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The Spectrum of Eigenvalues for certain subgraphs of the k -point Fixing Graph

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Abstract

Let $\mathcal{S}_n$ be the symmetric group on n -points. The k -point fixing graph $\mathcal{F}(n, k)$ is defined to be the graph with vertex set $\mathcal{S}_n$ and two vertices g, h of $\mathcal{F}(n, k)$ are joined if and only if gh^{-1} fixes exactly k points. In this paper, we give a recurrence formula for the eigenvalues of a class of regular subgraphs of $\mathcal{F}(n, k)$. By using this recurrence formula, we will determine the smallest eigenvalues for this class of regular subgraphs of $\mathcal{F}(n, 1)$ for sufficiently large n .

KEYWORDS: Arrangement graph, Cayley graphs, symmetric group.

2010 MSC: 05C50, 05A05

1 Introduction

Let G be a finite group and S be an inverse closed subset of G , i.e., $1 \notin S$ and $s \in S \Rightarrow s^{-1} \in S$. The *Cayley graph* $\Gamma(G, S)$ is the graph which has the elements of G as its vertices and two vertices $u, v \in G$ are joined by an edge if and only if $v = su$ for some $s \in S$.

A Cayley graph $\Gamma(G, S)$ is said to be *normal* if S is closed under conjugation. It is well known that the eigenvalues of a normal Cayley graph $\Gamma(G, S)$ can be expressed in terms of the irreducible characters of G .

Theorem 1.1 ([1, 5, 20, 21]). *The eigenvalues of a normal Cayley graph $\Gamma(G, S)$ are given by*

$$\eta_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s),$$

where χ ranges over all the irreducible characters of G . Moreover, the multiplicity of η_χ is $\chi(1)^2$.

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Let $\mathcal{S}_n$ be the symmetric group on $[n] = \{1, \dots, n\}$ and $S \subseteq \mathcal{S}_n$ be closed under conjugation. Since central characters are algebraic integers ([12, Theorem 3.7 on p. 36]) and that the characters of the symmetric group are integers ([12, 2.12 on p. 31] or [24, Corollary 2 on p. 103]), by Theorem 1.1, the eigenvalues of $\Gamma(\mathcal{S}_n, S)$ are integers.

Corollary 1.2. *The eigenvalues of a normal Cayley graph $\Gamma(\mathcal{S}_n, S)$ are integers.*

For $k \leq n$, a k -permutation of $[n]$ is an injective function from $[k]$ to $[n]$. So any k -permutation π can be represented by a vector $(i_1, \dots, i_k)$ where $\pi(j) = i_j$ for $j = 1, \dots, k$. Let $1 \leq r \leq k \leq n$. The (n, k, r) -arrangement graph $A(n, k, r)$ has all the k -permutations of $[n]$ as vertices and two k -permutations are adjacent if they differ in exactly r positions. Formally, the vertex set $V(n, k)$ and edge set $E(n, k, r)$ of $A(n, k, r)$ are

$$\begin{aligned} V(n, k) &= \{(p_1, p_2, \dots, p_k) \mid p_i \in [n] \text{ and } p_i \neq p_j \text{ for } i \neq j\}, \\ E(n, k, r) &= \{(p_1, p_2, \dots, p_k), (q_1, q_2, \dots, q_k)\} \mid p_i \neq q_i \text{ for } i \in R \text{ and} \\ &\quad p_j = q_j \text{ for all } j \in [k] \setminus R \text{ for some } R \subseteq [k] \text{ with } |R| = r\}. \end{aligned}$$

Note that $|V(n, k)| = n!/(n-k)!$ and $A(n, k, r)$ is a regular graph [3, Theorem 4.2]. In particular, $A(n, k, 1)$ is a $k(n-k)$ -regular graph. We note here that $A(n, k, 1)$ was first introduced in [4] as an interconnection network model for parallel computation. Furthermore, $A(n, k, 1)$ is called the partial permutation graph by Kravovski and Mohar in [13]. The eigenvalues of the arrangement graphs $A(n, k, 1)$ were first studied in [2] by using a method developed by Godsil and McKay [10]. A relation between the eigenvalues of $A(n, k, r)$ and certain Cayley graphs was given in [3].

The *derangement graph* Γ_n is the Cayley graph $\Gamma(\mathcal{S}_n, D_n)$ where D_n is the set of derangements in $\mathcal{S}_n$. That is, two vertices g, h of Γ_n are joined if and only if $g(i) \neq h(i)$ for all $i \in [n]$, or equivalently gh^{-1} fixes no point. Since D_n is closed under conjugation, by Corollary 1.2, the eigenvalues of the derangement graph are integers. The lower and upper bounds of the absolute values of these eigenvalues have been studied in [18, 19, 22]. Note that derangement graph is a kind of arrangement graph, i.e., $\Gamma_n = A(n, n, n)$.

Let $0 \leq k < n$ and $\mathcal{S}(n, k)$ be the set of all $\sigma \in \mathcal{S}_n$ such that σ fixes exactly k elements. Note that $\mathcal{S}(n, k)$ is an inverse closed subset of $\mathcal{S}_n$. The k -point fixing graph is defined to be

$$\mathcal{F}(n, k) = \Gamma(\mathcal{S}_n, \mathcal{S}(n, k)).$$

That is, two vertices g, h of $\mathcal{F}(n, k)$ are joined if and only if gh^{-1} fixes exactly k points. Note that the k -point fixing graph is also a kind of arrangement graph, i.e., $\mathcal{F}(n, k) = A(n, n, n-k)$. Furthermore, the 0-point fixing graph is the derangement graph, i.e., $\mathcal{F}(n, 0) = \Gamma_n = A(n, n, n)$.

Clearly, $\mathcal{F}(n, k)$ is vertex-transitive, so it is $|\mathcal{S}(n, k)|$ -regular and the largest eigenvalue of $\mathcal{F}(n, k)$ is $|\mathcal{S}(n, k)|$. Furthermore, $\mathcal{S}(n, k)$ is closed under conjugation. Therefore, by Corollary 1.2, the eigenvalues of the k -point fixing graph are integers. Eigenvalues of $\mathcal{F}(n, k)$ was first studied in [14]. The smallest eigenvalue of $\mathcal{F}(n, 1)$ was determined in [16]. The partition associated to the smallest eigenvalue of $\mathcal{F}(n, k)$ was determined in [17].

Let $1 \leq i \leq n$. For each $\sigma \in \mathcal{S}_n$, let $f_i(\sigma)$ denote the number of i -cycle appearing in the cyclic decomposition of σ . Let C_i be the subset of $\mathcal{S}_n$, in which, each element in C_i contains at least an i -cycle

in its cyclic decomposition, i.e.,

$$C_i = \{\sigma \in \mathcal{S}_n : f_i(\sigma) > 0\}.$$

Note that C_i is a union of conjugacy classes of $\mathcal{S}_n$. Let

$$C_{(i_1, i_2, \dots, i_s)} = \bigcup_{j=1}^s C_{i_j}.$$

Note that $\mathcal{S}_n \setminus C_{(i_1, i_2, \dots, i_s)}$ is also a union of conjugacy classes of $\mathcal{S}_n$.

Let $\Gamma_n^{(j)} = \Gamma(\mathcal{S}_n, \mathcal{S}_n \setminus C_{(1, 2, \dots, j)})$, where $1 \leq j < n$. Note that

$$\mathcal{S}_n \setminus C_{(1, 2, \dots, j)} \subseteq D_n = \mathcal{S}_n \setminus C_{(1)}.$$

Therefore $\Gamma_n^{(j)}$ is a subgraph of the derangement graph $\Gamma_n^{(1)} = \Gamma_n$. In fact,

$$\Gamma_n^{(n-1)} \leq_{\text{sub}} \Gamma_n^{(n-2)} \leq_{\text{sub}} \dots \leq_{\text{sub}} \Gamma_n^{(2)} \leq_{\text{sub}} \Gamma_n^{(1)},$$

where $H \leq_{\text{sub}} K$ means that H is a subgraph of K . The smallest eigenvalues of $\Gamma_n^{(j)}$ was determined in [15].

For $0 \leq k < n$, $2 \leq j < n$, set

$$\mathcal{S}^{(j)}(n, k) = \mathcal{S}(n, k) \setminus C_{(2, 3, \dots, j)},$$

and

$$\mathcal{F}^{(j)}(n, k) = \Gamma(\mathcal{S}_n, \mathcal{S}^{(j)}(n, k)).$$

Note that $\mathcal{S}^{(j)}(n, k)$ is a union of conjugacy classes of $\mathcal{S}_n$, and $\mathcal{F}^{(j)}(n, k)$ is a $|\mathcal{S}^{(j)}(n, k)|$ -regular Cayley graph. Furthermore,

$$\mathcal{S}^{(j)}(n, k) \subseteq \mathcal{S}(n, k) \quad \text{and if } k \geq 1, \quad \mathcal{S}(n, k) \setminus C_{(1)} = \emptyset.$$

Therefore $\mathcal{F}^{(j)}(n, k)$ is a subgraph of $\mathcal{F}(n, k)$. In fact,

$$\mathcal{F}^{(n-1)}(n, k) \leq_{\text{sub}} \mathcal{F}^{(n-2)}(n, k) \leq_{\text{sub}} \dots \leq_{\text{sub}} \mathcal{F}^{(2)}(n, k) \leq_{\text{sub}} \mathcal{F}(n, k).$$

Note also that

$$\begin{aligned} \mathcal{S}^{(j)}(n, 0) &= \mathcal{S}(n, 0) \setminus C_{(2, 3, \dots, j)} \\ &= D_n \setminus C_{(2, 3, \dots, j)} \\ &= (\mathcal{S}_n \setminus C_{(1)}) \setminus C_{(2, 3, \dots, j)} \\ &= \mathcal{S}_n \setminus C_{(1, 2, 3, \dots, j)}. \end{aligned}$$

Hence,

$$\mathcal{F}^{(j)}(n, 0) = \Gamma_n^{(j)}. \tag{1}$$

A partition λ of n , denoted by $\lambda \vdash n$, is a weakly decreasing sequence $\lambda_1 \geq \dots \geq \lambda_r$ with $\lambda_r \geq 1$ such that $\lambda_1 + \dots + \lambda_r = n$. We write $\lambda = (\lambda_1, \dots, \lambda_r)$. The *size* of λ , denoted by $|\lambda|$, is n and each λ_i is

called the i -th part of the partition. We also use the notation $(\mu_1^{a_1}, \dots, \mu_s^{a_s}) \vdash n$ to denote the partition where μ_i are the distinct nonzero parts that occur with multiplicity a_i . For example,

$$(5, 5, 4, 4, 2, 2, 2, 1) \longleftrightarrow (5^2, 4^2, 2^3, 1).$$

It is well known that both the conjugacy classes of $\mathcal{S}_n$ and the irreducible characters of $\mathcal{S}_n$ are indexed by partitions λ of $[n]$. Since $\mathcal{S}(n, k) \setminus C_{(2,3,\dots,j)}$ is closed under conjugation, the eigenvalue $\eta_{\chi_\lambda}^{(j)}(k)$ of $\mathcal{F}^{(j)}(n, k)$ can be denoted by $\eta_\lambda^{(j)}(k)$. Note here that $0 \leq k < n$ and $2 \leq j < n$. Throughout the paper, we shall use this notation.

The following theorem was proved in [15, Theorem 1.3].

Theorem 1.3. *Let j, n be positive integers and $j \leq n^\delta$ with $0 < \delta < \frac{2}{3}$. Then for sufficiently large n , the smallest eigenvalue of $\mathcal{F}^{(j)}(n, 0)$ is equal to*

$$\eta_{(n-1,1)}^{(j)}(0) = -\frac{d_n^{(j)}}{n-1}$$

where $d_n^{(j)} = |\mathcal{S}_n \setminus C_{(1,2,\dots,j)}|$. Furthermore, $\eta_\lambda^{(j)}(0) = -\frac{d_n^{(j)}}{n-1}$ if and only if $\lambda = (n-1, 1)$.

In this paper, we will prove the following theorem.

Theorem 1.4. *Let j, n be positive integers and $2 \leq j \leq n^\delta$ where $0 < \delta < \frac{1}{3}$. Then, for sufficiently large n , the smallest eigenvalue of $\mathcal{F}^{(j)}(n, 1)$ is equal to*

$$\eta_{(n-2,2)}^{(j)}(1) = -\frac{2}{n-3}d_{n-1}^{(j)}$$

where $d_n^{(j)} = |\mathcal{S}_n \setminus C_{(1,2,\dots,j)}|$. Furthermore, $\eta_\lambda^{(j)}(1) = -\frac{2}{n-3}d_{n-1}^{(j)}$ if and only if $\lambda = (n-2, 2)$.

The paper is organized as follows. In Section 2, we prove a recurrence formula for the eigenvalues of $\mathcal{F}^{(j)}(n, k)$ (Theorem 2.9). In Section 3, we will give the exact values of the eigenvalues $\eta_\lambda^{(j)}(1)$ for $\lambda \in \{(n), (1^n), (n-1, 1), (2, 1^{n-2}), (n-2, 2), (2^2, 1^{n-4}), (n-2, 1^2), (3, 1^{n-3})\}$ and for other λ we give an upper bound for $|\eta_\lambda^{(j)}(1)|$. In Section 4, we will prove Theorem 1.4.

2 Recurrence formula

Let $\lambda \vdash n$. For a box with coordinate (a, b) in the Ferrers diagram of λ , the *hook-length* $h_\lambda(a, b)$ is the size of the set of all the boxes with coordinate (i, j) where $i = a$ and $j \geq b$, or $i \geq a$ and $j = b$. The following lemma is well-known [9, 4.12 on p. 50].

Lemma 2.1. (Hook Formula)

$$f^\lambda = \chi_\lambda(1) = \frac{n!}{\prod h_\lambda(a, b)},$$

where the product is over all the boxes (a, b) in the Ferrers diagram of λ .

The following recurrence formula for the eigenvalues $\eta_\lambda(k)$ of $\mathcal{F}(n, k)$ corresponding to the partition λ of n was proved in [14, Theorem 3.7]. By using similar technique as in the proof of Theorem 2.2, we will prove Theorem 2.9.

Theorem 2.2. *Let $0 < k < n$ and $\lambda \vdash n$. Let $\mu_1, \dots, \mu_q$ be the only partitions of $(n-1)$ such that the Ferrers diagram of μ_i is obtained by removing 1 node from the right hand side of a row of the Ferrers diagram of λ . Then*

$$\eta_\lambda(k) = \frac{n}{kf^\lambda} \sum_{m=1}^q f^{\mu_m} \eta_{\mu_m}(k-1).$$

For each $\sigma \in \mathcal{S}_n$, we denote its conjugacy class by $\text{Con}_{\mathcal{S}_n}(\sigma)$, i.e., $\text{Con}_{\mathcal{S}_n}(\sigma) = \{\gamma^{-1}\sigma\gamma : \gamma \in \mathcal{S}_n\}$. Let $\mu \vdash n$ be the partition that represents $\text{Con}_{\mathcal{S}_n}(\sigma)$. We shall denote the size of $\text{Con}_{\mathcal{S}_n}(\sigma)$ by $N_{\mathcal{S}_n}(\mu)$.

Let $A \subseteq \mathcal{S}_n$ and $\alpha \in \mathcal{S}_n$. The set $\alpha^{-1}A\alpha$ is defined as

$$\alpha^{-1}A\alpha = \{\alpha^{-1}\sigma\alpha : \sigma \in A\}.$$

Let $0 \leq k < n$. Each $\beta \in \mathcal{S}_{n-k}$ can be considered as an element $\bar{\beta}$ of $\mathcal{S}_n$ by defining $\bar{\beta}(m) = \beta(m)$ for $1 \leq m \leq n-k$ and $\bar{\beta}(m) = m$ for $n-k+1 \leq m \leq n$. The $\bar{\beta}$ is called the *extension* of β to $\mathcal{S}_n$. The set of derangements $\mathcal{S}(n-k, 0)$ in $\mathcal{S}_{n-k}$ can be considered as a subset of $\mathcal{S}_n$ ($\overline{\mathcal{S}(n-k, 0)} = \{\bar{\sigma} : \sigma \in \mathcal{S}(n-k, 0)\}$).

Lemma 2.3. ([14, Lemma 3.1])

$$\mathcal{S}(n, k) = \bigcup_{\sigma \in \mathcal{S}_n} \sigma^{-1} \overline{\mathcal{S}(n-k, 0)} \sigma.$$

Let $\overline{\mathcal{S}^{(j)}(n-k, 0)} = \{\bar{\sigma} : \sigma \in \mathcal{S}^{(j)}(n-k, 0)\}$. Recall that $\mathcal{S}^{(j)}(n-k, 0)$ is the set of all derangements in $\mathcal{S}_{n-k}$ that do not consist of any t -cycle for $2 \leq t \leq j$, i.e.,

$$\mathcal{S}^{(j)}(n-k, 0) = \{\sigma \in D_{n-k} : f_t(\sigma) = 0 \ \forall 2 \leq t \leq j\}.$$

Lemma 2.4. *For $2 \leq j < n$,*

$$\mathcal{S}^{(j)}(n, k) = \bigcup_{\sigma \in \mathcal{S}_n} \sigma^{-1} \overline{\mathcal{S}^{(j)}(n-k, 0)} \sigma.$$

Proof. By Lemma 2.3,

$$\begin{aligned} \mathcal{S}^{(j)}(n, k) &= \mathcal{S}(n, k) \setminus C_{(2, 3, \dots, j)} \\ &= \left(\bigcup_{\sigma \in \mathcal{S}_n} \sigma^{-1} \overline{\mathcal{S}(n-k, 0)} \sigma \right) \setminus C_{(2, 3, \dots, j)} \\ &= \bigcup_{\sigma \in \mathcal{S}_n} \left(\left(\sigma^{-1} \overline{\mathcal{S}(n-k, 0)} \sigma \right) \setminus C_{(2, 3, \dots, j)} \right) \\ &= \bigcup_{\sigma \in \mathcal{S}_n} \sigma^{-1} \left(\overline{\mathcal{S}(n-k, 0)} \setminus C_{(2, 3, \dots, j)} \right) \sigma \\ &= \bigcup_{\sigma \in \mathcal{S}_n} \sigma^{-1} \overline{\mathcal{S}^{(j)}(n-k, 0)} \sigma. \end{aligned}$$

The second last equality in the above follows by noting that $\sigma^{-1}C_{(2, 3, \dots, j)}\sigma = C_{(2, 3, \dots, j)}$ for all $\sigma \in \mathcal{S}_n$. $\square$

By Lemma 2.4, there are $\sigma_{k1}, \sigma_{k2}, \dots, \sigma_{ks_k} \in \mathcal{S}^{(j)}(n-k, 0)$, such that

$$\mathcal{S}^{(j)}(n, k) = \bigcup_{i=1}^{s_k} \text{Con}_{\mathcal{S}_n}(\bar{\sigma}_{ki}). \quad (2)$$

and σ_{ki} is not conjugate to σ_{kp} in $\mathcal{S}_{n-k}$ for $i \neq p$. Furthermore,

$$\mathcal{S}^{(j)}(n-k, 0) = \bigcup_{i=1}^{s_k} \text{Con}_{\mathcal{S}_{n-k}}(\sigma_{ki}). \quad (3)$$

Note that $\chi_\lambda(\sigma) = \chi_\lambda(\beta)$ for all $\sigma \in \text{Con}_{\mathcal{S}_n}(\beta)$. Let $\text{Con}_{\mathcal{S}_n}(\beta)$ be represented by the partition $\varphi(\beta) \vdash n$. Then by Theorem 1.1 and Corollary 1.2, the eigenvalues of $\mathcal{F}^{(j)}(n, k)$ are integers given by

$$\eta_\lambda^{(j)}(k) = \frac{1}{f_\lambda} \sum_{i=1}^{s_k} N_{\mathcal{S}_n}(\varphi(\bar{\sigma}_{ki})) \chi_\lambda(\varphi(\bar{\sigma}_{ki})), \quad (4)$$

where $\chi_\lambda(\varphi(\bar{\sigma}_{ki})) = \chi_\lambda(\bar{\sigma}_{ki})$.

Let $1 \leq k < n$. Note that each $\bar{\sigma}_{ki}$ ($1 \leq i \leq s_k$) must consist of at least one 1-cycle in its cycle decomposition. Therefore $\varphi(\bar{\sigma}_{ki}) = (\nu_1, \nu_2, \dots, \nu_r) \vdash n$ and $\nu_r = 1$. Note that $\varphi(\bar{\sigma}_{ki}) - \hat{l}_{\varphi(\bar{\sigma}_{ki})} = (\nu_1, \nu_2, \dots, \nu_{r-1}) \vdash (n-1)$. We are now ready to state the following lemma which is a special case of [8, Theorem 3.4].

Lemma 2.5. *If the Ferrers diagrams obtained from λ by removing 1 node from the right hand side from any row of the diagram so that the resulting diagram will still be a partition of $(n-1)$ are those of $\mu_1, \dots, \mu_q$, then*

$$\chi_\lambda(\varphi(\bar{\sigma}_{ki})) = \sum_{m=1}^q \chi_{\mu_m}(\varphi(\bar{\sigma}_{ki}) - \hat{l}_{\varphi(\bar{\sigma}_{ki})}),$$

for all $1 \leq i \leq s_k$.

Example 2.6. Let $n = 7$ and $\lambda = (3, 3, 1)$, then

$$\begin{aligned} \chi_{(3,3,1)}((6, 1)) &= \chi_{(3,3)}((6)) + \chi_{(3,2,1)}((6)), \\ \chi_{(3,3,1)}((4, 2, 1)) &= \chi_{(3,3)}((4, 2)) + \chi_{(3,2,1)}((4, 2)), \\ \chi_{(3,3,1)}((3, 3, 1)) &= \chi_{(3,3)}((3, 3)) + \chi_{(3,2,1)}((3, 3)), \\ \chi_{(3,3,1)}((2, 2, 2, 1)) &= \chi_{(3,3)}((2, 2, 2)) + \chi_{(3,2,1)}((2, 2, 2)). \end{aligned}$$

Lemma 2.7. [25, (7.18) on p. 299] Let $\lambda = (n^{m_n}, \dots, 2^{m_2}, 1^{m_1}) \vdash n$ and $z_\lambda = \prod_{t=1}^n (t^{m_t} m_t!)$, then the size of the conjugacy class represented by λ is

$$N_{\mathcal{S}_n}(\lambda) = \frac{n!}{z_\lambda}.$$

Lemma 2.8. ([14, Lemma 3.6]) Let $\lambda = (\lambda_1, \dots, \lambda_r) \vdash (n-k)$ be a derangement, i.e., $\lambda_r \geq 2$. If

$$\nu = (\lambda, 1^k) \vdash n, \text{ and } \mu = (\lambda, 1^{k-1}) \vdash (n-1),$$

then

$$N_{\mathcal{S}_n}(\nu) = \frac{n}{k} N_{\mathcal{S}_{n-1}}(\mu).$$

Theorem 2.9. *Let $1 \leq k < n$, $2 \leq j < n$ and $\lambda \vdash n$. If the Ferrers diagrams obtained from λ by removing 1 node from the right hand side from any row of the diagram so that the resulting diagram will still be a partition of $(n-1)$ are those of $\mu_1, \dots, \mu_q$, then*

$$\eta_\lambda^{(j)}(k) = \frac{n}{kf^\lambda} \sum_{m=1}^q f^{\mu_m} \eta_{\mu_m}^{(j)}(k-1).$$

Proof. Suppose $k = 1$. By Equation (4),

$$\eta_\lambda^{(j)}(1) = \frac{1}{f^\lambda} \sum_{i=1}^{s_1} N_{\mathcal{S}_n}(\varphi(\bar{\sigma}_{1i})) \chi_\lambda(\varphi(\bar{\sigma}_{1i})).$$

Note that $\bar{\sigma}_{1i}$ consists of exactly one 1-cycle and it does not have any t -cycle for $2 \leq t \leq j$. So, $\varphi(\bar{\sigma}_{1i}) = (\nu_1, \nu_2, \dots, \nu_r) \vdash n$ with $\nu_r = 1$, $\nu_{r-1} \geq j+1$, and $\varphi(\bar{\sigma}_{1i}) - \hat{l}_{\varphi(\bar{\sigma}_{1i})} = (\nu_1, \nu_2, \dots, \nu_{r-1})$ is the partition of $(n-1)$ that represents $\text{Con}_{\mathcal{S}_{n-1}}(\sigma_{1i})$. By Lemma 2.5 and Lemma 2.8,

$$\begin{aligned} \eta_\lambda^{(j)}(1) &= \frac{1}{f^\lambda} \sum_{i=1}^{s_1} N_{\mathcal{S}_n}(\varphi(\bar{\sigma}_{1i})) \left(\sum_{m=1}^q \chi_{\mu_m}(\varphi(\bar{\sigma}_{1i}) - \hat{l}_{\varphi(\bar{\sigma}_{1i})}) \right) \\ &= \frac{1}{f^\lambda} \sum_{i=1}^{s_1} n N_{\mathcal{S}_{n-1}}(\varphi(\bar{\sigma}_{1i}) - \hat{l}_{\varphi(\bar{\sigma}_{1i})}) \left(\sum_{m=1}^q \chi_{\mu_m}(\varphi(\bar{\sigma}_{1i}) - \hat{l}_{\varphi(\bar{\sigma}_{1i})}) \right) \\ &= \frac{n}{f^\lambda} \sum_{m=1}^q \left(\sum_{i=1}^{s_1} N_{\mathcal{S}_{n-1}}(\varphi(\bar{\sigma}_{1i}) - \hat{l}_{\varphi(\bar{\sigma}_{1i})}) \chi_{\mu_m}(\varphi(\bar{\sigma}_{1i}) - \hat{l}_{\varphi(\bar{\sigma}_{1i})}) \right) \\ &= \frac{n}{f^\lambda} \sum_{m=1}^q f^{\mu_m} \eta_{\mu_m}^{(j)}(0), \end{aligned}$$

where the last equality follows from Equations (3) and (4). Thus, the theorem holds for $k = 1$.

Suppose $k > 1$. (We note here that the proof for $k > 1$ is similar to the proof for $k = 1$. The reason we distinguish them is to make the proof easier to comprehend.)

By Equation (4),

$$\eta_\lambda^{(j)}(k) = \frac{1}{f^\lambda} \sum_{i=1}^{s_k} N_{\mathcal{S}_n}(\varphi(\bar{\sigma}_{ki})) \chi_\lambda(\varphi(\bar{\sigma}_{ki})).$$

Note that $\bar{\sigma}_{ki}$ consists of exactly k 1-cycles and it does not have any t -cycle for $2 \leq t \leq j$. So, $\varphi(\bar{\sigma}_{ki}) = (\nu_1, \nu_2, \dots, \nu_r) \vdash n$ with $\nu_m = 1$ for $r-k+1 \leq m \leq r$ and $\nu_{r-k} \geq j+1$. Let $\bar{\bar{\sigma}}_{ki}$ be the extension of σ_{ki} to $\mathcal{S}_{n-1}$, i.e., $\bar{\bar{\sigma}}_{ki}(m) = \sigma_{ki}(m)$ for $1 \leq m \leq n-k$ and $\bar{\bar{\sigma}}_{ki}(m) = m$ for $n-k+1 \leq m \leq n-1$. Note that $\varphi(\bar{\sigma}_{ki}) - \hat{l}_{\varphi(\bar{\sigma}_{ki})} = (\nu_1, \nu_2, \dots, \nu_{r-1}) \vdash (n-1)$ is the partition of $(n-1)$ that represents $\text{Con}_{\mathcal{S}_{n-1}}(\bar{\bar{\sigma}}_{ki})$. Furthermore,

$$\mathcal{S}^{(j)}(n-1, k-1) = \bigcup_{i=1}^{s_k} \text{Con}_{\mathcal{S}_{n-1}}(\bar{\bar{\sigma}}_{ki}).$$

Therefore, by Theorem 1.1,

$$\eta_{\mu_m}^{(j)}(k-1) = \frac{1}{f^{\mu_m}} \sum_{i=1}^{s_k} N_{\mathcal{S}_{n-1}}(\varphi(\bar{\sigma}_{ki}) - \hat{l}_{\varphi(\bar{\sigma}_{ki})}) \chi_{\mu_m}(\varphi(\bar{\sigma}_{ki}) - \hat{l}_{\varphi(\bar{\sigma}_{ki})}).$$

By Lemmas 2.5 and 2.8,

$$\begin{aligned}
\eta_\lambda^{(j)}(k) &= \frac{1}{f^\lambda} \sum_{i=1}^{s_k} N_{\mathcal{S}_n}(\varphi(\bar{\sigma}_{ki})) \left(\sum_{m=1}^q \chi_{\mu_m}(\varphi(\bar{\sigma}_{ki}) - \hat{l}_{\varphi(\bar{\sigma}_{ki})}) \right) \\
&= \frac{1}{f^\lambda} \sum_{i=1}^{s_k} \frac{n}{k} N_{\mathcal{S}_{n-1}}(\varphi(\bar{\sigma}_{ki}) - \hat{l}_{\varphi(\bar{\sigma}_{ki})}) \left(\sum_{m=1}^q \chi_{\mu_m}(\varphi(\bar{\sigma}_{ki}) - \hat{l}_{\varphi(\bar{\sigma}_{ki})}) \right) \\
&= \frac{n}{k f^\lambda} \sum_{m=1}^q \left(\sum_{i=1}^{s_k} N_{\mathcal{S}_{n-1}}(\varphi(\bar{\sigma}_{ki}) - \hat{l}_{\varphi(\bar{\sigma}_{ki})}) \chi_{\mu_m}(\varphi(\bar{\sigma}_{1i}) - \hat{l}_{\varphi(\bar{\sigma}_{1i})}) \right) \\
&= \frac{n}{k f^\lambda} \sum_{m=1}^q f^{\mu_m} \eta_{\mu_m}^{(j)}(k-1).
\end{aligned}$$

Hence, the theorem holds for $k > 1$. □

3 Eigenvalues of $\mathcal{F}^{(j)}(n, 1)$

The terms and notations used here are standard. For undefined terms, we refer the reader to Sagan [23], Stanley [25] or Fulton and Harris [9]. The argument used here is similar to that of Ellis [6]. So, for the undefined terms, the reader may also refer to [6, Section 2.1, 2.2].

Let $\lambda \vdash n$. A λ -*tableau* is produced by placing the numbers $1, 2, \dots, n$ into the cells of the Young diagram of λ in some order. Two λ -tableaux are said to be *row-equivalent* if the corresponding rows of the two λ -tableaux contain the same elements. A λ -*tabloid* is a row-equivalence class of λ -tableaux (see [23, Section 2.1] for more details on the notion of λ -tableau).

We shall denote the Specht module and the permutation module corresponding to λ by S^λ and M^λ , respectively (see [23, Sections 2.1, 2.2 and 2.3] for more details on the notions of Specht module and the permutation module). The irreducible character of S^λ will be denoted by χ_λ and the character of M^λ will be denoted by ξ_λ . For each $\sigma \in \mathcal{S}_n$, $\xi_\lambda(\sigma)$ is the number of λ -tabloids fixed by σ .

A *semistandard λ -tableau* is produced by replacing each cell in the Young diagram of λ with a number between 1 and n , repetitions allowed, and such that the numbers in each row are non-decreasing and the numbers in each columns are strictly increasing. The *content* of a semistandard λ -tableau is $\beta = (\beta_1, \beta_2, \dots, \beta_n)$, where β_i equals to the number of times i appears in the semistandard λ -tableau.

Let $\lambda, \mu \vdash n$. The *Kostka number*, $K_{\lambda, \mu}$ is the number of semistandard λ -tableau with content μ .

Theorem 3.1. ([23, Theorem 2.11.2 on p. 85] Young's Rule) *For any $\beta \vdash n$,*

$$M^\beta \cong \bigoplus_{\lambda \vdash n} K_{\lambda, \beta} S^\lambda.$$

For example, $M^{(n-1, 1)}$, which corresponds to the natural permutation action of $\mathcal{S}_n$ on $[n]$, decomposes as

$$M^{(n-1, 1)} \cong S^{(n-1, 1)} \oplus S^{(n)}.$$

Therefore, $\xi_{(n-1,1)} = \chi_{(n-1,1)} + 1$ and for any $\sigma \in \mathcal{S}_n$,

$$\begin{aligned}\chi_{(n-1,1)}(\sigma) &= \#\{(n-1,1)\text{-tabloids fixed by } \sigma\} - 1 \\ &= \#\{\text{fixed points of } \sigma\} - 1.\end{aligned}$$

This implies that

$$\chi_{(n-1,1)}(\sigma) = -1, \quad (5)$$

for all $\sigma \in \mathcal{S}_n \setminus C_{(1,2,\dots,j)}$.

Lemma 3.2.

- (a) $M^{(n-2,2)} = S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,2)},$
- (b) $M^{(n-2,1^2)} = S^{(n)} \oplus 2S^{(n-1,1)} \oplus S^{(n-2,2)} \oplus S^{(n-2,1^2)}.$

Proof. (a) Let $\lambda \vdash n$ and $\mu = (n-2, 2)$. A semistandard λ -tableau with content μ is produced by replacing each cell in the Young diagram of λ with $(n-2)$ “1”s and two “2”s, such that the numbers in each row are non-decreasing and the numbers in each columns are strictly increasing. This means $K_{\lambda,\mu} \neq 0$ if and only if $\lambda \in \{(n), (n-1, 1), (n-2, 2)\}$. In fact, $K_{\lambda,\mu} = 1$ for all these λ 's. Hence, (a) follows from Theorem 3.1.

(b) Let $\lambda \vdash n$ and $\mu = (n-2, 1^2)$. A semistandard λ -tableau with content μ is produced by replacing each cell in the Young diagram of λ with $(n-2)$ “1”s, one “2” and one “3”, such that the numbers in each row are non-decreasing and the numbers in each columns are strictly increasing. This means $K_{\lambda,\mu} \neq 0$ if and only if $\lambda \in \{(n), (n-1, 1), (n-2, 2), (n-2, 1^2)\}$. In fact, $K_{\lambda,\mu} = 1$ for all these λ 's except $K_{\lambda,\mu} = 2$ when $\lambda = (n-1, 1)$. Hence, (b) follows from Theorem 3.1. $\square$

Note that if a $(n-2, 2)$ -tabloid is fixed by a $\sigma \in \mathcal{S}_n$, then σ must contain a 2-cycle or two 1-cycle in its cyclic decomposition. Every element in $\mathcal{S}_n \setminus C_{(1,2,\dots,j)}$ does not contain t -cycle for $1 \leq t \leq j$. Therefore, $\xi_{(n-2,2)}(\sigma) = 0$ for all $\sigma \in \mathcal{S}_n \setminus C_{(1,2,\dots,j)}$ where $2 \leq j < n$. By part (a) of Lemma 3.2 and Equation (5),

$$\chi_{(n-2,2)}(\sigma) = \xi_{(n-2,2)}(\sigma) - \chi_{(n-1,1)}(\sigma) - \chi_{(n)}(\sigma) = 0 - (-1) - 1 = 0. \quad (6)$$

Similarly, $\xi_{(n-2,1^2)}(\sigma) = 0$ for all $\sigma \in \mathcal{S}_n \setminus C_{(1,2,\dots,j)}$ where $2 \leq j < n$. So, by part (b) of Lemma 3.2, Equations (5) and (6),

$$\begin{aligned}\chi_{(n-2,1^2)}(\sigma) &= \xi_{(n-2,1^2)}(\sigma) - \chi_{(n-2,2)}(\sigma) - 2\chi_{(n-1,1)}(\sigma) - \chi_{(n)}(\sigma) \\ &= 0 - 0 - 2(-1) - 1 = 1.\end{aligned} \quad (7)$$

Let $\lambda \vdash n$ and $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_m)$ where λ'_i is the number of cells in the i th column of the Young diagram of λ . The λ' is called the *conjugate partition* of λ . Basically, λ' is the partition of n corresponding to the Young diagram obtained by interchanging rows with columns in the Young diagram of λ . For example, if $\lambda = (4, 3, 1)$ then $\lambda' = (3, 2, 2, 1)$.

Let $\epsilon = \chi_{1^n}$. Note that ϵ is the sign function for $\mathcal{S}_n$, i.e., $\epsilon(\sigma) = 1$ if σ is an even permutation and $\epsilon(\sigma) = -1$ if σ is an odd permutation. Let $\lambda \vdash n$ and λ' be its conjugate partition. It is well-known that $S^{\lambda'} \cong S^{(1^n)} \otimes S^\lambda$ (see [9, part (c) of 4.4 on p. 47]). As a consequence, we have the following lemma.

Lemma 3.3. *Let $\lambda \vdash n$ and λ' be its conjugate partition. Then*

$$\chi_{\lambda\lambda'} = \epsilon \cdot \chi_{\lambda}.$$

The following two equations follow from Lemma 3.3 and Equations (6) and (7), respectively.

$$\chi_{(2^2, 1^{n-4})}(\sigma) = \epsilon(\sigma) \cdot \chi_{(n-2, 2)}(\sigma) = 0; \quad (8)$$

$$\chi_{(3, 1^{n-3})}(\sigma) = \epsilon(\sigma) \cdot \chi_{(n-2, 1^2)}(\sigma) = \epsilon(\sigma); \quad (9)$$

for all $\sigma \in \mathcal{S}_n \setminus C_{(1, 2, \dots, j)}$ where $2 \leq j < n$

We shall use the following notations:

- (a) $e_n^{(j)}$ is the number of even permutations in $\mathcal{S}_n \setminus C_{(1, 2, \dots, j)}$;
- (b) $o_n^{(j)}$ is the number of odd permutations in $\mathcal{S}_n \setminus C_{(1, 2, \dots, j)}$;
- (c) $s_n^{(j)} = e_n^{(j)} - o_n^{(j)}$;
- (d) $d_n^{(j)} = |\mathcal{S}_n \setminus C_{(1, 2, \dots, j)}|$.

Lemma 3.4. *For $2 \leq j < n$, we have*

- (a) $\eta_{(n-2, 2)}^{(j)}(0) = 0$,
- (b) $\eta_{(n-2, 1^2)}^{(j)}(0) = \frac{2}{(n-1)(n-2)} d_n^{(j)}$,
- (c) $\eta_{(2^2, 1^{n-4})}^{(j)}(0) = 0$,
- (d) $\eta_{(3, 1^{n-3})}^{(j)}(0) = \frac{2}{(n-1)(n-2)} s_n^{(j)}$.

Proof. (a) By Theorem 1.1 and Equation (6),

$$\begin{aligned} \eta_{(n-2, 2)}^{(j)}(0) &= \frac{1}{f^{(n-2, 2)}} \sum_{\sigma \in \mathcal{S}^{(j)}(n, 0)} \chi_{(n-2, 2)}(\sigma) \\ &= \frac{1}{f^{(n-2, 2)}} \sum_{\sigma \in \mathcal{S}_n \setminus C_{(1, 2, \dots, j)}} \chi_{(n-2, 2)}(\sigma) = 0. \end{aligned}$$

(b) By Theorem 1.1 and Equation (7),

$$\begin{aligned} \eta_{(n-2, 1^2)}^{(j)}(0) &= \frac{1}{f^{(n-2, 1^2)}} \sum_{\sigma \in \mathcal{S}^{(j)}(n, 0)} \chi_{(n-2, 1^2)}(\sigma) \\ &= \frac{1}{f^{(n-2, 1^2)}} \sum_{\sigma \in \mathcal{S}_n \setminus C_{(1, 2, \dots, j)}} \chi_{(n-2, 1^2)}(\sigma) \\ &= \frac{2}{(n-1)(n-2)} \sum_{\sigma \in \mathcal{S}_n \setminus C_{(1, 2, \dots, j)}} 1 \\ &= \frac{2}{(n-1)(n-2)} d_n^{(j)}. \end{aligned}$$

(c) By Theorem 1.1 and Equation (8),

$$\begin{aligned}\eta_{(2^2, 1^{n-4})}^{(j)}(0) &= \frac{1}{f^{(2^2, 1^{n-4})}} \sum_{\sigma \in \mathcal{S}^{(j)}(n, 0)} \chi_{(2^2, 1^{n-4})}(\sigma) \\ &= \frac{1}{f^{(2^2, 1^{n-4})}} \sum_{\sigma \in \mathcal{S}_n \setminus C_{(1, 2, \dots, j)}} \chi_{(2^2, 1^{n-4})}(\sigma) = 0.\end{aligned}$$

(d) By Theorem 1.1 and Equation (9),

$$\begin{aligned}\eta_{(3, 1^{n-3})}^{(j)}(0) &= \frac{1}{f^{(3, 1^{n-3})}} \sum_{\sigma \in \mathcal{S}^{(j)}(n, 0)} \chi_{(3, 1^{n-3})}(\sigma) \\ &= \frac{2}{(n-1)(n-2)} \sum_{\sigma \in \mathcal{S}_n \setminus C_{(1, 2, \dots, j)}} \chi_{(3, 1^{n-3})}(\sigma) \\ &= \frac{2}{(n-1)(n-2)} \sum_{\sigma \in \mathcal{S}_n \setminus C_{(1, 2, \dots, j)}} \epsilon(\sigma) \\ &= \frac{2}{(n-1)(n-2)} s_n^{(j)}.\end{aligned}$$

□

Lemma 3.5. ([15, Lemma 2.2]) *For $1 \leq j < n$, we have*

- (a) $\eta_{(n)}^{(j)}(0) = d_n^{(j)}$,
- (b) $\eta_{(n-1, 1)}^{(j)}(0) = -\frac{d_n^{(j)}}{n-1}$,
- (c) $\eta_{(1^n)}^{(j)}(0) = s_n^{(j)}$,
- (d) $\eta_{(2, 1^{n-2})}^{(j)}(0) = -\frac{s_n^{(j)}}{n-1}$.

The following equation is a special case of Theorem 2.9

$$\eta_\lambda^{(j)}(1) = \frac{n}{f^\lambda} \sum_{m=1}^q f^{\mu_m} \eta_{\mu_m}^{(j)}(0). \quad (10)$$

Lemma 3.6. *For $2 \leq j < n$, the following table gives the eigenvalue $\eta_\lambda^{(j)}(1)$ for the partition λ :*

λ	$\eta_\lambda^{(j)}(1)$
(n)	$nd_{n-1}^{(j)}$
$(n-1, 1)$	0
(1^n)	$ns_{n-1}^{(j)}$
$(2, 1^{n-2})$	0
$(n-2, 2)$	$-\frac{2}{(n-3)}d_{n-1}^{(j)}$
$(n-2, 1^2)$	0
$(2^2, 1^{n-4})$	$-\frac{2}{(n-3)}s_{n-1}^{(j)}$
$(3, 1^{n-3})$	0

Proof. We shall use Lemmas 3.4, 3.5 and Equation (10) to calculate these eigenvalues.

$$(a) \quad \eta_{(n)}^{(j)}(1) = \frac{n}{f^{(n)}} \left[f^{(n-1)} \eta_{(n-1)}^{(j)}(0) \right] = n \eta_{(n-1)}^{(j)}(0) = n d_{n-1}^{(j)}.$$

(b)

$$\begin{aligned} \eta_{(n-1,1)}^{(j)}(1) &= \frac{n}{f^{(n-1,1)}} \left[f^{(n-2,1)} \eta_{(n-2,1)}^{(j)}(0) + f^{(n-1)} \eta_{(n-1)}^{(j)}(0) \right] \\ &= \frac{n}{n-1} \left[(n-2) \left(-\frac{d_{n-1}^{(j)}}{n-2} \right) + d_{n-1}^{(j)} \right] = 0. \end{aligned}$$

$$(c) \quad \eta_{(1^n)}^{(j)}(1) = \frac{n}{f^{(1^n)}} \left[f^{(1^{n-1})} \eta_{(1^{n-1})}^{(j)}(0) \right] = n s_{n-1}^{(j)}.$$

(d)

$$\begin{aligned} \eta_{(2,1^{n-2})}^{(j)}(1) &= \frac{n}{f^{(2,1^{n-2})}} \left[f^{(2,1^{n-3})} \eta_{(2,1^{n-3})}^{(j)}(0) + f^{(1^{n-1})} \eta_{(1^{n-1})}^{(j)}(0) \right] \\ &= \frac{n}{n-1} \left[(n-2) \left(-\frac{s_{n-1}^{(j)}}{n-2} \right) + s_{n-1}^{(j)} \right] = 0. \end{aligned}$$

(e)

$$\begin{aligned} \eta_{(n-2,2)}^{(j)}(1) &= \frac{n}{f^{(n-2,2)}} \left[f^{(n-2,1)} \eta_{(n-2,1)}^{(j)}(0) + f^{(n-3,2)} \eta_{(n-3,2)}^{(j)}(0) \right] \\ &= \frac{2}{(n-3)} \left[(n-2) \left(-\frac{d_{n-1}^{(j)}}{n-2} \right) + \frac{(n-1)(n-4)}{2} (0) \right] \\ &= -\frac{2}{(n-3)} d_{n-1}^{(j)}. \end{aligned}$$

(f)

$$\begin{aligned} \eta_{(n-2,1^2)}^{(j)}(1) &= \frac{n}{f^{(n-2,1^2)}} \left[f^{(n-2,1)} \eta_{(n-2,1)}^{(j)}(0) + f^{(n-3,1^2)} \eta_{(n-3,1^2)}^{(j)}(0) \right] \\ &= \frac{2n}{(n-1)(n-2)} \left[(n-2) \left(-\frac{d_{n-1}^{(j)}}{n-2} \right) + \frac{(n-2)(n-3)}{2} \left(\frac{2}{(n-2)(n-3)} d_{n-1}^{(j)} \right) \right] \\ &= 0. \end{aligned}$$

(g)

$$\begin{aligned}
\eta_{(2^2, 1^{n-4})}^{(j)}(1) &= \frac{n}{f_{(2^2, 1^{n-4})}} \left[f^{(2, 1^{n-3})} \eta_{(2, 1^{n-3})}^{(j)}(0) + f^{(2^2, 1^{n-5})} \eta_{(2^2, 1^{n-5})}^{(j)}(0) \right] \\
&= \frac{2}{n-3} \left[(n-2) \left(-\frac{s_{n-1}^{(j)}}{n-2} \right) + \frac{(n-1)(n-4)}{2} (0) \right] \\
&= -\frac{2}{(n-3)} s_{n-1}^{(j)}.
\end{aligned}$$

(h)

$$\begin{aligned}
\eta_{(3, 1^{n-3})}^{(j)}(1) &= \frac{n}{f_{(3, 1^{n-3})}} \left[f^{(2, 1^{n-3})} \eta_{(2, 1^{n-3})}^{(j)}(0) + f^{(3, 1^{n-4})} \eta_{(3, 1^{n-4})}^{(j)}(0) \right] \\
&= \frac{2n}{(n-1)(n-2)} \left[(n-2) \left(-\frac{s_{n-1}^{(j)}}{n-2} \right) + \frac{(n-2)(n-3)}{2} \left(\frac{2}{(n-2)(n-3)} s_{n-1}^{(j)} \right) \right] \\
&= 0.
\end{aligned}$$

□

Lemma 3.7. ([6, Lemma 2.4]) *For $n \geq 9$, the only Specht modules $\mathcal{S}^\lambda$ of dimension $f^\lambda < \binom{n-1}{2} - 1$ are as follows:*

- (a) $\mathcal{S}^{(n)}$ (the trivial representation) with dimension 1;
- (b) $\mathcal{S}^{(1^n)}$ (the sign representation) with dimension 1;
- (c) $\mathcal{S}^{(n-1, 1)}$ with dimension $n-1$;
- (d) $\mathcal{S}^{(2, 1^{n-2})} (\cong \mathcal{S}^{(1^n)} \otimes \mathcal{S}^{(n-1, 1)})$ with dimension $n-1$.

Lemma 3.8. ([16, Lemma 3.3]) *For $n \geq 13$, the only Specht modules $\mathcal{S}^\lambda$ of dimension $\binom{n-1}{2} - 1 \leq f^\lambda < \frac{1}{6}n(n-1)(n-5)$ are as follows:*

- (a) $\mathcal{S}^{(n-2, 2)}$ with dimension $\binom{n-1}{2} - 1$;
- (b) $\mathcal{S}^{(2^2, 1^{n-4})}$ with dimension $\binom{n-1}{2} - 1$;
- (c) $\mathcal{S}^{(n-2, 1^2)}$ with dimension $\binom{n-1}{2}$;
- (d) $\mathcal{S}^{(3, 1^{n-3})}$ with dimension $\binom{n-1}{2}$.

The eigenvalues $\eta_\lambda^{(j)}(1)$ with $f^\lambda < \frac{1}{6}n(n-1)(n-5)$ have been computed (see Lemmas 3.6, 3.7 and 3.8). In Lemma 3.11, we will give a bound of $|\eta_\lambda^{(j)}(1)|$ with $f^\lambda \geq \frac{1}{6}n(n-1)(n-5)$.

Lemma 3.9. ([6, Lemma 2.5]) *Let H be a graph on N vertices whose adjacency matrix A has eigenvalues $\eta_1 \geq \eta_2 \geq \dots \geq \eta_N$, then*

$$\sum_{i=1}^N \eta_i^2 = 2e(H),$$

where $e(H)$ is the number of edges in H .

Recall that for each $\sigma \in \mathcal{S}_n$, $f_i(\sigma)$ is the number of i -cycle appearing in the cyclic decomposition of σ .

Lemma 3.10. $|\mathcal{S}^{(j)}(n, 1)| = nd_{n-1}^{(j)}$.

Proof. Let

$$A_i = \{\alpha \in \mathcal{S}_n : f_t(\sigma) = 0 \ \forall 2 \leq t \leq j, \alpha(i) = i \text{ and } \alpha(m) \neq m \ \forall m \in [n] \setminus \{i\}\}.$$

Note that the restriction of A_i on $[n] \setminus \{i\}$ is the set of all the derangement of $[n] \setminus \{i\}$ that do not contain any t -cycle ($2 \leq t \leq j$) in its cyclic decomposition. Therefore, the restriction of A_i on $[n] \setminus \{i\}$ is the set of all permutations of $[n] \setminus \{i\}$ that do not contain any t -cycle ($1 \leq t \leq j$) in its cyclic decomposition. Thus, $|A_i| = d_{n-1}^{(j)}$. Since, $\mathcal{S}^{(j)}(n, 1) = \bigcup_{i=1}^n A_i$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, $|\mathcal{S}^{(j)}(n, 1)| = nd_{n-1}^{(j)}$. $\square$

Lemma 3.11. Let n and j be positive integer, $n \geq 6$, $2 \leq j < n$ and $\lambda \vdash n$. If the dimension of the Specht module is at least $\frac{1}{6}n(n-1)(n-5)$, then

$$|\eta_\lambda^{(j)}(1)| \leq 6\sqrt{\frac{d_{n-1}^{(j)}(n-2)(n-3)(n-4)(n-6)!}{(n-1)(n-5)}}.$$

Proof. By Theorem 1.1, Lemmas 3.9 and 3.10,

$$\sum_{\lambda \vdash n} \left(f^\lambda \eta_\lambda^{(j)}(1)\right)^2 = 2e\left(\mathcal{F}^{(j)}(n, 1)\right) = n! |\mathcal{S}^{(j)}(n, 1)| = n! \left(nd_{n-1}^{(j)}\right).$$

This implies that

$$|\eta_\lambda^{(j)}(1)| \leq \frac{\sqrt{n! \left(nd_{n-1}^{(j)}\right)}}{f^\lambda} \leq \frac{6\sqrt{n! \left(nd_{n-1}^{(j)}\right)}}{n(n-1)(n-5)} = 6\sqrt{\frac{d_{n-1}^{(j)}(n-2)(n-3)(n-4)(n-6)!}{(n-1)(n-5)}}.$$

$\square$

4 Proof of Theorem 1.4

Lemma 4.1. ([15, part (b) of Theorem 3.2]) Let j, n be positive integers and $j \leq n-1$. Then

$$d_n^{(j)} \geq \frac{n!}{3j}.$$

Lemma 4.2. ([15, part (c) of Theorem 3.5]) Let j, n be positive integers and $j < n$. Then for sufficiently large n ,

$$|s_n^{(j)}| < 8j^2(n-3)! \ln n.$$

Lemma 4.3. Let j, n be positive integers, $n \geq 32$, $2 \leq j \leq n-2$ and $\lambda \vdash n$. If the dimension of the Specht module is at least $\frac{1}{6}n(n-1)(n-5)$, then

$$|\eta_\lambda^{(j)}(1)| < |\eta_{(n-2,2)}^{(j)}(1)|.$$

Proof. By Lemma 3.11,

$$\left| \eta_{\lambda}^{(j)}(1) \right| \leq 6 \sqrt{\frac{d_{n-1}^{(j)}(n-2)(n-3)(n-4)(n-6)!}{(n-1)(n-5)}}.$$

By Lemma 3.6, $\left| \eta_{(n-2,2)}^{(j)}(1) \right| = \frac{2}{(n-3)} d_{n-1}^{(j)}$. So, it is sufficient to show that

$$\begin{aligned} & 6 \sqrt{\frac{d_{n-1}^{(j)}(n-2)(n-3)(n-4)(n-6)!}{(n-1)(n-5)}} < \frac{2}{(n-3)} d_{n-1}^{(j)} \\ \iff & \frac{9d_{n-1}^{(j)}(n-2)(n-3)(n-4)(n-6)!}{(n-1)(n-5)} < \frac{1}{(n-3)^2} \left[d_{n-1}^{(j)} \right]^2 \\ \iff & 9(n-2)(n-3)^3(n-4)(n-6)! < (n-1)(n-5)d_{n-1}^{(j)} \end{aligned}$$

Note that by Lemma 4.1, for $n \geq j+2$ (i.e. $n-2 \geq j$),

$$d_{n-1}^{(j)} \geq \frac{(n-1)!}{3j} \geq \frac{(n-1)!}{3(n-2)} = \frac{(n-1)(n-3)!}{3}.$$

Therefore, it is sufficient to show that

$$9(n-2)(n-3)^3(n-4)(n-6)! < \frac{(n-1)^2(n-5)(n-3)!}{3},$$

which is equivalent to

$$27(n-2)(n-3)^2 < (n-1)^2(n-5)^2. \quad (11)$$

Finally, note that (11) holds for $n = 32, 33$. For $n \geq 34$, we have

$$(n-1)^2(n-5)^2 > (n-2)(n-3)(n-5)^2 \geq 29(n-2)(n-3)(n-5) > 27(n-2)(n-3)^2.$$

Hence the lemma holds. $\square$

Lemma 4.4. *Let j, n be positive integers and $j \leq n^\delta$ with $0 < \delta < \frac{1}{3}$. Then for sufficiently large n ,*

$$\left| \eta_{\lambda}^{(j)}(1) \right| < \left| \eta_{(n-2,2)}^{(j)}(1) \right|,$$

for $\lambda \in \{(1^n), (2^2, 1^{n-4})\}$.

Proof. By Lemma 3.6, it is sufficient to show that $\frac{2}{(n-3)} d_{n-1}^{(j)} > n \left| s_{n-1}^{(j)} \right|$, i.e.,

$$d_{n-1}^{(j)} > \frac{n(n-3)}{2} \left| s_{n-1}^{(j)} \right|.$$

By Theorem 4.2,

$$\left| s_{n-1}^{(j)} \right| < 8j^2(n-4)! \ln(n-1).$$

By Theorem 4.1, $d_{n-1}^{(j)} \geq \frac{(n-1)!}{3j}$. Since $j \leq n^\delta$, $d_{n-1}^{(j)} \geq \frac{(n-1)!}{3n^\delta}$ and $\left| s_{n-1}^{(j)} \right| < 8n^{2\delta}(n-4)! \ln(n-1)$. Therefore, it is sufficient to show that

$$4n^{2\delta}n(n-3)(n-4)! \ln(n-1) < \frac{(n-1)!}{3n^\delta},$$

which is equivalent to

$$\frac{12n^{3\delta}n \ln(n-1)}{(n-1)(n-2)} < 1.$$

Now, $1 - 3\delta > 0$. So, for sufficiently large n ,

$$\frac{12n^{3\delta}n \ln(n-1)}{(n-1)(n-2)} = 12 \left(\frac{\ln(n-1)}{n^{1-3\delta}} \right) \left(\frac{n^2}{(n-1)(n-2)} \right) < 1.$$

Hence, the lemma holds. $\square$

Proof of Theorem 1.4. By Lemma 4.3,

$$\eta_{\lambda}^{(j)}(1) > \eta_{(n-2,2)}^{(j)}(1) = -\frac{2}{(n-3)}d_{n-1}^{(j)},$$

if $f^{\lambda} \geq \frac{1}{6}n(n-1)(n-5)$. It remains to show that $\eta_{\lambda}^{(j)}(1) > -\frac{2}{(n-3)}d_{n-1}^{(j)}$ for all $\lambda \vdash n$ with $f^{\lambda} < \frac{1}{6}n(n-1)(n-5)$. By Lemmas 3.7 and 3.8, there are exactly eight $\lambda \vdash n$ with $f^{\lambda} < \frac{1}{6}n(n-1)(n-5)$. The eigenvalues of these eight partitions have been calculated in Lemma 3.6. So, it is left to show that $\eta_{\lambda}^{(j)}(1) > -\frac{2}{(n-3)}d_{n-1}^{(j)}$ for $\lambda \in \{(1^n), (2^2, 1^{n-4})\}$, but this has been done in Lemma 4.4. Hence, for sufficiently large n ,

$$-\frac{2}{(n-3)}d_{n-1}^{(j)}$$

is the smallest eigenvalue of $\mathcal{F}^{(j)}(n, 1)$. Furthermore, $\eta_{\lambda}^{(j)}(1) = -\frac{2}{n-3}d_{n-1}^{(j)}$ if and only if $\lambda = (n-2, 2)$. This completes the proof of Theorem 1.4. $\square$

Let V be the vertex set of a graph Γ . A subset L of V is said to be an *independent set* in Γ if any two vertices in L are not adjacent to each other. Hoffman [11] observed the following useful bound on the size of an independent set in a regular graph (see also [7, Theorem 11]).

Theorem 4.5. (Hoffman Bound) *Let Γ be a d -regular graph with n vertices. Let A be the adjacency matrix of Γ with smallest eigenvalue τ . If I is an independent set in Γ , then*

$$|I| \leq \frac{n}{1 - \frac{d}{\tau}}.$$

Corollary 4.6. *The size of a largest independent set in $\mathcal{F}^{(j)}(n, 1)$ is at most*

$$2n(n-3)!.$$

Proof. By Lemma 3.10, Theorems 1.4 and 4.5, if I is an independent set in $\mathcal{F}(n, 1)$, then

$$\begin{aligned} |I| &\leq \frac{n!}{1 - \frac{nd_{n-1}^{(j)}}{-\frac{2}{(n-3)}d_{n-1}^{(j)}}} \\ &= \frac{2n!}{2 + n(n-3)} \\ &= 2n(n-3)!. \end{aligned}$$

$\square$

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