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Exotic fluorescence spectrum of a superconducting qubit driven simultaneously by longitudinal and transversal fields

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We study the fluorescence spectrum from a two-level system with a controllable energy separation modulated by the longitudinal field and excited by the transversal field. Combining the rotating-wave approximation and the Floquet theory, we derive a formal solution to the Bloch equations in the presence of both longitudinal and transversal driving fields and numerically evaluate the fluorescence spectrum. Exotic spectral features associated with the added longitudinal driving are found to be dramatically different from the standard Mollow triplet with a multipeak structure. The spectrum may be experimentally realized in the context of artificial atoms in a superconducting circuit.

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I. INTRODUCTION

The resonance fluorescence has been extensively studied in theory [1–9] and experiment [10–13]. In 1969, Mollow predicted with semiclassical theory the fluorescence spectrum for a two-level atom driven by a strong laser consisting of coherent (elastic scattering) and incoherent (inelastic scattering) parts [1]. In particular, the incoherent part exhibits three separated emission lines if the driving strength exceeds a threshold value. Such a profile is also known as the celebrated Mollow triplet. To physically understand the triplet generation, it is necessary to use the dressed-atom model [14], in which the triplet is interpreted as a result of the transition between specific dressed states. On the other hand, due to the limitation of natural atoms, the theories developed and experimented performed are limited in the context of the emitter with a fixed transition frequency. However, this limitation can be lifted after the artificial atoms became available. It has been recognized that quantum optics experiments can be implemented in superconducting circuits that simulate the natural atoms [15]. In particular, the resonance fluorescence has been experimentally studied in the context of a flux qubit [16], where Mollow's theory has been quantitatively verified.

Recently, an artificial atom with an additional longitudinal modulation described by a general Hamiltonian,

$$H(t) = -\frac{1}{2}B_z(t)\sigma_z - \frac{1}{2}B_x(t)\sigma_x, \quad (1)$$

can be realized by using an elaborated superconducting circuit [17]. Here, $\sigma_{z(x)}$ is the usual Pauli matrix. $B_{z(x)}(t)$ is a time-dependent driving force, which may be controlled through gate voltage and external magnetic flux. The time-dependent level separation resulting from the longitudinal modulation is an essential difference between an artificial system and a natural two-level atom. Recently, the related physics associated with the level separation modulation has been studied both in experiment and theory. Reference [18] reports an experimental study of motional averaging in the context of a transmon

with controllable energy separation. In Ref. [19], the probe-pump spectroscopy signals are calculated for a qubit pumped longitudinally. Generally speaking, the study of such systems has an important significance for complementarity to the traditional theory and quantum control.

In this paper, we investigate the spectrum of resonance fluorescence emitted from a qubit simultaneously driven by the longitudinal and transverse fields. The effects of the longitudinal driving on the spectrum is of primary interest and has not been studied yet. Using the rotating-wave approximation (RWA) for the transverse field and the Floquet theory, we solve the Bloch equations in the presence of both the longitudinal and transversal fields and derive the fluorescence spectrum in the steady-state limit. We find that as the relative strength of the longitudinal driving increases, the incoherent fluorescence spectrum varies from the celebrated Mollow triplet to an exotic multipeak structure. The new spectral feature is explained in terms of the transitions of the two-mode Floquet states, which are analytically derived using a unitary transformation approach. In addition, the effects of longitudinal driving on the splitting and width of emission lines are illustrated. Our calculations use the driving parameters in the experimentally accessible range and results obtained may be verified experimentally in a superconducting qubit.

The paper is organized as follows. In Sec. II, we introduce the procedure of the Floquet method to solve the Bloch equations. In Sec. III, we derive the fluorescence spectrum and present the main results. Finally, the conclusion is given in Sec. IV.

II. MODEL AND FLOQUET THEORY

A. Bloch equations

We consider a superconducting qubit driven simultaneously by longitudinal and transversal fields, which can be described by the Hamiltonian below ($\hbar = 1$):

$$H(t) = \frac{1}{2}[\omega_0 + \Omega_z \cos(\omega_z t)]\sigma_z + \Omega_x \cos(\omega_x t)\sigma_x, \quad (2)$$

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where ω_0 is the static energy splitting of the two levels, and $\Omega_{x(z)}$ and $\omega_{x(z)}$ are the driving strength and the frequency of the transversal (longitudinal) field, respectively. When the driving fields are relatively weak, i.e., $\omega_0 \gg \Omega_{x(z)}$ and ω_z (this is the experimentally accessible parameters regime [18]), we can include the dissipation from the reservoir, with which the qubit interacts, by adding phenomenologically dissipative terms in the Liouville equation, i.e., the equation of motion for the qubit is given by a Lindblad master equation:

$$\frac{d}{dt}\rho(t) = -i[H(t), \rho(t)] - \frac{\kappa}{2}[\sigma_+\sigma_-\rho(t) + \rho(t)\sigma_+\sigma_- - 2\sigma_-\rho(t)\sigma_+], \quad (3)$$

where $\rho(t)$ is the reduced density matrix of the qubit, κ is the radiative decay rate, and $\sigma_{\pm} = (\sigma_x \pm i\sigma_y)/2$ are the usual raising and lowering operators for the qubit.

In this work, we consider the resonance case of $\omega_x = \omega_0$, which allows us to invoke the usual RWA in Eq. (2). With the RWA, the Hamiltonian becomes

$$H(t) = \frac{1}{2}[\omega_0 + \Omega_z \cos(\omega_z t)]\sigma_z + \frac{\Omega_x}{2}(\sigma_+ e^{-i\omega_x t} + \sigma_- e^{i\omega_x t}). \quad (4)$$

Now it is convenient to transform Eq. (3) into a frame that rotates at frequency ω_x along the z axes, and the master equation takes the form

$$\frac{d}{dt}\tilde{\rho}(t) = -i[\tilde{H}(t), \tilde{\rho}(t)] - \frac{\kappa}{2}[\sigma_+\sigma_-\tilde{\rho}(t) + \tilde{\rho}(t)\sigma_+\sigma_- - 2\sigma_-\tilde{\rho}(t)\sigma_+], \quad (5)$$

where

$$\tilde{\rho}(t) = \exp(i\omega_x \sigma_z t/2)\rho(t)\exp(-i\omega_x \sigma_z t/2) \quad (6)$$

and

$$\tilde{H}(t) = \frac{\Omega_x}{2}\sigma_x + \frac{\Omega_z}{2}\cos(\omega_z t)\sigma_z \quad (7)$$

are the reduced density matrix and the Hamiltonian in the rotating frame, respectively. Denoting $\langle \tilde{\sigma}(t) \rangle = \text{Tr}[\sigma \tilde{\rho}(t)]$ as the expectation value of operator σ in the state $\tilde{\rho}(t)$, we can rewrite Eq. (5) in a matrix form,

$$i \frac{d}{dt} \begin{pmatrix} \langle \tilde{\sigma}_+(t) \rangle \\ \langle \tilde{\sigma}_-(t) \rangle \\ \langle \tilde{\sigma}_z(t) \rangle \end{pmatrix} = M(t) \begin{pmatrix} \langle \tilde{\sigma}_+(t) \rangle \\ \langle \tilde{\sigma}_-(t) \rangle \\ \langle \tilde{\sigma}_z(t) \rangle \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ -\kappa \end{pmatrix}, \quad (8)$$

where the coefficient matrix takes the form

$$M(t) = \begin{pmatrix} -\Omega_z \cos(\omega_z t) - i\frac{\kappa}{2} & 0 & \frac{\Omega_x}{2} \\ 0 & \Omega_z \cos(\omega_z t) - i\frac{\kappa}{2} & -\frac{\Omega_x}{2} \\ \Omega_x & -\Omega_x & -i\kappa \end{pmatrix}. \quad (9)$$

It is evident that $M(t) = M(t + 2\pi/\omega_z)$ has a periodicity of $2\pi/\omega_z$, which allows us to use the Floquet theory to solve Eq. (8).

B. Floquet theory and the solution to the master equation

In this section, we shall solve Eq. (8) by using the Floquet theory. According to the Floquet theory, we assume that the

homogeneous part of Eq. (8) possesses the formal solution as follows [20,21]:

$$\begin{pmatrix} \langle \tilde{\sigma}_+(t) \rangle \\ \langle \tilde{\sigma}_-(t) \rangle \\ \langle \tilde{\sigma}_z(t) \rangle \end{pmatrix} = \sum_{\gamma=1}^3 |\phi_{\gamma}(t)\rangle \langle \varphi_{\gamma}(t')| \begin{pmatrix} \langle \tilde{\sigma}_+(t') \rangle \\ \langle \tilde{\sigma}_-(t') \rangle \\ \langle \tilde{\sigma}_z(t') \rangle \end{pmatrix} e^{-i\varepsilon_{\gamma}(t-t')}, \quad (10)$$

where $|\phi_{\gamma}(t)\rangle$ and $\langle \varphi_{\gamma}(t')|$ are 3×1 and 1×3 vectors to be determined below, respectively, and ε_{γ} is the characteristic exponent. In addition, $|\phi_{\gamma}(t + 2\pi/\omega_z)\rangle = |\phi_{\gamma}(t)\rangle$ and $\langle \varphi_{\gamma}(t' + 2\pi/\omega_z)| = \langle \varphi_{\gamma}(t')|$ are periodic in time with the same periodicity as $M(t)$. By differentiating the formal solution with respect to t , we obtain

$$\begin{aligned} i \frac{d}{dt} \begin{pmatrix} \langle \tilde{\sigma}_+(t) \rangle \\ \langle \tilde{\sigma}_-(t) \rangle \\ \langle \tilde{\sigma}_z(t) \rangle \end{pmatrix} &= i \sum_{\gamma=1}^3 \left[\frac{d|\phi_{\gamma}(t)\rangle}{dt} - i\varepsilon_{\gamma}|\phi_{\gamma}(t)\rangle \right] \\ &\times \langle \varphi_{\gamma}(t')| e^{-i\varepsilon_{\gamma}(t-t')} \begin{pmatrix} \langle \tilde{\sigma}_+(t') \rangle \\ \langle \tilde{\sigma}_-(t') \rangle \\ \langle \tilde{\sigma}_z(t') \rangle \end{pmatrix} \\ &= M(t) \sum_{\gamma=1}^3 |\phi_{\gamma}(t)\rangle \langle \varphi_{\gamma}(t')| e^{-i\varepsilon_{\gamma}(t-t')} \begin{pmatrix} \langle \tilde{\sigma}_+(t') \rangle \\ \langle \tilde{\sigma}_-(t') \rangle \\ \langle \tilde{\sigma}_z(t') \rangle \end{pmatrix}, \quad (11) \end{aligned}$$

which leads to

$$[M(t) - i\partial_t]|\phi_{\gamma}(t)\rangle = \varepsilon_{\gamma}|\phi_{\gamma}(t)\rangle. \quad (12)$$

Since $|\phi_{\gamma}(t)\rangle$ is periodic in time, we can expand it as

$$|\phi_{\gamma}(t)\rangle = \sum_{\beta=1}^3 \sum_{n=-\infty}^{\infty} \exp(in\omega_z t) \phi_{n,\beta}^{(\gamma)} |\beta\rangle, \quad (13)$$

where $\phi_{n,\beta}^{(\gamma)}$ is the Fourier coefficient, and $|\beta\rangle$ ($\beta = 1, 2, 3$) represents the 3×1 unit basis vector in which the β th element equals 1 and the others are zero, for instance,

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Substituting the expansion into Eq. (12), we obtain

$$\sum_{\beta=1}^3 \sum_{n=-\infty}^{\infty} [M(t) + n\omega_z - \varepsilon_{\gamma}] \exp(in\omega_z t) \phi_{n,\beta}^{(\gamma)} |\beta\rangle = 0. \quad (14)$$

Multiplying $\langle \alpha| \exp(-im\omega_z t)$ ($\alpha = 1, 2, 3$) from the left on both sides of the above equation and performing the integration $\frac{\omega_z}{2\pi} \int_0^{2\pi/\omega_z} (dt)$, we arrive at a stationary equation for the Fourier

coefficients of $|\phi_\gamma(t)\rangle$,

$$\sum_{\beta=1}^3 \sum_{n=-\infty}^{\infty} \mathcal{M}_{\alpha\beta}^{(m-n)} \phi_{n,\beta}^{(\gamma)} = \varepsilon_\gamma \phi_{m,\alpha}^{(\gamma)}, \quad (15)$$

where

$$\mathcal{M}_{\alpha\beta}^{(m-n)} = \frac{\omega_z}{2\pi} \int_0^{2\pi/\omega_z} \langle \alpha | M(t) + n\omega_z | \beta \rangle e^{i(n-m)\omega_z t} dt \quad (16)$$

can be arranged as an infinite matrix with its row (column) specified by α and m (β and n), which takes the form

$$\mathcal{M} = \begin{pmatrix} \ddots & & & & \\ & D^{(n+1)} & C & & \\ & C & D^{(n)} & C & \\ & & C & D^{(n-1)} & \\ & & & & \ddots \end{pmatrix}, \quad (17)$$

where

$$D^{(n)} = \begin{pmatrix} n\omega_z - i\frac{\kappa}{2} & 0 & \frac{\Omega_x}{2} \\ 0 & n\omega_z - i\frac{\kappa}{2} & -\frac{\Omega_x}{2} \\ \Omega_x & -\Omega_x & n\omega_z - i\kappa \end{pmatrix}, \quad (18)$$

$$C = \begin{pmatrix} -\frac{\Omega_z}{2} & 0 & 0 \\ 0 & \frac{\Omega_z}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (19)$$

Now we face an eigenvalue problem in linear algebra: $\mathcal{M}|\phi^{(\gamma)}\rangle = \varepsilon_\gamma |\phi^{(\gamma)}\rangle$ and $|\phi^{(\gamma)}\rangle \equiv \sum_{\beta=1}^3 \sum_{n=-\infty}^{\infty} \phi_{n,\beta}^{(\gamma)} |\beta\rangle |n\rangle$, where $|n\rangle$ merely represents the n th Fourier component $\exp(in\omega_z t)$. In spite of an infinite rank of the matrix $\mathcal{M}$, it is feasible to introduce an appropriate truncation and diagonalize $\mathcal{M}$ with a desired accuracy. The accuracy can be verified easily by increasing the dimension of truncated matrix and comparing the results with those of a smaller truncated one.

Similarly, we can differentiate the formal solution with respect to t' and get

$$\langle \varphi_\gamma(t') | [M(t') + i\overleftarrow{\partial}_{t'}] = \varepsilon_\gamma \langle \varphi_\gamma(t') |. \quad (20)$$

By using the Fourier expansion

$$\langle \varphi_\gamma(t') | = \sum_{\alpha=1}^3 \sum_{m=-\infty}^{\infty} \langle \alpha | \varphi_{m,\alpha}^{(\gamma)} \exp(-im\omega_z t'), \quad (21)$$

one readily derives the stationary equation for the Fourier coefficients of $\langle \varphi_\gamma(t') |$:

$$\sum_{\alpha=1}^3 \sum_{m=-\infty}^{\infty} \varphi_{m,\alpha}^{(\gamma)} \mathcal{M}_{\alpha\beta}^{(m-n)} = \varepsilon_\gamma \varphi_{n,\beta}^{(\gamma)}, \quad (22)$$

which corresponds to $\langle \langle \varphi^{(\gamma)} | \mathcal{M} = \varepsilon_\gamma \langle \langle \varphi^{(\gamma)} |$ with $\langle \langle \varphi^{(\gamma)} | \equiv \sum_{\alpha=1}^3 \sum_{m=-\infty}^{\infty} \varphi_{m,\alpha}^{(\gamma)} \langle \alpha | (m |$. We should emphasize that $\mathcal{M}$ is a non-Hermite matrix leading to $\langle \langle \varphi^{(\gamma)} | \neq (\langle \phi^{(\gamma)} |)^\dagger$. This is also the reason why $\varphi_{m,\alpha}^{(\gamma)}$ should be determined by means of Eq. (22). In this work, we use the numerical method to find the matrix P and its inverse such that $P^{-1} \mathcal{M} P = \Lambda$, where Λ is a diagonal matrix whose diagonal entries are, of course, the characteristic exponents. Provided that P is known, we can obtain $|\phi^{(\gamma)}\rangle$ from the corresponding column (specified by the column or the row of ε_γ in Λ) of P , and $\langle \langle \varphi^{(\gamma)} |$ from the corresponding row of P^{-1} . Finally, we obtain $|\phi_\gamma(t)\rangle$ by $(t|\phi^{(\gamma)})$ with $(t|n) = \exp(in\omega_z t)$, while $\langle \varphi_\gamma(t) |$ is obtained from $\langle \langle \varphi^{(\gamma)} | t$ with $(m|t) = \exp(-im\omega_z t)$.

We now discuss briefly the properties of the characteristic exponents. For the model considered here with a finite κ , it is found by numerical calculation that the characteristic exponents are complex and can be sorted into three groups with the forms

$$\varepsilon_{1,n} \equiv n\omega_z - i\Gamma_1, \quad \varepsilon_{2(3),n} \equiv n\omega_z \pm \Omega' - i\Gamma_2. \quad (23)$$

Here n is an integer, and Ω' and $\Gamma_{1,2}$ are positive real numbers. Within each group, one finds that the imaginary parts of the characteristic exponents are unique while their real parts differ by integer multiples of ω_z , which are derived from the periodic structure of $\mathcal{M}$ similar to the Floquet matrix [20]. Moreover, it can be verified that the sum of the three types of characteristic exponents satisfies the relation $\sum_\gamma \varepsilon_{\gamma,n} \bmod \omega_z = \frac{\omega_z}{2\pi} \int_0^{2\pi/\omega_z} \text{Tr}[M(t)] dt = -2i\kappa$, which results from Abel's identity [21]. In principle, in the calculation it is arbitrary to choose three characteristic exponents from the diagonal matrix Λ as long as the three characteristic exponents satisfy the aforementioned sum rule, which leads to the unique physical results. In this paper, we choose three exponents with real parts in the interval of $(-\omega_z/2, \omega_z/2]$. The physical significance of these exponents is discussed in the next section.

After obtaining $|\phi_\gamma(t)\rangle$ and $\langle \varphi_\gamma(t) |$, we can derive the solution to the inhomogeneous Eq. (8). According to the theory of linear differential equations, we have the solution to Eq. (8) in a vector form:

$$\begin{aligned} \begin{pmatrix} \langle \tilde{\sigma}_+(t) \rangle \\ \langle \tilde{\sigma}_-(t) \rangle \\ \langle \tilde{\sigma}_z(t) \rangle \end{pmatrix} &= \sum_{\gamma=1}^3 |\phi_\gamma(t)\rangle \langle \varphi_\gamma(t') | \begin{pmatrix} \langle \tilde{\sigma}_+(t') \rangle \\ \langle \tilde{\sigma}_-(t') \rangle \\ \langle \tilde{\sigma}_z(t') \rangle \end{pmatrix} e^{-i\varepsilon_\gamma(t-t')} + \sum_{\gamma=1}^3 \int_{t'}^t |\phi_\gamma(t)\rangle \langle \varphi_\gamma(s) | \begin{pmatrix} 0 \\ 0 \\ -\kappa \end{pmatrix} e^{-i\varepsilon_\gamma(t-s)} ds \\ &= \sum_{\gamma=1}^3 \sum_{n,m=-\infty}^{\infty} \begin{pmatrix} \phi_{n,1}^{(\gamma)} \\ \phi_{n,2}^{(\gamma)} \\ \phi_{n,3}^{(\gamma)} \end{pmatrix} \left\{ [\varphi_{m,1}^{(\gamma)} \langle \tilde{\sigma}_+(t') \rangle + \varphi_{m,2}^{(\gamma)} \langle \tilde{\sigma}_-(t') \rangle + \varphi_{m,3}^{(\gamma)} \langle \tilde{\sigma}_z(t') \rangle] e^{in\omega_z t - im\omega_z t'} \right. \\ &\quad \left. \times e^{-i\varepsilon_\gamma(t-t')} - \kappa \varphi_{m,3}^{(\gamma)} e^{i(n-m)\omega_z t} \left[\frac{1 - e^{i(m\omega_z - \varepsilon_\gamma)(t-t')}}{i(\varepsilon_\gamma - m\omega_z)} \right] \right\}. \end{aligned} \quad (24)$$

FIG. 1. The population difference as a function of time t calculated from Eq. (3) (dotted line with $\Omega_z = 0$ and red dashed line with $\Omega_z \neq 0$) and Eq. (24) (blue, solid). The driving parameters are (a) $\omega_z = 0.01\omega_0$ and $\Omega_z/\omega_z = 0.1$, (b) $\omega_z = 0.01\omega_0$ and $\Omega_z/\omega_z = 1.5$, (c) $\omega_z = 0.05\omega_0$ and $\Omega_z/\omega_z = 0.1$, and (d) $\omega_z = 0.05\omega_0$ and $\Omega_z/\omega_z = 1.5$. The other parameters are set as $\omega_x = \omega_0 = 10^3\kappa$ and $\Omega_x = 20\kappa$.

It is evident that the last term in the first line of Eq. (24) results from the inhomogeneous part of the Bloch equations and determines the steady-state behavior of the system. With this expression (24) at hand, we are now in a position to illustrate its accuracy by comparing its predictions with those of Eq. (3) solved by the Runge-Kutta method. To check the dynamics of population difference, we set the initial time $t' = 0$ and $\langle \sigma_z(0) \rangle = 1$. Physically, this means that the system is prepared in the excited state.

Figures 1(a)–1(d) display the dynamics of the population difference. When performing numerical calculation, we use the fixed truncation number $N_{tr} = 10$ instead of ∞ for the sum over n and m in Eq. (24). In all plots, we find that Eq. (24) provides almost the same results as those of Eq. (3) numerically solved by the Runge-Kutta method, i.e., solid and dashed lines coincide very well. Such an agreement confirms two facts: one is that our formal solution (24) can provide reliable results under a relatively small truncation; the other is that the traditional RWA used in the formal solution is applicable in our model. In addition, we can see the role of longitudinal driving on the population difference dynamics. It is evident that when $\Omega_z/\omega_z = 0.1$, the dynamics without influence of the longitudinal driving ($\Omega_z = 0$) are similar as those with the influence ($\Omega_z \neq 0$) [see Figs. 1(a) and 1(c)]. However, if Ω_z/ω_z increases to 1.5, one finds that the longitudinal driving makes the dynamics to dramatically deviate from the result of $\Omega_z = 0$, implying that the effect of longitudinal driving becomes significant only if the ratio Ω_z/ω_z is sufficiently large. In general, we can perform further numerical simulations to estimate the threshold ratio that the longitudinal driving starts to cause significant modification. Roughly speaking, the modification apparently appears when $\Omega_z/\omega_z > 0.5$.

To end this section, we remark on the validity of the present method. Our treatment is based on the RWA for the

transverse field and the Floquet theory. When the RWA is valid for the relatively weak transverse field ($\Omega_x/\omega_0 \ll 1$) and on- or near-resonant cases ($\omega_x \sim \omega_0$), the Floquet method is generally applicable to arbitrary values of the longitudinal driving strength and the frequency.

III. FLUORESCENCE SPECTRUM

We now proceed to derive the fluorescence spectrum for the superconducting qubit by using Eq. (24). In the rotating frame, the steady fluorescence spectrum is proportional to the real part of the Fourier transform of the first-order correlation function [22]:

$$S(\omega) \propto \frac{1}{\pi} \text{Re} \int_0^\infty \lim_{t' \rightarrow \infty} g^{(1)}(\tau; t') e^{i(\omega_0 - \omega)\tau} d\tau, \quad (25)$$

where

$$\begin{aligned} g^{(1)}(\tau; t') &\equiv \langle \tilde{\sigma}_+(t' + \tau) \tilde{\sigma}_-(t') \rangle \\ &= \text{Tr}_{S+B} [\tilde{U}^\dagger(t' + \tau) \sigma_+ \tilde{U}(t' + \tau) \\ &\quad \times \tilde{U}^\dagger(t') \sigma_- \tilde{U}(t') \tilde{\rho}(0) \otimes \rho_B] \end{aligned} \quad (26)$$

is the first-order correlation function. $\tilde{\rho}(0)$ and ρ_B are the qubit state and the reservoir vacuum state, respectively. Here $\tilde{U}(t)$ is the unitary evolution operator for the composite system of the qubit and the reservoir in the rotating frame. We use the quantum regression theory to evaluate this two-time correlation function [23]. In the Markovian approximation, with $t \equiv t' + \tau$, we have

$$\begin{aligned} \langle \tilde{\sigma}_+(t) \tilde{\sigma}_-(t') \rangle &= \text{Tr}_{S+B} [\sigma_+ \tilde{U}(t, t') \sigma_- \tilde{\rho}(t') \rho_B \tilde{U}^\dagger(t, t')] \\ &= \text{Tr}_S \{ \sigma_+ \text{Tr}_B [\tilde{U}(t, t') \sigma_- \tilde{\rho}(t') \rho_B \tilde{U}^\dagger(t, t')] \}. \end{aligned} \quad (27)$$

The second line of the above equation indicates that the two-time correlation function is equivalent to the single time expectation value of σ_+ in the reduced state evolving from the initial state $\sigma_- \tilde{\rho}(t')$. In other words, we can obtain the expression for $\langle \tilde{\sigma}_+(t) \tilde{\sigma}_-(t') \rangle$ from $\langle \tilde{\sigma}_+(t) \rangle$ by replacing the initial condition $\tilde{\rho}(t')$ with $\sigma_- \tilde{\rho}(t')$, which leads to $(\langle \tilde{\sigma}_+(t') \rangle, \langle \tilde{\sigma}_-(t') \rangle, \langle \tilde{\sigma}_z(t') \rangle) \rightarrow (\frac{1 + \langle \tilde{\sigma}_z(t') \rangle}{2}, 0, -\langle \tilde{\sigma}_-(t') \rangle)$ and $\text{Tr}[\rho(t')] \rightarrow \langle \tilde{\sigma}_-(t') \rangle$. Therefore, the two-time correlation function can be straightforwardly derived as

$$\begin{aligned} \langle \tilde{\sigma}_+(t) \tilde{\sigma}_-(t') \rangle &= \sum_{\gamma=1}^3 \sum_{n,m=-\infty}^{\infty} \phi_{n,1}^{(\gamma)} e^{in\omega_z t - im\omega_z t'} e^{-i\varepsilon_\gamma(t-t')} \\ &\quad \times \left[\varphi_{m,1}^{(\gamma)} \frac{1 + \langle \tilde{\sigma}_z(t') \rangle}{2} - \varphi_{m,3}^{(\gamma)} \langle \tilde{\sigma}_-(t') \rangle \right] \\ &\quad - \kappa \sum_{\gamma=1}^3 \sum_{n,m=-\infty}^{\infty} \phi_{n,1}^{(\gamma)} \varphi_{m,3}^{(\gamma)} e^{i(n-m)\omega_z t} \\ &\quad \times \left[\frac{1 - e^{i(m\omega_z - \varepsilon_\gamma)(t-t')}}{i(\varepsilon_\gamma - m\omega_z)} \right] \langle \tilde{\sigma}_-(t') \rangle. \end{aligned} \quad (28)$$

In the steady-state limit ($t' \rightarrow \infty$, fixing $t - t' = \tau$), we can simplify the above expression as follows:

$$\begin{aligned}
 g^{(1)}(\tau) &= \lim_{t' \rightarrow \infty} \langle \tilde{\sigma}_+(t' + \tau) \tilde{\sigma}_-(t') \rangle \\
 &= \sum_{\gamma, \beta} \sum_{n, m, k, l} \frac{\kappa^2 \phi_{m,1}^{(\gamma)} \phi_{m-n,3}^{(\gamma)} \phi_{k,2}^{(\beta)} \phi_{l,3}^{(\beta)} \delta_{n,l-k}}{(l\omega_z - \varepsilon_\beta)[\varepsilon_\gamma - (m-n)\omega_z]} e^{in\omega_z \tau} \\
 &\quad + \sum_{\gamma, n} \frac{\phi_{n,1}^{(\gamma)} \phi_{n,1}^{(\gamma)}}{2} e^{i(n\omega_z - \varepsilon_\gamma)\tau} + \kappa \sum_{\gamma, \beta} \sum_{n, m, k, l} \phi_{n,1}^{(\gamma)} \\
 &\quad \times \delta_{n,m+l-k} \left\{ \phi_{m,3}^{(\gamma)} \phi_{k,2}^{(\beta)} \left[1 - \frac{\kappa}{i(\varepsilon_\gamma - m\omega_z)} \right] \right. \\
 &\quad \left. - \frac{\phi_{m,1}^{(\gamma)} \phi_{k,3}^{(\beta)}}{2} \right\} \frac{\phi_{l,3}^{(\beta)} e^{i(n\omega_z - \varepsilon_\gamma)\tau}}{i(\varepsilon_\beta - l\omega_z)}. \quad (29)
 \end{aligned}$$

The sum over the Greek index runs from 1 to 3, while the Latin variable takes values from $-\infty$ to ∞ .

Upon substituting Eq. (29) into Eq. (25) and integrating with respect to τ , we obtain the fluorescence spectrum, $S(\omega) = S_{\text{coh}}(\omega) + S_{\text{inc}}(\omega)$, where the so-called coherent part is given by

$$\begin{aligned}
 S_{\text{coh}}(\omega) &= \kappa^2 \sum_{\gamma, \beta} \sum_{n, m, k, l} \frac{\phi_{m,1}^{(\gamma)} \phi_{m-n,3}^{(\gamma)} \phi_{k,2}^{(\beta)} \phi_{l,3}^{(\beta)} \delta_{n,l-k}}{(l\omega_z - \varepsilon_\beta)[\varepsilon_\gamma - (m-n)\omega_z]} \\
 &\quad \times \delta(\omega - \omega_0 - n\omega_z) \quad (30)
 \end{aligned}$$

and the incoherent part by

$$\begin{aligned}
 S_{\text{inc}}(\omega) &= \frac{1}{\pi} \text{Re} \left\{ \sum_{\gamma, n} \frac{1}{i(\omega - \omega_0 - n\omega_z + \varepsilon_\gamma)} \phi_{n,1}^{(\gamma)} \right. \\
 &\quad \times \left[\frac{\phi_{n,1}^{(\gamma)}}{2} + \sum_{\beta, m, k, l} \phi_{m,3}^{(\gamma)} \phi_{k,2}^{(\beta)} \phi_{l,3}^{(\beta)} \frac{\kappa}{i(\varepsilon_\beta - l\omega_z)} \right. \\
 &\quad \times \delta_{n,m+l-k} \left(1 - \frac{\kappa}{i(\varepsilon_\gamma - m\omega_z)} \right) \\
 &\quad \left. \left. - \frac{\kappa}{2} \sum_{\beta, m, k, l} \phi_{m,1}^{(\gamma)} \phi_{k,3}^{(\beta)} \phi_{l,3}^{(\beta)} \frac{\delta_{n,m+l-k}}{i(\varepsilon_\beta - l\omega_z)} \right] \right\}. \quad (31)
 \end{aligned}$$

By using these results, in principle one can obtain the spectral line shape numerically and understand the spectral properties by analyzing the quantities in the formal spectrum. In particular, the mean intensities per unit time of the coherent and incoherent parts can be obtained as follows:

$$\begin{aligned}
 I_{\text{coh}} &= \int_{-\infty}^{\infty} d\omega S_{\text{coh}}(\omega) \\
 &= \sum_{\gamma, \beta} \sum_{n, m, k, l} \frac{\kappa^2 \phi_{m,1}^{(\gamma)} \phi_{m-n,3}^{(\gamma)} \phi_{k,2}^{(\beta)} \phi_{l,3}^{(\beta)} \delta_{n,l-k}}{(l\omega_z - \varepsilon_\beta)[\varepsilon_\gamma - (m-n)\omega_z]} \\
 &= \overline{\langle \tilde{\sigma}_+(\infty) \rangle \langle \tilde{\sigma}_-(\infty) \rangle}, \quad (32)
 \end{aligned}$$

$$\begin{aligned}
 I_{\text{inc}} &= \int_{-\infty}^{\infty} d\omega [S(\omega) - S_{\text{coh}}(\omega)] \\
 &= \overline{\langle \tilde{\sigma}_+(\infty) \tilde{\sigma}_-(\infty) \rangle} - \overline{\langle \tilde{\sigma}_+(\infty) \rangle \langle \tilde{\sigma}_-(\infty) \rangle}, \quad (33)
 \end{aligned}$$

FIG. 2. The mean intensities per unit time of the coherent component (blue, solid) and incoherent component (red, dashed) fluorescence as a function of Ω_x for $\omega_x = \omega_0 = 10^3 \kappa$. The longitudinal driving parameters are set as $\omega_z = 0.01 \omega_0$ and (a) $\Omega_z/\omega_z = 0.1$ and (b) $\Omega_z/\omega_z = 1.8$.

where $\overline{\langle \tilde{\sigma}_+(\infty) \tilde{\sigma}_-(\infty) \rangle} \equiv [1 + \overline{\langle \tilde{\sigma}_z(\infty) \rangle}]/2$ and the overline indicates the time average.

In Figs. 2(a) and 2(b), we display I_{coh} and I_{inc} as a function of Ω_x for $\omega_z = 0.01 \omega_0$ and two values of the ratio Ω_z/ω_z . If $\Omega_z/\omega_z = 0.1$, I_{coh} and I_{inc} are found to behave in a similar manner as those that follow from Mollow's prediction [1]. As the transverse driving gains in strength, the coherent part decreases rapidly while the incoherent part becomes dominant. However, if $\Omega_z/\omega_z = 1.8$, both I_{coh} and I_{inc} are modified dramatically by the longitudinal driving. The coherent part I_{coh} survives and is enhanced in the range of $2\kappa < \Omega_x < 15\kappa$ as compared to the case of $\Omega_z/\omega_z = 0.1$. On the other hand, the incoherent part is found to be suppressed significantly as compared to the case of $\Omega_z/\omega_z = 0.1$. This phenomenon shows that the spectrum can be significantly modified in the presence of a sufficiently large ratio of Ω_z/ω_z .

Using the numerically calculated formal spectrum $S(\omega)$, we can now illustrate new spectral features caused by the longitudinal driving. In Figs. 3(a)–3(d) we show the line shapes of the incoherent part of the fluorescence for various longitudinal driving parameters and fixed transverse driving $\Omega_x = 5\kappa$. As the ratio Ω_z/ω_z increases, the line shape morphs from a typical Mollow triplet to a multipeak structure. Interestingly, the splitting between the central peak at ω_0 and its

FIG. 3. The incoherent fluorescence spectrum as a function of $\Delta = \omega - \omega_0$ for $\omega_x = \omega_0 = 10^3\kappa$, $\Omega_x = 5\kappa$, and $\omega_z = 0.01\omega_0$. The ratio Ω_z/ω_z is set as (a) 0.1, (b) 0.5, (c) 0.75, and (d) 1.5.

nearest-neighbor sidebands is found to decrease significantly with the increase of Ω_z/ω_z . In addition, it turns out the intensity of the central component around ω_0 decreases while those of the components around $n\omega_z$ ($n = \pm 1, \pm 2, \dots$) increases instead. In what follows, we carry out a comprehensive study of the effect of longitudinal driving on the spectrum based on our formal incoherent spectrum given by Eq. (31).

In the incoherent part, Eq. (31) shows that for fixed γ and n , the Lorentzian emission line is contributed from the real part of $1/[i(\omega - \omega_0 - n\omega_z + \varepsilon_\gamma)]$. It is evident that the Lorentzian emission line is centered at $\omega_0 + n\omega_z - \text{Re}[\varepsilon_\gamma]$ with the FWHM given by $2|\text{Im}[\varepsilon_\gamma]|$. Hence, it is not difficult to imagine that the sum over γ and n can give rise to the multipeak spectrum. To understand physically how the emission lines arises, it is instructive to recall the dressed-atom model, where the emission naturally arises as a result of the transition between the dressed states [14]. Similarly, the spectrum calculated in this work can be explained in a similar manner. The key task is to find out the dressed states and their transitions. Alternatively, it is more appropriate to use the Floquet states in the semiclassical model [24].

For the model we consider, the Floquet states (the combined state of the qubit and driving) and quasienergies can be derived as follows:

$$|u_{\pm, n_z, n_x}(t)\rangle = \exp\left[\frac{i\omega_x t}{2}(2n_x + 1 - \sigma_z)\right] |\tilde{u}_{\pm, n_z}(t)\rangle, \quad (34)$$

$$\epsilon_{\pm, n_z, n_x} = \frac{1}{2}(\omega_x + \omega_z \pm E) + n_z\omega_z + n_x\omega_x,$$

where $|\tilde{u}_{\pm, n_z}(t)\rangle$ and E are defined in Eqs. (A13) and (A10), respectively. The details of derivation can be found in the Appendix. When finding these states, we can figure out their transitions and understand the generation of the

emission lines. Roughly speaking, the transition such as $|u_{\pm, n_z+n, N+1}(t)\rangle \rightarrow |u_{\pm, n_z, N}(t)\rangle$ gives rise to an emission line at frequency $\epsilon_{\pm, n_z+n, N+1} - \epsilon_{\pm, n_z, N} = \omega_0 + n\omega_z$, where n is an integer and we recall that $\omega_x = \omega_0$. The other transition such as $|u_{\mp, n_z+n, N+1}(t)\rangle \rightarrow |u_{\pm, n_z, N}(t)\rangle$ gives rise to an emission line at frequency $\epsilon_{\mp, n_z+n, N+1} - \epsilon_{\pm, n_z, N} = \omega_0 + n\omega_z \mp E$. In Fig. 4, we show the schematic diagram of transitions of the Floquet states leading to the generation of the emission lines. On the other hand, by these observations and recalling that the position of the emission line is formally given by $\omega_0 + n\omega_z - \text{Re}[\varepsilon_\gamma]$, we conclude that the real parts of the characteristic exponents shifted in the range of $(-\omega_z/2, \omega_z/2]$ and take on three values: $0, \pm E \bmod \omega_z$. Below we examine whether these values derived from the viewpoint of the transitions of the two-mode Floquet states are consistent with the numerical results.

FIG. 4. Schematic diagram represents transitions of the two-mode Floquet states. Each color of arrows represents a frequency determined by the difference between the two two-mode quasienergies.

FIG. 5. The real (a) and imaginary (b) parts of ε_1 (purple, dash-dotted), ε_2 [blue solid line for finite κ and red circle for $\kappa = 0$ obtained by $\omega_z - E$ with E given by Eq. (A10)], and ε_3 (red, dashed) as a function of ratio Ω_z/ω_z for $\omega_z = 0.01\omega_0$ and $\Omega_x = 5\kappa$. The other parameters are set as $\omega_x = \omega_0 = 10^3\kappa$.

In Fig. 5(a), we show the real parts of the characteristic exponents as a function of ratio Ω_z/ω_z for fixed transverse driving strength, which are obtained by numerically diagonalizing the matrix $\mathcal{M}$. One readily notes that the real parts of characteristic exponents take on zero and two nonzero values with opposite signs. We also find that the analytical result

FIG. 6. The weights of the components of incoherent fluorescence spectrum centered at $\omega_0 - \omega_z$ ($W_{1,-1}$, blue solid line), ω_0 ($W_{1,0}$, purple dash-dotted line), and $\omega_0 + \omega_z$ ($W_{1,1}$, red dashed line) as a function of ratio Ω_z/ω_z for $\omega_z = 0.01\omega_0$ and $\Omega_x = 5\kappa$. The other parameters are set as $\omega_x = \omega_0 = 10^3\kappa$.

$\omega_z - E$ (red circle) as a function of Ω_z/ω_z indeed agrees with the numerical result of $\text{Re}[\varepsilon_2]$. (We use $\omega_z - E$ not $\pm E$ because E is out of the interval $(-\omega_z/2, \omega_z/2]$ while $\text{Re}[\varepsilon_\gamma]$ has been shifted in the interval.) The agreement confirms that the emission lines can be interpreted as a result of the transitions of the two-mode Floquet states we derived. On the other hand, we find that $|\text{Re}[\varepsilon_{2,3}]|$ decreases with the increase of the ratio Ω_z/ω_z . This behavior is reflected in the spectrum in which the larger the ratio Ω_z/ω_z , the smaller the splitting between the central peak (centered at $\omega_0 + n\omega_z$) and its satellite sidebands [Figs. 3(c) and 3(d)]. We can understand the role of longitudinal driving on the exponents' real part by the analytical expression of E . When $\Omega_x < \omega_z$, the first-order Bessel function $J_0(\xi\Omega_z/\omega_z)$ decreases and $\xi \rightarrow 1$ as the ratio Ω_z/ω_z increases. It follows that E tends to ω_z , resulting in a decrease in the exponent's real part. In addition, one finds that $\text{Re}[\varepsilon_{2,3}] \rightarrow \pm 5\kappa$ in the limit of $\Omega_z/\omega_z \rightarrow 0$. Thus, the two sidebands appear at frequencies $\omega_0 \pm \Omega_x$ when $\Omega_z = 0$, which are the same as the Mollow spectrum sidebands (under the same transverse driving condition) [1].

We now examine the imaginary parts of the characteristic exponents, which determines the FWHM of the emission lines. In Fig. 5(b), the imaginary parts of the characteristic exponents are plotted as a function of the ratio Ω_z/ω_z . It is clear that there are only two independent values for the three exponents for each fixed ratio Ω_z/ω_z . The imaginary part of the exponent with a vanishing real part takes on a smaller absolute value than the imaginary parts for the other two exponents with nonzero real parts. This relation shall be reflected in the spectrum in which the FWHM of the line at frequency $\omega_0 + n\omega_z$ is narrower than those of the lines at frequencies $\omega_0 + n\omega_z - \text{Re}[\varepsilon_{2,3}]$. Furthermore, we find that as the ratio Ω_z/ω_z increases, $-\text{Im}[\varepsilon_{2,3}]$ decreases and $-\text{Im}[\varepsilon_1]$ increases, which means that the longitudinal driving can cause narrowing and broadening of the spectral emission lines at frequencies $\omega_0 + n\omega_z - \text{Re}[\varepsilon_{2,3}]$ and $\omega_0 + n\omega_z$, respectively. On the other hand, in the limit of $\Omega_z/\omega_z \rightarrow 0$, one finds that $-\text{Im}[\varepsilon_1] \rightarrow 0.5\kappa$ and $-\text{Im}[\varepsilon_{2,3}] \rightarrow 0.75\kappa$, leading to the FWHM of the emission line at ω_0 being κ and those of the emission lines at $\omega_0 \pm \Omega_x$ being 1.5κ , which recovers the FWHM of the Mollow spectrum (under the same transverse driving condition) [1].

Next, we investigate the weights of the emission lines centered around $\omega_0 + n\omega_z$ as a function of the ratio Ω_z/ω_z . From the formal spectrum, we can obtain the weight of each Lorentzian emission line:

$$W_{\gamma,n} = \text{Re} \left\{ \phi_{n,1}^{(\gamma)} \left[\frac{1}{2} \phi_{n,1}^{(\gamma)} + \sum_{\beta,m,k,l} \phi_{m,3}^{(\gamma)} \phi_{k,2}^{(\beta)} \frac{\kappa}{i(\varepsilon_\beta - l\omega_z)} \right. \right. \\ \times \phi_{l,3}^{(\beta)} \left(1 - \frac{\kappa}{i(\varepsilon_\gamma - m\omega_z)} \right) \delta_{n,m+l-k} \\ \left. \left. - \frac{\kappa}{2} \sum_{\beta,m,k,l} \phi_{m,1}^{(\gamma)} \phi_{k,3}^{(\beta)} \phi_{l,3}^{(\beta)} \frac{\delta_{n,m+l-k}}{i(\varepsilon_\beta - l\omega_z)} \right] \right\}. \quad (35)$$

In Fig. 6, we calculate the three weights $W_{1,n}$ with $n = 0, \pm 1$ as a function of the ratio Ω_z/ω_z , corresponding to the weights of the Lorentzian lines at $\omega_0, \omega_0 \pm \omega_z$. In the limit $\Omega_z/\omega_z \rightarrow 0$,

FIG. 7. The incoherent fluorescence spectrum as a function of $\Delta = \omega - \omega_0$ for $\omega_x = \omega_0 = 10^3\kappa$, $\omega_z = 0.02\omega_0$, and $\Omega_z/\omega_z = 1$. The Ω_x is set as (a) 10κ and (b) 20κ .

it is not surprising to see that $W_{1,\pm 1} \rightarrow 0$, indicating that the emission lines at $\omega_0 + n\omega_z$ disappear. With the increase of Ω_z/ω_z , we find that the $W_{1,\pm 1}$ increases from zero to a maximum and $W_{1,0}$ decreases monotonously. In particular, $W_{1,\pm 1}$ can even exceed $W_{1,0}$ when the ratio is large enough. In other words, the emission line at ω_0 is suppressed while the emission lines at $\omega_0 \pm \omega_z$ become significantly enhanced for a moderately large ratio Ω_z/ω_z [see Fig. 3(d)]. To end this section, we examine the spectral line shape by increasing the longitudinal driving frequency ω_z . In Figs. 7(a) and 7(b) we show the spectra for $\omega_z = 0.02\omega_0$ with moderately strong transverse driving: $\Omega_x = 10\kappa$ and $\Omega_x = 20\kappa$, respectively. One finds that the emission lines are well separated as compared to the case of $\omega_z = 0.01\omega_0$. When further increasing ω_z up to $0.05\omega_0$, in Figs. 8(a) and 8(b) we are able to observe the well-separated individual Mollow triplets centered around ω_0 and $\omega_0 \pm \omega_z$. Interestingly, one finds that the central triplet has equal sidebands while the two satellite triplets have two unequal sidebands. These results indicate that it is possible to clearly observe an exotic fluorescence spectrum provided that the qubit is exposed to the relatively strong ($\Omega_z/\omega_z > 0.5$ but $\Omega_z/\omega_0 \ll 1$) and slow ($\omega_z \ll \omega_0$) longitudinal driving.

IV. CONCLUSION

In this work we have studied the fluorescence spectrum from a superconducting qubit driven simultaneously by the

FIG. 8. The incoherent fluorescence spectrum as a function of $\Delta = \omega - \omega_0$ for $\omega_x = \omega_0 = 10^3\kappa$, $\omega_z = 0.05\omega_0$, and $\Omega_z/\omega_z = 1$. The Ω_x is set as (a) 10κ and (b) 20κ .

longitudinal and transverse fields. To solve the time evolution and derive the spectrum for the qubit, we first invoke the RWA for the transverse field and then transform the equation of motion into a frame rotating at frequency ω_x , yielding a set of Bloch equations with time dependence contributed only from the longitudinal field. The time-dependent Bloch equations are solved by the Floquet theory, yielding the formal solution expressed in terms of the Fourier coefficients. The validity of the solution under an appropriate truncation has been verified by comparison with the Runge-Kutta method. Using the formal solution to the Bloch equations, we have derived the fluorescence spectrum and carried out numerical calculation to illustrate the exotic fluorescence spectrum with the modulation of the longitudinal driving. We have found that the spectrum exhibits a rich multipeak structure and dramatically deviates from the standard Mollow triplet. The multipeak spectrum can be physically interpreted as a result of the transitions between the specific two-mode Floquet states for the combined system of the qubit and the longitudinal and transverse driving fields. In addition, it turns out that the longitudinal driving is able to modulate the splitting between the main peak and its sidebands, as well as the linewidth. Our results show that the influence of the longitudinal driving on the spectrum depends on the ratio Ω_z/ω_z . Moreover, the exotic spectral features become apparent with the increase of the ratio.

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APPENDIX: THE DERIVATION OF TWO-MODE FLOQUET STATES AND QUASIENERGIES

We first construct the single-mode Floquet state by solving the time-dependent Schrödinger equation in the rotating frame $i\partial_t|\tilde{\psi}(t)\rangle = \tilde{H}(t)|\tilde{\psi}(t)\rangle$ in the absence of dissipation, where $\tilde{H}(t)$ is given in Eq. (7) and the ket used here and following represents the state in the Hilbert space for the qubit. It is feasible to perform unitary transformation as [25]

$$|\Psi(t)\rangle = e^{X(t)}|\tilde{\psi}(t)\rangle, \quad (\text{A1})$$

where $X(t) = i\frac{\Omega_z}{2\omega_z}\xi \sin(\omega_z t)\sigma_z$. Here the parameter ξ is constrained between 0 and 1 and will be determined self-consistently later. By the unitary transformation, the equation of motion is transformed as

$$\frac{d}{dt}|\Psi(t)\rangle = h(t)|\Psi(t)\rangle, \quad (\text{A2})$$

where

$$h(t) = e^{X(t)}\tilde{H}(t)e^{-X(t)} + i\partial_t X(t) \quad (\text{A3})$$

is the transformed Hamiltonian. One readily obtains the explicit form for $h(t)$ as follows:

$$\begin{aligned} h(t) &= \frac{\Omega_x}{2} \left\{ \cos\left[\frac{\Omega_z \xi \sin(\omega_z t)}{\omega_z}\right] \sigma_x - \sin\left[\frac{\Omega_z \xi \sin(\omega_z t)}{\omega_z}\right] \sigma_y \right\} \\ &\quad + \frac{\Omega_z}{2} (1 - \xi) \cos(\omega_z t) \sigma_z \\ &= \frac{\Omega_x}{2} \left[J_0\left(\frac{\xi \Omega_z}{\omega_z}\right) + \sum_{n=1}^{\infty} 2J_{2n}\left(\frac{\xi \Omega_z}{\omega_z}\right) \cos(2n\omega_z t) \right] \sigma_x \\ &\quad - \Omega_x \sum_{n=1}^{\infty} J_{2n-1}\left(\frac{\xi \Omega_z}{\omega_z}\right) \sin[(2n-1)\omega_z t] \sigma_y \\ &\quad + \frac{\Omega_z}{2} (1 - \xi) \cos(\omega_z t) \sigma_z, \end{aligned} \quad (\text{A4})$$

where we used the identity $\exp[iz \sin(\omega_z t)] = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\omega_z t}$, with $J_n(z)$ being the Bessel function of the first kind. To proceed, we retain the slow-varying terms with the harmonic $n\omega_z$ ($n \leq 1$) and omit the fast-varying terms, which leads to

$$\begin{aligned} h(t) &= \frac{\Omega_x}{2} J_0\left(\frac{\xi \Omega_z}{\omega_z}\right) \sigma_x - \Omega_x J_1\left(\frac{\xi \Omega_z}{\omega_z}\right) \sin(\omega_z t) \sigma_y \\ &\quad + \frac{\Omega_z}{2} (1 - \xi) \cos(\omega_z t) \sigma_z. \end{aligned} \quad (\text{A5})$$

When ξ is self-consistently determined via the equation [25]

$$\Omega_x J_1\left(\frac{\xi \Omega_z}{\omega_z}\right) = \frac{\Omega_z}{2} (1 - \xi), \quad (\text{A6})$$

we can rewrite $h(t)$ as follows:

$$\begin{aligned} h(t) &= \frac{\Omega_x}{2} J_0\left(\frac{\xi \Omega_z}{\omega_z}\right) \sigma_x + \frac{\tilde{\Omega}_z}{2} \left[e^{-i\omega_z t} \frac{(\sigma_z - i\sigma_y)}{2} \right. \\ &\quad \left. + e^{i\omega_z t} \frac{(\sigma_z + i\sigma_y)}{2} \right], \end{aligned} \quad (\text{A7})$$

where

$$\tilde{\Omega}_z = \Omega_z (1 - \xi). \quad (\text{A8})$$

The validity of this treatment was discussed in detail in our previous work. Generally speaking, this treatment is fully justified when $\Omega_z/\omega_z < 2$, regardless of ω_z or $\Omega_z/\omega_z < 6$ with $\Omega_x \ll \omega_z$ [25]. To proceed, we further perform a rotation operation along the x axes with rotating frequency ω_z , which leads to

$$\begin{aligned} h &= \exp(i\omega_z t \sigma_x / 2) [h(t) - i\partial_t] \exp(-i\omega_z t \sigma_x / 2) \\ &= \frac{1}{2} \left[\Omega_x J_0\left(\frac{\xi \Omega_z}{\omega_z}\right) - \omega_z \right] \sigma_x + \frac{\tilde{\Omega}_z}{2} \sigma_z. \end{aligned} \quad (\text{A9})$$

This is a static 2×2 matrix and can be diagonalized yielding the eigenstates $|\Psi_{\pm}\rangle$ and eigenenergies $\pm \frac{1}{2}E$:

$$E = \left\{ \left[\Omega_x J_0\left(\frac{\xi \Omega_z}{\omega_z}\right) - \omega_z \right]^2 + \tilde{\Omega}_z^2 \right\}^{1/2}. \quad (\text{A10})$$

By these observation, we can construct Floquet states in the rotating frame as follows:

$$\begin{aligned} |\tilde{\psi}_{\pm}(t)\rangle &= \tilde{U}(t)|\Psi_{\pm}\rangle \\ &= e^{-X(t)} \exp\left(\frac{-i\omega_z t \sigma_x}{2}\right) e^{-iht} |\Psi_{\pm}\rangle \\ &= e^{\mp iEt/2} e^{-X(t)} \exp\left(\frac{-i\omega_z t \sigma_x}{2}\right) |\Psi_{\pm}\rangle \\ &= \exp\left\{-i\left[\frac{1}{2}(\omega_z \pm E) + n_z \omega_z\right]t\right\} \\ &\quad \times e^{i\omega_z t(n_z \pm \frac{1}{2}) - X(t)} e^{-\frac{i}{2}\omega_z t \sigma_x} |\Psi_{\pm}\rangle \\ &\equiv e^{-i\epsilon_{\pm, n_z} t} |\tilde{u}_{\pm, n_z}(t)\rangle, \end{aligned} \quad (\text{A11})$$

where

$$\epsilon_{\pm, n} \equiv \frac{1}{2}(\omega_z \pm E) + n_z \omega_z, \quad (\text{A12})$$

$$|\tilde{u}_{\pm, n_z}(t)\rangle \equiv e^{i\omega_z t(n_z \pm \frac{1}{2}) - X(t)} e^{-\frac{i\omega_z t \sigma_x}{2}} |\Psi_{\pm}\rangle. \quad (\text{A13})$$

Transforming back into the original frame, we have

$$\begin{aligned} |\psi_{\pm}(t)\rangle &= \exp\left(-\frac{i\omega_x \sigma_z t}{2}\right) |\tilde{\psi}(t)\rangle \\ &= e^{-i\epsilon_{\pm, n_z, n_x} t} |u_{\pm, n_z, n_x}(t)\rangle, \end{aligned} \quad (\text{A14})$$

where $|u_{\pm, n_z, n_x}(t)\rangle$ are the two-mode Floquet states associated with quasienergies $\epsilon_{\pm, n_z, n_x}$ in the laboratory frame.

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