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THE REYNOLDS NUMBER AS THE PARAMETER IN THE STRONG SOLUTION TO THE NAVIER-STOKES EQUATION

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Abstract. It has been shown previously that the strong solution to the Navier-Stokes equation for an incompressible fluid can be written in the form of an integral mapping that involves the Green's function of the heat equation. In this work, it is demonstrated that the Reynolds number is the main parameter that controls the solution stability.

INTRODUCTION

The Navier-Stokes equation, together with the continuity equation, is the governing equation of fluid motion valid as long as the continuum hypothesis holds. For a wide variety of flow phenomena encountered in science and engineering, the assumption of constant properties is adequate. However, even when simplified by this assumption the Navier-Stokes equation still presents a challenge, namely the nonlinear convective term, which makes the solving procedure very cumbersome, if possible at all.

It is commonly believed that the Navier-Stokes equation indeed governs *any* fluid motion, including turbulence [1]. If it is so, no additional hypothesis should be made in order to describe turbulence by means of the Navier-Stokes equation. However, since no general method for solving the Navier-Stokes equation exists nowadays, it has become usual to resort to various models that often are inconsistent with the phenomenon in question. Thus, for example, the turbulence concept of eddy viscosity emerges independent of any physical properties of the fluid. The advantage of splitting the velocity components into time averaged and fluctuating parts is doubtful, for it leads to the closure problem of the Navier-Stokes equation. Besides, it is not completely clear how the averaging should take place in reality, for depending on the scale of the time step used for averaging the result can differ significantly.

It was demonstrated that the Navier-Stokes equation reduces to the Einstein-Kolmogorov (reaction-diffusion) equation under the assumption of the existence of narrow probability wave packets [2].

In addition, it was demonstrated that, without any further assumptions, if the Schrödinger equation is valid to describe a physical state of a fluid particle, the continuity equation implies the existence of the velocity potential [3]. This potential is nothing else but the phase of the corresponding wave function. Hence, the Cole-Hopf transformation [4], [5], in which the velocity function is expressed through the corresponding Schrödinger wave function, reduces the Navier-Stokes equation to a reaction-diffusion equation in the form of the Einstein-Kolmogorov equation. The solution to the resulting equation, and, by consequence, to the Navier-Stokes equation, can then be written in terms of the Green's function of the heat equation and is given in the form of an integral mapping. Such a form of the solution makes bifurcation period doubling possible, i.e. solutions for transition and turbulent flow regimes in spite of the existence of the velocity potential.

In this paper, it is demonstrated that the Reynolds number is the main parameter that controls the solution stability.

THE STRONG SOLUTION TO THE NAVIER-STOKES EQUATION

The existence of the velocity potential is a direct consequence from the derivation of the continuity equation from the Schrödinger equation. This implies that the Cole-Hopf transformation is applicable to the Navier-Stokes equation for an incompressible flow and allows reducing the Navier-Stokes equation to the Einstein-Kolmogorov equation, in which the reaction term depends on the pressure. The solution to the resulting equation, and to the Navier-Stokes equation as well, can then be written in terms of the Green's function of the heat equation and is given in the form of an integral mapping. Such a form of the solution makes bifurcation period doubling possible, i.e. solutions to transition and turbulent flow regimes in spite of the existence of the velocity potential.

It was shown that, if the function ψ is the Schrödinger probability wave function and ν is the kinematic viscosity of the fluid in question, the transformation

$$\mathbf{v}(\mathbf{x}) = -2\nu \frac{\nabla \psi}{\psi} \quad (1)$$

can be used, to linearize the Navier-Stokes equation.

The latter expression is known as the Cole-Hopf transformation [4], [5]. It was discovered by Hopf and Cole independently and is used to linearize the Burgers equation [6].

Upon substituting eqn. (1) into the Navier-Stokes equation written for the velocity field $\mathbf{v}$ of an incompressible flow

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} \quad (2)$$

one obtains a reaction-diffusion equation, known as the Einstein-Kolmogorov equation [7]

$$\frac{\partial \psi}{\partial t} - \nu \nabla^2 \psi = \frac{\Delta p}{2\mu} \psi \quad (3)$$

where μ is the dynamic viscosity of the fluid in question, p stands for pressure and Δp denotes the pressure difference between the local pressure and a certain reference pressure, e.g. the ambient pressure. A detailed derivation of eqn. (3) from (2), by means of (1), is shown in [3].

Notice that the corresponding transformations for the initial and boundary conditions are as follows

$$\psi(\mathbf{x}, 0) = C \exp\left(-i \frac{m}{\hbar} \int \mathbf{v}(\mathbf{x}, 0) d\mathbf{x}\right) \quad (4)$$

and

$$\psi = \exp\left(-\frac{1}{2\nu} \int \mathbf{v} \cdot d\mathbf{x}\right) \quad (5)$$

respectively. In the latter expressions, C is an arbitrary constant, m is the mass of the particle associated with the quantum wave packet, and $\hbar$ is the reduced Planck's constant.

The strong solution to the Navier-Stokes equation (2) can now be written in terms of the Green's function of the non-homogeneous diffusion equation [8, p. 409]:

$$\begin{aligned}\psi(\mathbf{x}, t) = & \int_0^t \iiint \wp(\mathbf{x}, t; \xi, \tau) \frac{\Delta p}{2\mu} \psi(\mathbf{x}, \tau) d^3\xi d\tau + \iiint \wp(\mathbf{x}, t; \xi, 0) \psi(\xi, 0) d^3\xi \\ & + \nu \int_0^t \oint \left[\wp(\mathbf{x}, t; \xi, \tau) \nabla_{\xi} \psi - \psi(\xi, \tau) \nabla_{\xi} \wp(\mathbf{x}, t; \xi, \tau) \right] \cdot \mathbf{n} dS d\tau\end{aligned}\quad (6)$$

where $\wp(\mathbf{x}, t; \xi, \tau)$ is the Green's function of the diffusion equation for the domain of interest.

THE REYNOLDS NUMBER AS THE MAIN PARAMETER THAT CONTROLS THE SOLUTION STABILITY

Notice that the parameter $\Delta p/2\mu$ in the first integral of (6) is nothing else but $2\text{Re}/t$, where Re denotes the Reynolds number. Hence, as expected, Re is the parameter that quantifies the flow regime. Notice here also that, because $2\text{Re}/t = 8\nu[\nabla\psi]^2/\psi$, the pressure field can be fully excluded from the set of equations governing the velocity field. In other words, the wave function, ψ , if determined, fully defines the flow velocity.

Thus, equation (6) assumes the form

$$\begin{aligned}\psi(\mathbf{x}, t) = & \frac{2\text{Re}}{t} \int_0^t \iiint \wp(\mathbf{x}, t; \xi, \tau) \psi(\mathbf{x}, \tau) d^3\xi d\tau + \iiint \wp(\mathbf{x}, t; \xi, 0) \psi(\xi, 0) d^3\xi \\ & + \nu \int_0^t \oint \left[\wp(\mathbf{x}, t; \xi, \tau) \nabla_{\xi} \psi - \psi(\xi, \tau) \nabla_{\xi} \wp(\mathbf{x}, t; \xi, \tau) \right] \cdot \mathbf{n} dS d\tau\end{aligned}\quad (7)$$

Furthermore, as the first integral in (6) remains bounded for any t and Re is finite, it follows that as $t \rightarrow \infty$, the first term in the right side of (6) becomes negligible in comparison with the second term. It means, in turn, that, provided a long enough time is given to a turbulent flow to exist, the flow re-laminarizes. This conclusion, drawn from observing (6), is consistent with the one drawn in [9], where it is claimed that the lifetimes of turbulence in pipe flow do not diverge at a finite critical Reynolds number. All the available data indicate that turbulence in pipe flows decays, and that eventually the flow will always re-laminarize. Similarly, lifetimes for the Couette flow and for a shear flow model do not appear to diverge, suggesting that a finite lifetime of turbulence may be a universal property of this class of flows.

The Feigenbaum theorem allows one to determine the local critical value of the Reynolds number [10]. Indeed,

$$\lim_{n \rightarrow \infty} \frac{\text{Re}_{n-1} - \text{Re}_{n-2}}{\text{Re}_n - \text{Re}_{n-1}} = \lim_{n \rightarrow \infty} \frac{v_{n-1} - v_{n-2}}{v_n - v_{n-1}} = \delta \quad (8)$$

where $\delta = 4.669\dots$ is the Feigenbaum constant.

Upon noticing from (1) that $\mathbf{v} \sim \nabla(\ln \psi)$, it follows that the value of the local critical Reynolds number for the transition to turbulence to begin is

$$\text{Re}_{\text{cr}} = e^{\delta} \approx 100 \quad (9)$$

The later result is in an excellent accord with the result obtained from the buckling model of the laminar-turbulent transition [11, 12, 13].

SOME GENERAL REMARKS

The present work establishes a fundamental link between the Schrödinger equation, which governs energy transport processes on quantum scales and the Navier-Stokes equation, which models phenomena of viscous fluid flow. It seems the very fact that the Navier-Stokes equation reduces to the Einstein-Kolmogorov equation (a non-homogeneous heat equation) is of fundamental importance, too. Indeed, if a fluid flow is seen from the topological perspective, it can be perceived as a space, whose curvature varies (in the case of turbulent flows, tremendously) with time and location. Yet, provided a sufficiently long time is given to the flow to exist, the solution, obtained here, predicts re-laminarization of the flow. In other words, in the course of time, smoothing of the flow space takes place; a direct consequence of the fact that the solution is written in terms of the Green's function of the heat equation. This, in turn, provides a striking similarity with the recent work by Perelman [13], [14], [15], who proved the Poincaré-Perelman theorem, in particular, using the fact that the evolution equation for the metric tensor implies the evolution equation for the curvature tensor, Rm , of the form

$$\frac{\partial Rm}{\partial t} = \nabla^2 Rm + Q \quad (10)$$

where Q is a certain quadratic expression of the curvatures. In our opinion, a similarity between the approaches in [14], [15], [16] and this work is not a coincidence, but reflects the very fundamental properties of matter and the governing laws of nature.

CONCLUSIONS

It has been shown previously that the existence of the velocity potential is a direct consequence from the derivation of the continuity equation from the Schrödinger equation. This implies that the Cole-Hopf transformation is applicable to the Navier-Stokes equation for an incompressible flow and allows reducing the Navier-Stokes equation to the Einstein-Kolmogorov equation, in which the reaction term depends on the pressure. The solution to the resulting equation, and to the Navier-Stokes equation as well, can then be written in terms of the Green's function of the heat equation and is given in the form of an integral mapping. Such a form of the solution makes bifurcation period doubling possible, i.e. solutions to transition and turbulent flow regimes in spite of the existence of the velocity potential.

In this work, it is demonstrated that the Reynolds number is the main parameter that controls the solution stability.

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