal{G}_k^-(t)|} \right]$ and the dynamical phase

$$\begin{aligned} \phi_k^{\text{dyn}}(t) = & - \int_0^t ds \frac{\langle \psi_k^{i-}(s) | \mathcal{H}_k^f | \psi_k^{i-}(s) \rangle}{\langle \psi_k^{i-}(s) | \psi_k^{i-}(s) \rangle} \\ & + \frac{i}{2} \ln \left[\frac{\langle \psi_k^{i-}(t) | \psi_k^{i-}(t) \rangle}{\langle \psi_k^{i-}(0) | \psi_k^{i-}(0) \rangle} \right]. \end{aligned} \quad (4)$$

For a spin-1/2 system, the geometric phase $\phi_k^G(t)$ represents half of the solid angle subtended by an area on the Bloch sphere, whose boundary is formed by the trajectory of the time-evolving state in the projective Hilbert space from time zero to t , plus a geodesic connecting the two open ends of this physical trajectory [95]. For a cyclic state, $\phi_k^G(t)$ reduces to the Aharonov-Anandan (AA) geometric phase [52]. As a side note, the adiabatic counterpart of the AA phase, namely, the Berry phase, is more familiar and has been used to probe the branching point of time-independent systems [38]. Remarkably, $\phi_k^G(t)$ defined here is *real* and *gauge invariant* [52,88]. These are the critical properties for us to introduce the real-valued, quantized topological winding number $v_D(t)$ for non-Hermitian systems. Furthermore, the winding number of $\phi_k^G(t)$ in k -space will experience a quantized jump at every critical time t_c when the system passes through a DQPT point [96]. As will be shown, the value of $v_D(t)$ will change monotonically in time if the initial state is topologically trivial and the quench crosses an isolated EP.

III. MODEL

We are going to consider DQPTs following non-Hermitian quenches in a lattice model. Specifically, we let $\mathcal{H}_k^f = \mathcal{H}_k^i + i \frac{\gamma}{2} \sigma_z$, where $\mathcal{H}_k^i = h_x(k) \sigma_x + h_z(k) \sigma_z$ is Hermitian. The initial state $|\psi_k^i\rangle$ at each k is chosen to be the lower eigenstate of the prequench Hamiltonian $\mathcal{H}_k^i$ with energy $-d_k^i = -\sqrt{h_x^2(k) + h_z^2(k)}$. To demonstrate our theory explicitly, we focus on the non-Hermitian lattice model introduced in Ref. [70], which may be realizable in optical resonators. In momentum representation, the postquench model Hamiltonian is $H^f = \sum_k \mathcal{H}_k^f |k\rangle \langle k|$, where

$$\mathcal{H}_k^f = h_x(k) \sigma_x + \left[h_z(k) + i \frac{\gamma}{2} \right] \sigma_z, \quad (5)$$

with $h_x(k) = \mu + r \cos k$ and $h_z(k) = r \sin k$. Note that $\mathcal{H}_k^f$ has a complex spectrum. According to the periodic table of non-Hermitian Hamiltonians [97], $\mathcal{H}_k^f$ belongs to the topological class BDI and can be characterized by integer winding numbers [88]. When the non-Hermitian hopping amplitude γ satisfies $\mu - r < \frac{\gamma}{2} < \mu + r$ or $\mu - r < -\frac{\gamma}{2} < \mu + r$, the trajectory of vector $\mathbf{h}(k) = [h_x(k), h_z(k)]$ versus quasimomentum k will encircle one of the two EPs at $\mathbf{h}(k) = (\pm \frac{\gamma}{2}, 0)$. In this case, the system is topologically nontrivial and characterized by a winding number defined in a 4π Brillouin zone [70].

In the following, we study dynamics following a quench in which the vector $\mathbf{h}(k)$ encircles zero and one EP for both topologically trivial and nontrivial initial states [98]. Other cases with more general quench processes and initial conditions are also discussed in the Appendixes [88].

IV. RESULTS

Figure 1 gives a schematic of quench cases studied in this section. In cases 1, 2, and 3, the trajectory of $\mathbf{h}(k)$ does not encircle the zero of the h_x - h_z plane, implying that the initial state is topologically trivial. In case 4, the trajectory of $\mathbf{h}(k)$ encircles the zero of the h_x - h_z plane, and the initial state is thus topologically nontrivial, characterized by a quantized winding number. We now study the line of LYF zeros $z_{n,k}^-$, the rate function of return probability $g^-(t)$, geometric phase $\phi_k^G(t)$, and winding number $v_D(t)$ for nonunitary dynamics following the four qualitatively different quench cases. These quantities provide us with key information of DQPTs in the non-Hermitian lattice.

Results of $z_{n,k}^-$, $g^-(t)$, $\phi_k^G(t)$, and $v_D(t)$ for quench cases 1 and 2 are shown in Figs. 2 and 3, respectively. In Fig. 2(a), we see that each line of LYF zeros $z_{n,k}^-$ for $n = 1, \dots, 4$ has a unique crossing on the imaginary time axis i.e., $[\operatorname{Re}(z_{n,k}^-) = 0]$, yielding a positive critical time of DQPT at $t_{c,n}^- = -i z_{n,k_c}^-$ with a unique critical momentum k_c . This is true for all $n > 0$, whereas there are no positive critical times for $n \leq 0$. The same pattern of LYF zeros is also observed for quench case 2 as shown in Fig. 3(a). In Figs. 2(b) and 3(b), the rate functions behave nonanalytically at critical times $t_{c,n}^-$ found in Figs. 2(a) and 3(a), which is the defining feature of DQPTs. In Figs. 2(c) and 3(c), the pattern of geometric phase $\phi_k^G(t)$ versus time t and quasimomentum k is shown. We see that

FIG. 1. A sketch of the four quench cases. The open triangle denotes the zero of the h_x - h_z plane. The two solid triangles represent EPs of the Hamiltonian $\mathcal{H}_k^f$. The dashed circle represents the vector $\mathbf{h}(k) = [h_x(k), h_z(k)]$. (a) Case 1: The initial state is topologically nontrivial; $\mathbf{h}(k)$ encircles the EP $(\frac{\gamma}{2}, 0)$. (b) Case 2: The initial state is topologically trivial; $\mathbf{h}(k)$ encircles the EP $(-\frac{\gamma}{2}, 0)$. (c) Case 3: The initial state is topologically trivial; $\mathbf{h}(k)$ does not encircle any EPs. (d) Case 4: The initial state is topologically nontrivial; $\mathbf{h}(k)$ encircles the EP $(\frac{\gamma}{2}, 0)$.

the winding number of $\phi_k^G(t)$ in k -space increases every time when the evolution passes through a critical time. This pattern is accurately accounted for by the winding number defined in Eq. (3) for both quench cases, as shown in Figs. 2(d) and 3(d). Two notable features deserve to be mentioned. First, the value of $\nu_D(t)$ is always quantized and changes monotonically in time if the initial state is topologically trivial and the postquench Hamiltonian vector $\mathbf{h}(k)$ encircles one of the EPs. This not only provides us with a clear window to look into the topological signature of an isolated EP in the system's dynamics, but also suggests a promising approach to distinguish bulk properties of different non-Hermitian topological phases separated by EPs. Second, the quench across the EP in case 1 induces a positive and monotonic increasing $\nu_D(t)$ in time, while the quench across the EP in case 2 induces a negative and monotonic decreasing $\nu_D(t)$. This difference originates from the fact that the trajectory of vector $\mathbf{h}(k)$ in cases 1 and 2 crosses the branch cut connecting the two EPs along opposite directions. We therefore find a way to distinguish the two EPs from their topological response to a simple quench in non-Hermitian systems, complementing other existing approaches focusing on quasiadiabatic evolution [49,50]. In addition, the qualitative feature of DQPTs found here should be robust to perturbations that preserve the discrete symmetries of $\mathcal{H}_k^f$ [88], with the actual values of critical time $t_{c,n}^-$ still dependent on the system parameters.

In Fig. 4, we show results of $z_{n,k}^-$ and $g^-(t)$ for quench case 3. In this case, no EPs are encircled by the vector $\mathbf{h}(k)$, and lines of LYF zeros have no crossings on the imaginary time axis as shown in Fig. 4(a). This indicates that there are no DQPTs in this case and the winding number remains zero in the

FIG. 2. DQPTs for non-Hermitian quench case 1. System parameters are $\mu = 0.6$ and $r = 0.3$ for both pre- and postquench Hamiltonians, and $\gamma = 1$. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}^-$ with $n = 1, 2, 3, 4$. (b) Rate function of the return probability [99]. The first four DQPTs happen around $t \approx 4.7, 8.7, 12.8$, and 16.9 . (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of topological winding number $\nu_D(t)$.

nonunitary evolution, as also supported by results reported in Fig. 4(b) and calculations of $\nu_D(t)$. More generally, for a vector $\mathbf{h}(k)$ encircling no EPs, each line of LYF zeros could form a closed trajectory, crossing over the imaginary time axis an even number of times as demonstrated in the Appendices D and E

FIG. 3. DQPTs for non-Hermitian quench case 2. System parameters are $\mu = -0.6$ and $r = 0.3$ for both pre- and postquench Hamiltonians, and $\gamma = 1$. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}^-$ with $n = 1, 2, 3, 4$. (b) Rate function of the return probability [99]. The first four DQPTs happen around $t \approx 4.7, 8.7, 12.8$, and 16.9 . (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of topological winding number $\nu_D(t)$.

FIG. 4. DQPTs for non-Hermitian quench case 3. System parameters are $\mu = 0.6$ and $r = 0.3$ for both pre- and postquench Hamiltonians, and $\gamma = 2$. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}^-$ with $n = 1, 2, 3, 4$. (b) Rate function of the return probability [99].

[88]. This results in DQPTs with a winding number oscillating in time, implying that the pre- and postquench systems belong to the same topological class.

Finally, we present results of $z_{n,k}^-$, $g^-(t)$, $\phi_k^G(t)$, and $\nu_D(t)$ for quench case 4 with a topologically nontrivial initial state in Fig. 5. Different from the other three cases with trivial initial states, each line of LYF zeros now features two types of crossing points on the imaginary time axis. One type of them appears once for each line of LYF zeros regularly, while the other type contains a pair of critical times separated by a vanishingly small time window. DQPTs corresponding to the latter type of critical times may then represent the memory of an evolving state about its initial topology. Notably, as time progresses this memory is lost rapidly, insofar as the

FIG. 5. DQPTs for non-Hermitian quench case 4. System parameters are $\mu = 0.3$ and $r = 0.6$ for both pre- and postquench Hamiltonians, and $\gamma = 1$. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}^-$ with $n = 1, 2, 3, 4$. (b) Rate function of the return probability [99]. The dip at $t < 5$ is smooth and does not correspond to a DQPT. The first four topological DQPTs happen around $t \approx 5.1, 8.2, 12.5$, and 16.7 . A pair of accidental DQPTs happen around $t \approx 10.6$. (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of topological winding number $\nu_D(t)$.

difference between the two critical times within each pair is exponentially small. These arguments are further supported by the time evolution of winding number shown in Fig. 5(d), where a pair of accidental DQPTs caused by the initial-state topology appear around $t \approx 10.6$. The insensitivity of DQPTs in nonunitary evolution to the initial-state topology has two implications. First, by choosing simple initial states, DQPTs due to quenching across an EP can be demonstrated analytically in a transparent manner [88]. Furthermore, the non-Hermitian quench may provide us with a useful strategy to generate topologically nontrivial dynamical phases of matter from easy-to-prepare initial states.

More numerical and analytic results of DQPTs for other kinds of non-Hermitian quenches and initial conditions are presented in the Appendices D, E, and F [88], all supporting our observations here.

In a photonic setup, the non-Hermitian quench may be engineered by introducing gain or loss uniformly to half of the system along the propagation direction of the wave. Then the signature of DQPTs may be identified by measuring and collecting phase factors of each k -mode in space. The quantum walk setup reported in Ref. [62] could also be a candidate to experimentally study DQPTs in non-Hermitian systems.

In a more recent experiment using a reconfigurable array of nanomechanical oscillators [100], dynamical quantum phase transitions following a quench have been observed at a time scale as long as 0.04 s. The lattice system is formed by eight nanomechanical oscillators fabricated by 100-nm-thick silicon nitride coated with a thin layer of gold. By incorporating dissipations, this system can also be used to study dynamical phenomena in non-Hermitian lattices [100] and therefore represents a good candidate to demonstrate the predictions of our work in the near term.

V. CONCLUSION AND DISCUSSION

In summary, we have discovered DQPTs in non-Hermitian lattices by quenching across EPs. Two types of DQPTs were observed and distinguished by the parity of critical times on each line of LYF zeros. Topological DQPTs following the quench across an isolated EP are characterized by an odd number of critical times along each line of the LYF zeros, and accompanied by a monotonously changing topological winding number in time. Initial states with nontrivial topology can also leave transient signatures in the postquench dynamics, resulting in accidental DQPTs. Whether these qualitative findings hold for more general dynamical protocols, like linear ramp or periodic driving, and in higher physical dimensions deserves further studies. The possible connection of DQPTs with the universal Kibble-Zurek scaling [101,102] is also an interesting question.

The concept of DQPT is initially introduced as an organizing principle to study dynamics in closed nonequilibrium many-body systems. This work makes it evident that the extension of DQPT to open quantum systems [103–106] is a promising avenue. The always real geometric phase and the associated winding number exploited here, together with their generalizations to density matrix formalism [35,36,88] and other relevant quantities [107–109] including the Uhlmann phase of density matrices observed recently [110–113], are

expected to work together to fully decode geometric and topological order in the nonequilibrium dynamics of many-body open quantum systems.

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APPENDIX A: SYMMETRY AND TOPOLOGICAL CLASSIFICATION OF THE NON-HERMITIAN POSTQUENCH HAMILTONIAN $\mathcal{H}_k^f$

In this Appendix, we analyze the symmetry and topological classification of the non-Hermitian postquench Hamiltonian. This will be helpful for understanding the connection between DQPTs and topological phase transitions.

The Hamiltonian $\mathcal{H}_k^f$ introduced in Eq. (5) of the main text has a complex spectrum for any nonzero non-Hermitian term γ , and it does not possess the $\mathcal{PT}$ symmetry. Under the rotation $\sigma_z \rightarrow \sigma_y$, it becomes $\mathcal{H}'_k^f = h_x(k)\sigma_x + [h_z(k) + i\gamma/2]\sigma_y$, where $h_x(k) = \mu + r \cos(k)$ and $h_z(k) = r \sin(k)$. It is not hard to see that this Hamiltonian admits (i) the time-reversal symmetry $\mathcal{T} = K$, where K means taking the complex conjugation, i.e., $\mathcal{T}\mathcal{H}'_k^f\mathcal{T}^{-1} = \mathcal{H}'_{-k}^f$; (ii) the particle-hole (charge conjugate) symmetry $\mathcal{C} = \sigma_z K$, i.e., $\mathcal{C}\mathcal{H}'_k^f\mathcal{C}^{-1} = -\mathcal{H}'_{-k}^f$; and (iii) the chiral (sublattice) symmetry $\Gamma = \sigma_z$, i.e., $\Gamma\mathcal{H}'_k^f\Gamma = -\mathcal{H}'_k^f$.

Therefore, $\mathcal{H}'_k^f$ and also $\mathcal{H}_k^f$ fall under the Altland-Zirnbauer class BDI in a periodic table proposed by Ref. [97] for non-Hermitian Hamiltonians. According to Ref. [97], each topological phase of $\mathcal{H}_k^f$ ($\mathcal{H}_k^{f\dagger}$) is characterized by a winding number $w_1 \in \mathbb{Z}$ ($w_2 \in \mathbb{Z}$), defined as

$$w_{1(2)} = \sum_{n=1}^2 \int_{-\pi}^{\pi} \frac{dk}{2\pi} \partial_k \arg[E_n(k)], \quad (\text{A1})$$

where $E_1(k)$ and $E_2(k)$ are the two complex energy bands of $\mathcal{H}_k^f$ ($\mathcal{H}_k^{f\dagger}$). For the Hermitian prequench Hamiltonian, these winding numbers are all zero by definition.

The DQPTs presented in Fig. 2 of the main text may then be related to the difference of winding number w_1 in the pre- and postquench Hamiltonians. Direct calculation using Eq. (A1) indeed gives $w_1 = 0$ (1) for $\mathcal{H}_k^i$ ($\mathcal{H}_k^f$) in this case. Furthermore, since taking the Hermitian conjugate of $\mathcal{H}_k^f$ flips the sign of γ , and therefore exchanges the two exceptional points at $\pm\gamma/2$ in the parameter plane h_x - h_z , the DQPTs presented in Fig. 3 of the main text may also be related to the difference of winding number w_2 in the pre- and postquench Hamiltonians. Direct calculation using Eq. (A1) gives $w_2 = 0$ (-1) for $\mathcal{H}_k^i$ ($\mathcal{H}_k^f$) in this case, which further justifies our viewpoint.

To sum up, the non-Hermitian Hamiltonian $\mathcal{H}_k^f$ belongs to the BDI class and can be characterized by an integer winding number [97]. The change of the winding number through a topological phase transition in the system described by $\mathcal{H}_k^f$ ($\mathcal{H}_k^{f\dagger}$) is closely related to the DQPTs shown in Fig. 2 (Fig. 3) of our main text.

APPENDIX B: FORMALISM OF DQPTS FOR ONE-DIMENSIONAL, MULTIPLE-BAND NON-HERMITIAN LATTICES

In this Appendix, we generalize the formalism of DQPTs in one-dimensional, two-band non-Hermitian lattices to multiple-band systems.

Consider a one-dimensional lattice system of $\mathcal{N}$ bulk bands subject to the same form of global quench protocol as introduced in the main text: $H = H^f\theta(t) + H^i\theta(-t)$. The pre- and postquench Hamiltonians H^i and H^f can both be non-Hermitian. In the momentum representation, we have $H^{i/f} = \sum_k \mathcal{H}_k^{i/f} |k\rangle\langle k|$, where $k \in [0, 2\pi)$ is the quasimomentum. $\mathcal{H}_k^{i/f}$ satisfies the following eigenvalue equations:

$$\mathcal{H}_k^{i/f} |\psi_k^{i/f,\ell}\rangle = d_{\ell,k}^{i/f} |\psi_k^{i/f,\ell}\rangle, \quad (\text{B1})$$

where $\ell = 1, \dots, \mathcal{N}$ are band indices. Assuming that the spectrum $\pm d_{\ell,k}^i$ of prequench Hamiltonian $\mathcal{H}_k^i$ is gapped at every k on the complex plane, we consider the initial state which fills the ℓ th complex-valued “energy band” of $\mathcal{H}_k^i$. For such an initial state, the return amplitude is given by $G_\ell(t) = \prod_k \mathcal{G}_{\ell,k}(t)$, where $\mathcal{G}_{\ell,k}(t) = \langle \psi_k^{i,\ell} | e^{-i\mathcal{H}_k^f t} | \psi_k^{i,\ell} \rangle$ is the return amplitude at each quasimomentum k . The rate function of return probability in the thermodynamic limit then follows as

$$g_\ell(t) = -\frac{1}{2\pi} \int_0^{2\pi} dk \ln[|\mathcal{G}_{\ell,k}(t)|^2]. \quad (\text{B2})$$

A DQPT happens when a line of LYF zeros crosses the imaginary time axis at a critical momentum k_c , yielding a critical time t_c^ℓ at which $\mathcal{G}_{\ell,k_c}(t_c) = 0$. For a multiple-band system, a compact analytic expression for t_c^ℓ is not available in general. Numerically, one can identify t_c^ℓ by investigating the zeros of $|\mathcal{G}_{\ell,k}(t)|^2$ at each given time t . The topological characterization of DQPTs in multiple-band systems is the same as that for a two-band system. The time-dependent winding number for the initial state considered here is given by

$$v_D^\ell(t) = \frac{1}{2\pi} \int_0^{2\pi} dk [\partial_k \phi_{\ell,k}^G(t)], \quad (\text{B3})$$

where the geometric phase $\phi_{\ell,k}^G(t) \equiv \phi_{\ell,k}(t) - \phi_{\ell,k}^{\text{dyn}}(t)$, with the total phase $\phi_{\ell,k}(t) = -i \ln \left[\frac{\mathcal{G}_{\ell,k}(t)}{|\mathcal{G}_{\ell,k}(t)|} \right]$ and the dynamical phase

$$\begin{aligned} \phi_{\ell,k}^{\text{dyn}}(t) = & - \int_0^t ds \frac{\langle \psi_k^{i,\ell}(s) | \mathcal{H}_k^f | \psi_k^{i,\ell}(s) \rangle}{\langle \psi_k^{i,\ell}(s) | \psi_k^{i,\ell}(s) \rangle} \\ & + \frac{i}{2} \ln \left[\frac{\langle \psi_k^{i,\ell}(t) | \psi_k^{i,\ell}(t) \rangle}{\langle \psi_k^{i,\ell}(0) | \psi_k^{i,\ell}(0) \rangle} \right]. \end{aligned} \quad (\text{B4})$$

As mentioned in the main text and demonstrated in Appendix C, the geometric phase thus defined is real, gauge invariant, and thus an experimentally measurable quantity.

APPENDIX C: REALITY AND GAUGE INVARIANCE OF THE GEOMETRIC PHASE

In this Appendix, we show that the geometric phase $\phi_k^G(t)$ introduced in the main text is real and gauge invariant, representing a quantity that is measurable experimentally.

Consider a general initial state $|\psi_k^i\rangle$, which may or may not be a right eigenvector of the prequench Hamiltonian $\mathcal{H}_k^i$. The return amplitude for such an initial state is given by $\mathcal{G}_k(t) = \langle \psi_k^i | e^{-i\mathcal{H}_k^f t} | \psi_k^i \rangle$. The total phase of the return amplitude is

$$\phi_k(t) = -i \ln \left[\frac{\mathcal{G}_k(t)}{|\mathcal{G}_k(t)|} \right]. \quad (\text{C1})$$

By definition this is a real quantity since we can always write $\mathcal{G}_k(t) = |\mathcal{G}_k(t)|e^{i\phi_k(t)}$, with $\phi_k(t)$ being its phase angle. To obtain the form of dynamical phase given in the main text, we introduce the time-dependent normalized state as

$$|\varphi_k^i(t)\rangle \equiv \frac{|\psi_k^i(t)\rangle}{\sqrt{\langle \psi_k^i(t) | \psi_k^i(t) \rangle}}, \quad (\text{C2})$$

where $|\psi_k^i(t)\rangle \equiv e^{-i\mathcal{H}_k^f t} |\psi_k^i\rangle$ represents the unnormalized time-evolving state. Since $\langle \varphi_k^i(t) | \varphi_k^i(t) \rangle = 1$, we have $\frac{d}{dt} \langle \varphi_k^i(t) | \varphi_k^i(t) \rangle = 0$, implying that $\langle \varphi_k^i(t) | \frac{d}{dt} | \varphi_k^i(t) \rangle$ is purely imaginary. A real dynamical phase is then defined in terms of

the normalized state $|\varphi_k^i(t)\rangle$ as

$$\phi_k^{\text{dyn}}(t) \equiv -i \int_0^t ds \langle \varphi_k^i(s) | \frac{d}{ds} | \varphi_k^i(s) \rangle. \quad (\text{C3})$$

In terms of the unnormalized time-evolving state $|\psi_k^i(t)\rangle$, we can also write $\phi_k^{\text{dyn}}(t)$ as

$$\phi_k^{\text{dyn}}(t) = - \int_0^t ds \frac{\langle \psi_k^i(s) | i \frac{d}{ds} | \psi_k^i(s) \rangle}{\langle \psi_k^i(s) | \psi_k^i(s) \rangle} + \frac{i}{2} \ln \left[\frac{\langle \psi_k^i(t) | \psi_k^i(t) \rangle}{\langle \psi_k^i(0) | \psi_k^i(0) \rangle} \right]. \quad (\text{C4})$$

Using the Schrödinger equation $i \frac{d}{dt} |\psi_k^i(t)\rangle = \mathcal{H}_k^f |\psi_k^i(t)\rangle$ satisfied by $|\psi_k^i(t)\rangle$, we further have

$$\phi_k^{\text{dyn}}(t) = - \int_0^t ds \frac{\langle \psi_k^i(s) | \mathcal{H}_k^f | \psi_k^i(s) \rangle}{\langle \psi_k^i(s) | \psi_k^i(s) \rangle} + \frac{i}{2} \ln \left[\frac{\langle \psi_k^i(t) | \psi_k^i(t) \rangle}{\langle \psi_k^i(0) | \psi_k^i(0) \rangle} \right], \quad (\text{C5})$$

which is the expression for dynamical phase given in the main text. Note that in the Hermitian limit $\langle \psi_k^i(t) | \psi_k^i(t) \rangle = \langle \psi_k^i(0) | \psi_k^i(0) \rangle = 1$, and we are left with the familiar dynamical phase in unitary evolution. Another equivalent expression for $\phi_k^{\text{dyn}}(t)$ is [95]

$$\phi_k^{\text{dyn}}(t) = - \int_0^t ds \text{Re} \left[\frac{\langle \psi_k^i(s) | \mathcal{H}_k^f | \psi_k^i(s) \rangle}{\langle \psi_k^i(s) | \psi_k^i(s) \rangle} \right], \quad (\text{C6})$$

which can be directly deduced from Eq. (C5) since $\frac{i}{2} \ln \left[\frac{\langle \psi_k^i(t) | \psi_k^i(t) \rangle}{\langle \psi_k^i(0) | \psi_k^i(0) \rangle} \right]$ is purely imaginary.

The geometric phase $\phi_k^G(t)$ is given by the difference between the (real) total phase $\phi_k(t)$ and (real) dynamical phase $\phi_k^{\text{dyn}}(t)$:

$$\begin{aligned} \phi_k^G(t) &\equiv \phi_k(t) - \phi_k^{\text{dyn}}(t) \\ &= -i \ln \left(\frac{\langle \psi_k^i(0) | \psi_k^i(t) \rangle}{|\langle \psi_k^i(0) | \psi_k^i(t) \rangle|} \right) + \int_0^t ds \frac{\langle \psi_k^i(s) | \mathcal{H}_k^f | \psi_k^i(s) \rangle}{\langle \psi_k^i(s) | \psi_k^i(s) \rangle} - \frac{i}{2} \ln \left[\frac{\langle \psi_k^i(t) | \psi_k^i(t) \rangle}{\langle \psi_k^i(0) | \psi_k^i(0) \rangle} \right]. \end{aligned} \quad (\text{C7})$$

So it is also real. To see its gauge invariance, we consider the following local gauge transformation:

$$|\psi_k^i(t)\rangle \rightarrow \mathbf{g}_k(t) |\chi_k^i(t)\rangle. \quad (\text{C8})$$

Under this transformation, the first term on the right-hand side (RHS) of Eq. (C7) changes as

$$-i \ln \left(\frac{\langle \psi_k^i(0) | \psi_k^i(t) \rangle}{|\langle \psi_k^i(0) | \psi_k^i(t) \rangle|} \right) \rightarrow -i \ln \left(\frac{\langle \chi_k^i(0) | \chi_k^i(t) \rangle}{|\langle \chi_k^i(0) | \chi_k^i(t) \rangle|} \right) - i \ln \left(\frac{\mathbf{g}_k^*(0) \mathbf{g}_k(t)}{\sqrt{\mathbf{g}_k^*(0) \mathbf{g}_k(0) \mathbf{g}_k^*(t) \mathbf{g}_k(t)}} \right), \quad (\text{C9})$$

the second term on the RHS of Eq. (C7) becomes

$$\int_0^t ds \frac{\langle \psi_k^i(s) | i \frac{d}{ds} | \psi_k^i(s) \rangle}{\langle \psi_k^i(s) | \psi_k^i(s) \rangle} \rightarrow \int_0^t ds \frac{\langle \chi_k^i(s) | i \frac{d}{ds} | \chi_k^i(s) \rangle}{\langle \chi_k^i(s) | \chi_k^i(s) \rangle} + i \ln \frac{\mathbf{g}_k(t)}{\mathbf{g}_k(0)}, \quad (\text{C10})$$

and the third term on the RHS of Eq. (C7) transforms to

$$-i \ln \left[\sqrt{\frac{\langle \psi_k^i(t) | \psi_k^i(t) \rangle}{\langle \psi_k^i(0) | \psi_k^i(0) \rangle}} \right] \rightarrow -i \ln \left[\sqrt{\frac{\langle \chi_k^i(t) | \chi_k^i(t) \rangle}{\langle \chi_k^i(0) | \chi_k^i(0) \rangle}} \right] - i \ln \left[\sqrt{\frac{\mathbf{g}_k^*(t) \mathbf{g}_k(t)}{\mathbf{g}_k^*(0) \mathbf{g}_k(0)}} \right]. \quad (\text{C11})$$

Summing up extra terms induced by the gauge transformation, we find

$$-i \ln \left(\frac{\mathbf{g}_k^*(0)\mathbf{g}_k(t)}{\sqrt{\mathbf{g}_k^*(0)\mathbf{g}_k(0)\mathbf{g}_k^*(t)\mathbf{g}_k(t)}} \right) + i \ln \frac{\mathbf{g}_k(t)}{\mathbf{g}_k(0)} - i \ln \left[\sqrt{\frac{\mathbf{g}_k^*(t)\mathbf{g}_k(t)}{\mathbf{g}_k^*(0)\mathbf{g}_k(0)}} \right] = i \ln \left[\frac{\mathbf{g}_k(t)}{\mathbf{g}_k(0)} \frac{\sqrt{\mathbf{g}_k^*(0)\mathbf{g}_k(0)\mathbf{g}_k^*(t)\mathbf{g}_k(t)}}{\mathbf{g}_k^*(0)\mathbf{g}_k(t)} \sqrt{\frac{\mathbf{g}_k^*(0)\mathbf{g}_k(0)}{\mathbf{g}_k^*(t)\mathbf{g}_k(t)}} \right] = 0. \quad (\text{C12})$$

Therefore,

$$\begin{aligned} \phi_k^G(t) &= -i \ln \left(\frac{\langle \psi_k^i(0) | \psi_k^i(t) \rangle}{|\langle \psi_k^i(0) | \psi_k^i(t) \rangle|} \right) + \int_0^t ds \frac{\langle \psi_k^i(s) | \mathcal{H}_k^f | \psi_k^i(s) \rangle}{\langle \psi_k^i(s) | \psi_k^i(s) \rangle} - \frac{i}{2} \ln \left[\frac{\langle \psi_k^i(t) | \psi_k^i(t) \rangle}{\langle \psi_k^i(0) | \psi_k^i(0) \rangle} \right] \\ &= -i \ln \left(\frac{\langle \chi_k^i(0) | \chi_k^i(t) \rangle}{|\langle \chi_k^i(0) | \chi_k^i(t) \rangle|} \right) + \int_0^t ds \frac{\langle \chi_k^i(s) | \mathcal{H}_k^f | \chi_k^i(s) \rangle}{\langle \chi_k^i(s) | \chi_k^i(s) \rangle} - \frac{i}{2} \ln \left[\frac{\langle \chi_k^i(t) | \chi_k^i(t) \rangle}{\langle \chi_k^i(0) | \chi_k^i(0) \rangle} \right], \end{aligned} \quad (\text{C13})$$

implying that the geometric phase $\phi_k^G(t)$ is gauge invariant and thus experimentally measurable [114].

The geometric meaning of $\phi_k^G(t)$ is discussed in Ref. [95]. For a spin-1/2 system, it represents half of the solid angle subtended by an area on the Bloch sphere. The boundary of this area is formed by the physical trajectory of the state evolved from time zero to t in the parameter space, together with a geodesic connecting the two open ends of this physical trajectory. For nonunitary evolution of a cyclic state, $\phi_k^G(t)$ reduces to the real Aharonov-Anandan phase reported in Ref. [52].

APPENDIX D: MORE EXAMPLES OF DQPTS FOR NON-HERMITIAN QUENCHES ON HERMITIAN PREQUENCH HAMILTONIANS

In this Appendix, we give more computational examples of DQPTs following non-Hermitian quenches to the lattice model studied in the main text (see Fig. 6). The line of LYF zeros, rate function of return probability, geometric phase, and topological winding number will be presented for the following values of system parameters, where (μ_i, r_i, γ_i) and (μ_f, r_f, γ_f) are parameter value sets of $\mathcal{H}_k^i$ and $\mathcal{H}_k^f$, respectively. Different cases are numbered as continuations of the study in the main text:

Case	(μ_i, r_i, γ_i)	(μ_f, r_f, γ_f)
5	(0.6, 0.3, 0)	(0.6, 0.3, 0.5)
6	(0.2, 0.6, 0)	(0.2, 0.6, 0.5)
7	(-0.3, 0.6, 0)	(-0.3, 0.6, 1)
8	(-0.3, 0.6, 0)	(-0.3, 0.6, 2)

In case 5 (Fig. 7), we observe that even though there are still signatures of DQPTs, each line of LYF zeros crosses the imaginary time axis either zero or two times. The corresponding winding number changes in time in an oscillating manner, similar to the accidental DQPTs found in Hermitian systems [14].

In case 6 (Fig. 8), the vector $\mathbf{h}(k) = [h_x(k), h_z(k)]$ encircles both EPs after the quench. Each line of LYF zeros crosses the imaginary time axis an even number of times. The corresponding winding number changes in time in an oscillating fashion and the DQPTs are also accidental [14]. Notably, the

topological signature of the initial state appears as a sharp tail on each line of the LYF zeros, corresponding to a pair of DQPTs (with opposite change of winding numbers) happening within a relatively short time window.

In case 7 (Fig. 9), the initial state is topologically nontrivial while the vector $\mathbf{h}(k)$ encircles the EP at $(-\frac{\gamma}{2}, 0)$ after the quench. The resulting DQPTs are similar to those presented in case 4 of the main text. There are topological DQPTs accompanied by transient ones due to memory effects of the initial state. The transient ones decay quickly and only topological DQPTs related to crossing of the EP remain after an initial time window.

In case 8 (Fig. 10), the initial state is topologically nontrivial while the vector $\mathbf{h}(k)$ does not encircle any EPs. From the lines of LYF zeros, we see that only transient DQPTs related

FIG. 6. A sketch of the four quench cases studied in Appendix D. The open triangle denotes the zero. The two solid triangles represent EPs. The dashed circle represents the vector $\mathbf{h}(k) = [h_x(k), h_z(k)]$. (a) Case 5: The initial state is topologically trivial; $\mathbf{h}(k)$ does not encircle any EPs. (b) Case 6: The initial state is topologically nontrivial; $\mathbf{h}(k)$ encircles both two EPs. (c) Case 7: The initial state is topologically nontrivial; $\mathbf{h}(k)$ encircles one EP at $(-\frac{\gamma}{2}, 0)$. (d) Case 8: The initial state is topologically nontrivial; $\mathbf{h}(k)$ does not encircle any EPs.

FIG. 7. DQPTs for non-Hermitian quench case 5. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}^-$ with $n = 1, 2, 3, 4$. (b) Rate function of the return probability [99]. The first five DQPTs happen around $t \approx 9.2, 12, 14.8, 17.3$, and 17.8 . (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of topological winding number $\nu_D(t)$.

FIG. 9. DQPTs for non-Hermitian quench case 7. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}^-$ with $n = 1, 2, 3, 4$. (b) Rate function of the return probability [99]. The first four topological DQPTs happen around $t \approx 5.1, 8.2, 12.5$, and 16.7 . A pair of accidental DQPTs happens around $t \approx 10.6$. (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of topological winding number $\nu_D(t)$.

to the topological property of initial states appear. With the development of time, these DQPTs are harder and harder to observe due to their very short time windows.

APPENDIX E: DQPTS WITH NON-HERMITIAN PREQUENCH HAMILTONIANS

In the main text, we choose the prequench Hamiltonian H^i to be Hermitian. In this Appendix, we show that our general

FIG. 8. DQPTs for non-Hermitian quench case 6. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}^-$ with $n = 1, 2, 3, 4$. (b) Rate function of the return probability [99]. The first six DQPTs happen around $t \approx 11.5, 12.8, 15.3, 16.4, 16.8$, and 19.2 . (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of topological winding number $\nu_D(t)$.

FIG. 10. DQPTs for non-Hermitian quench case 8. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}^-$ with $n = 1, 2, 3, 4$. (b) Rate function of the return probability [99]. A pair of accidental DQPTs happens around $t \approx 3.9$. (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of topological winding number $\nu_D(t)$.

FIG. 11. A sketch of the four quench cases studied in Appendix E. The two open (solid) triangles represent EPs of the prequench (postquench) Hamiltonian. The dashed circle represents the trajectory of vector $\mathbf{h}(k) = [h_x(k), h_z(k)]$. (a) Case 1 and (c) case 3: No EPs are introduced inside the trajectory of $\mathbf{h}(k)$ through the quench. (b) Case 2: The EP at $(\frac{\gamma}{2}, 0)$ is introduced inside the trajectory of $\mathbf{h}(k)$ through the quench. (d) Case 4: Both EPs at $(\pm\frac{\gamma}{2}, 0)$ are introduced inside the trajectory of $\mathbf{h}(k)$ through the quench.

conclusions for DQPTs hold even if H^i is also non-Hermitian and the initial state fills one of its complex energy bands.

The formalism of the theory as presented in the main text remains the same for both Hermitian and non-Hermitian prequench Hamiltonians. So here we only present computational results for some typical parameter values summarized in the following table:

Case	(μ_i, r_i, γ_i)	(μ_f, r_f, γ_f)
1	(1.0, 0.3, 0.5)	(1.0, 0.3, 1.0)
2	(1.0, 0.3, 0.5)	(1.0, 0.3, 2.0)
3	(1.0, 0.3, 0.5)	(1.0, 0.3, 3.0)
4	(0.1, 0.6, 2.0)	(0.1, 0.6, 0.5)

Here (μ_i, r_i, γ_i) and (μ_f, r_f, γ_f) represent prequench and postquench system parameters, respectively. A sketch of the four quench cases is shown in Fig. 11. Results for the lines of LYF zeros, rate function, geometric phase, and topological winding number are presented for these four quench cases.

In case 1 (Fig. 12), no EPs are introduced inside the trajectory of $\mathbf{h}(k)$ through the quench. In this case, each line of LYF zeros has an even number of crossings with the imaginary time axis, and the topological winding number increases in time in an oscillating manner. Similar observations are made in case 5 of Appendix D, where the initial state is the topologically trivial ground state of a Hermitian H^i . So these cases also belong to the same topological class regardless of whether the initial Hamiltonian is Hermitian or not. Note that the transient memory effects are not seen here.

FIG. 12. DQPTs for quench case 1 in Appendix E. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}$ with $n = 1, 2, 3, 4, 5, 6, 7$. (b) Rate function of the return probability [99]. DQPTs happen around $t \approx 8.0, 9.9, 11.7, 12.0, 14.4, 16.8, 18.6$, and 19.3 . (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of the topological winding number $\nu_D(t)$.

In case 2 (Fig. 13), one EP at $(\frac{\gamma}{2}, 0)$ is introduced inside the trajectory of $\mathbf{h}(k)$ through the quench. In this case, each line of LYF zeros forms an open curve with a unique crossing on the imaginary time axis, and the topological winding number increases monotonically in time. This is similar to case 1 reported in the main text, where the initial state is topologically

FIG. 13. DQPTs for quench case 2 in Appendix E. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}$ with $n = 1, 2, 3, 4, 5$. (b) Rate function of the return probability [99]. DQPTs happen around $t \approx 3.7, 7.5, 11.3, 15.1$, and 18.9 . (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of the topological winding number $\nu_D(t)$.

FIG. 14. DQPTs for quench case 3 in Appendix E. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}$ with $n = 1, 2, 3, 4$. (b) Rate function of the return probability [99]. (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of topological winding number $\nu_D(t)$.

trivial. So these two cases belong to the same topological class regardless of whether the initial Hamiltonian is Hermitian or not.

In case 3 (Fig. 14), no EPs are introduced inside the trajectory of $\mathbf{h}(k)$ through the quench. Different from case 1, in this case every line of LYF zeros has no crossings with the imaginary time axis, and the topological winding number remains zero at all times. Similar observations are made in case 3 of the main text, where the initial state is the topologically trivial ground state of a Hermitian H^i . So these cases also belong to the same topological class regardless of whether the initial Hamiltonian is Hermitian or not.

In case 4 (Fig. 15), two EPs at $(\pm \frac{\gamma}{2}, 0)$ are introduced inside the trajectory of $\mathbf{h}(k)$ through the quench. Similar to case 1, each line of LYF zeros has an even number of crossings with the imaginary time axis, and the topological winding number increases in an oscillating manner in time. So it belongs to the same topological class as case 1 (Fig. 12).

APPENDIX F: DQPTS FOLLOWING NON-HERMITIAN QUENCHES FOR INFINITE-TEMPERATURE INITIAL STATES

In this Appendix, we show that the dynamics of an infinite-temperature initial state following a non-Hermitian quench is able to capture all essential features of DQPTs in non-Hermitian lattices. This provides with us a simple reference state for future experimental studies.

Our results in the main text showed that the memory of the system to its initial state does not last long. This suggests that if our aim is to probe the topological signature of an EP in the postquench dynamics, we have more degrees of freedom to choose the initial state. In particular, we here consider an infinite-temperature initial state described by a density matrix

FIG. 15. DQPTs for quench case 4 in Appendix E. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}$ with $n = 0, 1, 2, 3$. (b) Rate function of the return probability [99]. DQPTs happen around $t \approx 2.8, 4.7, 7.6, 12.0, 12.4, 17.2$, and 19.2 . (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of topological winding number $\nu_D(t)$.

$\rho_k^i = \frac{1}{2}\sigma_0$ at each quasimomentum k , where σ_0 is a 2×2 identity matrix. The return amplitude for such an initial state is given by

$$\mathcal{G}_k(t) = \text{Tr}(e^{-i\mathcal{H}_k^f t} \rho_k^i) = \cos(d_k^f t), \quad (\text{F1})$$

FIG. 16. A sketch of the four quench regions. The open triangle denotes the zero. The two solid triangles represent EPs of the postquench Hamiltonian. The dashed circle represents the postquench Hamiltonian vector $\mathbf{h}(k) = [h_x(k), h_z(k)]$. (a) Region 1: $\mathbf{h}(k)$ encircles one EP at $(+\frac{\gamma}{2}, 0)$. (b) Region 2: $\mathbf{h}(k)$ encircles one EP at $(-\frac{\gamma}{2}, 0)$. (c) Region 3: $\mathbf{h}(k)$ does not encircle any EPs. (d) Region 4: $\mathbf{h}(k)$ encircles both EPs at $(\pm \frac{\gamma}{2}, 0)$.

FIG. 17. DQPTs for non-Hermitian quench region 1. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}$ with $n = 0, 1, 2, 3, 4$. (b) Rate function of the return probability [115]. DQPTs happen around $t = t_{c,0}^0, 3t_{c,0}^0, 5t_{c,0}^0, 7t_{c,0}^0$, and $9t_{c,0}^0$. (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of topological winding number $\nu_D(t)$.

where for the non-Hermitian lattice model studied in the main text, we have

$$d_k^f = \sqrt{\mu^2 + 2\mu r \cos(k) + r^2 - \frac{\gamma^2}{4}} + i\gamma r \sin(k). \quad (\text{F2})$$

At a DQPT point, $\mathcal{G}_k(t_c) = 0$, which directly yields two possible critical momenta $k_c = 0, \pi$. The corresponding critical times reside in the real range of $t_{c,n}^0$ (for $k_c = 0$) and $t_{c,n}^\pi$ (for $k_c = \pi$), given by

$$t_{c,n}^0 = \frac{(2n+1)\pi}{2\sqrt{(\mu+r+\frac{\gamma}{2})(\mu+r-\frac{\gamma}{2})}},$$

$$t_{c,n}^\pi = \frac{(2n+1)\pi}{2\sqrt{(\mu-r+\frac{\gamma}{2})(\mu-r-\frac{\gamma}{2})}}, \quad (\text{F3})$$

where $n \in \mathbb{N}$. Following the main text, we let $r > 0$ and $\gamma > 0$. Referring to the relative location between the vector $\mathbf{h}(k) = [h_x(k), h_z(k)] = [\mu + r \cos(k), r \sin(k)]$ and the EPs, we can classify the postquench dynamics into four different regions:

(1) $\mu + r > \frac{\gamma}{2}$, $-\frac{\gamma}{2} < \mu - r < \frac{\gamma}{2}$: the vector $\mathbf{h}(k)$ only encircles the EP at $(\frac{\gamma}{2}, 0)$, and the critical time is given by $t_{c,n}^0$.

(2) $\mu - r < -\frac{\gamma}{2}$, $-\frac{\gamma}{2} < \mu + r < \frac{\gamma}{2}$: the vector $\mathbf{h}(k)$ only encircles the EP at $(-\frac{\gamma}{2}, 0)$, and the critical time is given by $t_{c,n}^\pi$.

(3) $|\mu \pm r| < \frac{\gamma}{2}$: the vector $\mathbf{h}(k)$ does not encircle any EPs, and there are no real critical times [i.e., both $t_{c,n}^0$ and $t_{c,n}^\pi$ given by Eq. (F3) are purely imaginary].

(4) $\mu - r < -\frac{\gamma}{2}$ and $\mu + r > \frac{\gamma}{2}$, or $\mu \pm r < -\frac{\gamma}{2}$, or $\mu \pm r > \frac{\gamma}{2}$: the vector $\mathbf{h}(k)$ either encircles both EPs $(\pm\frac{\gamma}{2}, 0)$, or does not encircle any one of them but avoids the branch cut

FIG. 18. DQPTs for non-Hermitian quench region 2. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}$ with $n = 0, 1, 2, 3, 4$. (b) Rate function of the return probability [115]. DQPTs happen around $t = t_{c,0}^\pi, 3t_{c,0}^\pi, 5t_{c,0}^\pi, 7t_{c,0}^\pi$, and $9t_{c,0}^\pi$. (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of topological winding number $\nu_D(t)$.

connecting the two EPs. The corresponding critical times are $t_{c,n}^0$ and $t_{c,n}^\pi$.

In the following, we check the signals of DQPTs with computational examples in these four regions. The system parameters considered in the examples are summarized in the following table:

Region	(μ, r, γ)
1	(0.6, 0.3, 1)
2	(-0.6, 0.3, 1)
3	(0.3, 0.6, 2)
4	(0.3, 0.6, 0.2)

When presenting the results, we refer to the region number instead of explicit parameter values. A sketch of the considered quench processes is shown in Fig. 16.

The line of LYF zeros is related to critical times by $z_{n,k}^{0/\pi} = it_{c,n}^{0/\pi}$. For the four dynamical regions, the lines of LYF zeros at different n are shown in corresponding figures. In regions 1 (Fig. 17) and 2 (Fig. 18), each line of LYF zeros has exactly one crossing on the imaginary time axis, implying dynamical topological phase transitions with monotonically increasing winding numbers in the postquench nonunitary dynamics. In region 3 (Fig. 19), all lines of LYF zeros avoid the imaginary time axis and no DQPTs are expected. In region 4 (Fig. 20), each line of LYF zeros crosses the imaginary time axis twice. DQPTs will happen in this region with oscillating-in-time winding numbers, and therefore also be regarded as accidental [14]. These observations are supported by the calculation of winding numbers in these four regions. Note that for the infinite-temperature initial state, we define

FIG. 19. DQPTs for non-Hermitian quench region 3. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}$ with $n = 0, 1, 2, 3, 4$. (b) Rate function of the return probability [115]. There are no DQPTs in this case. (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of topological winding number $\nu_D(t)$.

the dynamical phase factor as

$$\phi_k^{\text{dyn}}(t) = - \int_0^t ds \text{Re} \left[\frac{\text{Tr}(e^{i\mathcal{H}_k^f t} \mathcal{H}_k^f e^{-i\mathcal{H}_k^f t})}{\text{Tr}(e^{i\mathcal{H}_k^f t} e^{-i\mathcal{H}_k^f t})} \right]. \quad (\text{F4})$$

FIG. 20. DQPTs for non-Hermitian quench region 4. (a) Real and imaginary parts of lines of LYF zeros $z_{n,k}$ with $n = 0, 1, 2, 3, 4, 5$. (b) Rate function of the return probability [115]. Ticks on t axis refer to critical time of DQPTs. From left to right they are $t = t_{c,0}^0, t_{c,1}^0, t_{c,2}^0, t_{c,3}^0, t_{c,4}^0, t_{c,5}^0, t_{c,1}^\pi$, and $t_{c,5}^\pi$. (c) Geometric phase $\phi_k^G(t)$ versus quasimomentum k and time t . (d) Time evolution of topological winding number $\nu_D(t)$.

The definitions of total phase $\phi_k(t)$, geometric phase $\phi_k^G(t)$, and winding number $\nu_D(t)$ then follow the main text.

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