Investigation of nonlinear ultrasonic guided waves in open waveguides based on perfectly matched layers

Zuo, Peng; Liu, Yang; Zhou, Yu; Fan, Zheng

2018

Zuo, P., Liu, Y., Zhou, Y., & Fan, Z. (2018). Investigation of nonlinear ultrasonic guided waves in open waveguides based on perfectly matched layers. *The Journal of the Acoustical Society of America*, 144(3), 1572-1588. doi:10.1121/1.5055210

<https://hdl.handle.net/10356/89687>

<https://doi.org/10.1121/1.5055210>

© 2018 Acoustical Society of America. This paper was published in *The Journal of the Acoustical Society of America* and is made available as an electronic reprint (preprint) with permission of Acoustical Society of America. The published version is available at: [<http://dx.doi.org/10.1121/1.5055210>]. One print or electronic copy may be made for personal use only. Systematic or multiple reproduction, distribution to multiple locations via electronic or other means, duplication of any material in this paper for a fee or for commercial purposes, or modification of the content of the paper is prohibited and is subject to penalties under law.

Downloaded on 26 Apr 2023 18:10:46 SGT

Investigation of nonlinear ultrasonic guided waves in open waveguides based on perfectly matched layers

Peng Zuo,¹ Yang Liu,² Yu Zhou,³ and Zheng Fan^{1,a)}

¹*School of Mechanical and Aerospace Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 639798, Singapore*

²*Schlumberger-Doll Research, One Hampshire Street, Cambridge, Massachusetts 02139, USA*

³*Advanced Remanufacturing and Technology Center (ARTC), 3 Clean Tech Loop, CleanTech Two, Singapore 637143, Singapore*

(Received 7 April 2018; revised 21 August 2018; accepted 30 August 2018; published online 26 September 2018)

Nonlinear ultrasonic guided waves have been investigated widely in closed waveguides such as plates, pipes, etc. However, the description of nonlinear ultrasonic guided waves remains challenging for open waveguides, as energy may leak into the surrounding medium. In this work, the properties of nonlinear ultrasonic guided waves in open waveguides are investigated. Mathematical framework is first established based on real reciprocity relation and modal expansion with perfectly matched layers. Numerical models are then implemented, including nonlinear semi-analytical finite element (SAFE) method to predict the properties of nonlinear ultrasonic guided waves, and time domain finite element models to simulate the nonlinear guided wave propagation and cross validate the predictions from the nonlinear SAFE method. Two examples, an aluminum plate attached to an elastomer and an aluminum plate with water loaded on one side, are studied to demonstrate the proposed methods and reveal some interesting phenomena that only exist in open waveguides. It is interesting to find out that the amplitude of the attenuated second harmonic wave in immersed waveguides can keep constant with propagation distance, only if the primary wave is non-leaky, which may bring potential non-destructive test applications for underwater inspections. Such a feature is validated by well-designed experiments in one-sidedly immersed plates.

© 2018 Acoustical Society of America. <https://doi.org/10.1121/1.5055210>

[LC]

Pages: 1572–1588

I. INTRODUCTION

Nonlinear ultrasonic guided wave, combining the advantages of nonlinear ultrasound and ultrasonic guided waves, has been attractive for non-destructive test (NDT) for a few decades as it is sensitive to the incipient damage and can propagate for a long distance for large area inspections. The nonlinear ultrasonic guided wave was first investigated in plate-like structures.^{1–7} Two conditions (usually called internal resonant conditions) were summarized using the perturbation method and modal expansion analysis for the generation of the linearly increased second harmonic wave. Then the theoretical analysis of nonlinear ultrasonic guided waves was extended from plate-like structures to cylindrical rods and shells^{8–10} where although much more guided modes exist, they have the same internal resonant conditions as in plates. It is known that analytical solutions are only available in plate-like structures and rod/shell structures due to their relatively simple boundary conditions when solving the nonlinear Navier equations. Hence, numerical methods are necessary to understand modal properties of nonlinear ultrasonic guided waves in complex waveguides with arbitrary cross sections. For such cases, the semi-analytical finite element (SAFE) method was used to analyze modal properties of the waveguide together with nonlinear analysis to predict the properties of the harmonic waves.^{11,12}

However, all of the studies discussed above are only limited to closed waveguides (waveguides in vacuum), where all the energy is confined in the waveguides. There are more challenging cases, typically existing in civil engineering and geophysics, in which the waveguides are embedded/immersed in another infinite medium (known as open waveguides) such as embedded fibers, underwater pipes, etc. In these cases, interactions between the waveguide and the surrounding material may lead to energy in the waveguide leaking into the surrounding medium, which further distorts the behavior of the primary and second harmonic waves. Thus, the amplitude of the second harmonic wave may not be linearly cumulative as in closed waveguides. In order to seek for potential applications of nonlinear ultrasonic guided waves in open waveguides, understanding the behavior of the second harmonic wave in such structures is therefore a necessary prerequisite.

It is known that the modal expansion method is commonly used in closed waveguides to derive the amplitude of the second harmonic wave,^{3,4} stemmed from the fact that the modes, including propagating modes, non-propagating modes, and evanescent modes, satisfy the orthogonality relationship.¹³ However, the application of this method to open waveguides is more intricate since the orthogonality relationship in these structures is obscure because of the unboundedness of the surrounding material. Theoretically, the unboundedness of the surrounding material in open

^{a)}Electronic mail: zfan@ntu.edu.sg

waveguides leads to three types of modes forming.^{14–16} They are trapped modes where energy concentrates in the waveguide core structure, leaky modes, which propagate with energy leaking into the surrounding medium, and radiation modes that resonate in the surrounding material. In mathematics, trapped modes and radiation modes form the modal basis of an open waveguide, but the continuum radiation modes are difficult to handle in mathematical manipulation as a large number of sampled radiation modes have to be included in the expansion.^{17,18} In electromagnetic fields, the discrete leaky modes have been proposed to approximate the contribution of the radiation modes to the modal expansion,¹⁹ even though they do not belong to the modal basis due to their amplitude growing to infinity in the transverse direction.¹⁵ In order to circumvent this problem, two methods have been suggested.¹⁹ One approach is to truncate the leaky modes within a restricted space very close to the waveguide core structure so that they are approximately orthogonal. The other method is to retain the full wave field of leaky modes but replace the orthogonality relation with a more general, mathematically exact orthogonality by deforming the real coordinates into the complex plane. Even though the two methods have shown to be satisfied sensibly not only on open waveguides for optical applications,^{20,21} but also on embedded elastic waveguides for acoustic scattering studies,²² the application of orthogonality for leaky modes in open waveguides still remains challenging.

More recently, an approach has been proposed to address the applicability of leaky modes in open waveguides by introducing a finite perfectly matched layer (PML) to truncate the infinite surrounding medium.^{18,23} With PMLs, the leaky wave can be totally damped in a limited space, efficiently simulating the infinite surrounding medium, so that the infinite open waveguide is converted to a finite one. In addition, the finite PML transforms the continuum radiation modes existing in the infinite open waveguides into discrete sets of PML modes, which only resonate in the PML.¹⁸ Following this, an orthogonality relationship has been derived, which can be applied to any types of modes, including trapped modes, leaky modes, and PML modes.²³ Hence a new modal basis has been defined by these three types of modes, and leaky modes are applicable in modal expansion to describe any wave field in open waveguides through superposition of the modal basis.

This paper starts with the mathematical framework by using real reciprocity relation and modal expansion with PMLs. Numerical models are then implemented, including the nonlinear SAFE method to predict the properties of nonlinear ultrasonic guided waves and finite element (FE) models to simulate the propagation of nonlinear ultrasonic guided waves and cross validate the predictions from the nonlinear SAFE method. Following the numerical models, two examples are given to illustrate the proposed methods and reveal some interesting properties that only occur in the open waveguides, including an aluminum plate attached to a semi-infinite half-space of an elastomer and an aluminum plate with water loaded on one side. Finally, well designed experiments are carried out to validate the properties of the nonlinear ultrasonic guided wave in one-sidedly immersed

plates, showing potential NDT applications of nonlinear ultrasonic guided waves for underwater inspections.

II. MATHEMATICAL FRAMEWORK OF NONLINEAR ULTRASONIC GUIDED WAVE IN OPEN WAVEGUIDES

In this section, the theoretical foundation of nonlinear ultrasonic guided wave propagation in open waveguides is developed through the theory of the finite elasticity and nonlinear elastic material models. In finite elasticity, the description of the motion of a material body has been given by considering the relation between two configurations, reference configuration and current configuration, in which each material particle of the body is named with a marker labeled by position vector $\mathbf{X}$ in the reference configuration and $\mathbf{x} = \varphi(\mathbf{X}, t)$ in the current configuration at a given time t . The change in shape of the infinitesimal volume elements can be measured by the deformation gradient tensor, defined by

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}. \quad (1)$$

The Green-Lagrange strain tensor is defined in terms of deformation gradient tensor

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}), \quad (2)$$

where $\mathbf{F}^T$ represents transposition of $\mathbf{F}$, and $\mathbf{I}$ is the identity tensor. In the nonlinear elastic material model, the Landau-Lifshitz hyperelastic constitutive equation is usually used to describe the nonlinear elastic properties of the solid, which can be given through a strain energy density function

$$w = \frac{1}{2} \lambda I_1^2 + \mu I_2 + \frac{1}{3} C I_1^3 + B I_1 I_2 + \frac{1}{3} A I_3, \quad (3)$$

where λ and μ are Lamé constants; A , B , and C are third-order elastic constants, which introduce the nonlinear elastic properties of the solid; I_1 , I_2 , and I_3 are defined by $I_1 = E_{ii}$, $I_2 = E_{ij}E_{ji}$, and $I_3 = E_{ij}E_{jk}E_{ki}$, representing the first three invariants of the Green-Lagrange tensor. The second Piola-Kirchhoff (P-K) stress tensor can be derived from the energy density function by

$$\mathbf{S} = \frac{\partial w}{\partial \mathbf{E}}. \quad (4)$$

The first P-K stress and second P-K stress are related through

$$\mathbf{s} = \mathbf{F} \mathbf{S}. \quad (5)$$

In this scenario, the momentum balance equation of the solid in the reference configuration can be written as

$$\rho_0 \ddot{\mathbf{u}} = \nabla \cdot \mathbf{s}, \quad (6)$$

where ρ_0 is the density of the material in the reference configuration; $\mathbf{u}$ is the displacement; ∇ is the gradient operator with respect to the reference configuration, and the “dot” represents the derivative of time.

Mathematical derivations of the nonlinear ultrasonic guided waves start with infinite open waveguides. As shown in Fig. 1(a), the open waveguide consists of two main parts, the solid core structure, occupying Ω_A , and the infinite surrounding material in Ω_B , which could be a solid or an inviscid fluid. In this paper, only the nonlinearity properties of the core structure are considered, thus, the core is assumed to hold the Landau-Lifshitz constitutive equation; the infinite surrounding material is assumed to be homogeneous, isotropic, and elastic.

The governing equations and boundary conditions of the open waveguide can be formulated in terms of the displacement $\mathbf{u}$,

$$\rho_A \frac{\partial^2 \mathbf{u}_A}{\partial t^2} = (\lambda_A + 2\mu_A) \nabla(\nabla \cdot \mathbf{u}_A) - \mu_A \nabla \times (\nabla \times \mathbf{u}_A) + \bar{\mathbf{f}} \quad \text{in } \Omega_A, \quad (7)$$

$$\rho_B \frac{\partial^2 \mathbf{u}_B}{\partial t^2} = (\lambda_B + 2\mu_B) \nabla(\nabla \cdot \mathbf{u}_B) - \mu_B \nabla \times (\nabla \times \mathbf{u}_B) \quad \text{in } \Omega_B, \quad (8)$$

$$(\mathbf{s}_{AL} + \bar{\mathbf{s}}) \cdot \mathbf{n}_A = -\mathbf{s}_B \cdot \mathbf{n}_B^{\text{inn}}, \quad \mathbf{u}_A = \mathbf{u}_B \quad \text{on } \Gamma_{A/B}, \quad (9)$$

$$\mathbf{u}_B \rightarrow 0, \quad \mathbf{s}_B \rightarrow 0 \quad \text{at infinity}, \quad (10)$$

where $\mathbf{u}_A$ and $\mathbf{u}_B$ represent the displacement in Ω_A and Ω_B , respectively; λ_A and μ_A are the Lamé constants of core structure in Ω_A with λ_B and μ_B being the Lamé constants of the surrounding material in Ω_B ; $\mathbf{s}_B$ is the first P-K stress in Ω_B ; and $\mathbf{s}_{AL}$ represents the corresponding linear part of the first P-K stress in Ω_A , given by

$$\mathbf{s}_{AL} = \frac{\lambda_A}{2} \text{tr}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \mathbf{I} + \mu_A (\nabla \mathbf{u} + (\nabla \mathbf{u})^T). \quad (11)$$

$\bar{\mathbf{f}}$ and $\bar{\mathbf{s}}$ collect all nonlinear terms, and can be explicitly expressed with tensor components as

$$\begin{aligned} \bar{f}_i = & \left(\mu + \frac{A}{4} \right) (u_{l,kk} u_{l,i} + u_{l,kk} u_{i,l} + 2u_{i,lk} u_{l,k}) \\ & + \left(\lambda + \mu + \frac{A}{4} + B \right) (u_{l,ik} u_{l,k} + u_{k,lk} u_{i,l}) \\ & + (\lambda + B) (u_{i,kk} u_{l,l}) + \left(\frac{A}{4} + B \right) \\ & \times (u_{k,lk} u_{l,i} + u_{k,ik} u_{k,l}) + (B + 2C) (u_{k,ik} u_{l,l}) + O(u_i^3), \end{aligned} \quad (12)$$

$$\begin{aligned} \bar{s}_{ij} = & \left(\frac{\lambda}{2} u_{k,l} u_{k,l} + C u_{k,k} u_{l,l} \right) \delta_{ij} + B u_{k,k} u_{j,i} + \frac{A}{4} u_{j,k} u_{k,i} \\ & + \frac{B}{2} (u_{k,l} u_{k,l} + u_{k,l} u_{l,k}) \delta_{ij} \\ & + (\lambda + B) u_{k,k} u_{i,j} + \left(\mu + \frac{A}{4} \right) \\ & \times (u_{i,k} u_{j,k} + u_{k,i} u_{k,j} + u_{i,k} u_{k,j}) + O(u_i^3). \end{aligned} \quad (13)$$

The perturbation theory is used to solve the nonlinear wave propagation equations in which the displacement field $\mathbf{u}$ is divided into two parts¹⁰

$$\mathbf{u}_A = \mathbf{u}_{A1} + \mathbf{u}_{A2}, \quad (14)$$

$$\mathbf{u}_B = \mathbf{u}_{B1} + \mathbf{u}_{B2}, \quad (15)$$

where $\mathbf{u}_{A1}$ and $\mathbf{u}_{B1}$ are the primary displacement fields; $\mathbf{u}_{A2}$ and $\mathbf{u}_{B2}$ represent the secondary displacement fields, which are much smaller than the primary fields. Substituting Eqs. (14) and (15) into Eqs. (7)–(10), the nonlinear boundary value problem can be divided into two linear boundary value

FIG. 1. Schematics of (a) an infinite open waveguide and (b) the corresponding finite open waveguide with PML. $\Gamma_{A/B}$ and Γ_B are the interface between the waveguide core structure and the surrounding material and the outer surface of the finite open waveguide, respectively; $\mathbf{n}_A$ and $\mathbf{n}_B^{\text{inn}}$ represent the outward-pointing unit normal vector of the core structure and the surrounding material on the interface, respectively, holding a relationship as $\mathbf{n}_A = -\mathbf{n}_B^{\text{inn}}$; $\mathbf{n}_B^{\text{out}}$ is the outward-pointing unit normal vector of the finite open waveguide on Γ_B ; d_x and d_y denote the start position of the PML in the x and y directions, respectively; h_x and h_y represent the length of the PML in the x and y directions, respectively.

problems. In the first-order perturbation equations, the governing equations and boundary conditions become

$$\rho_A \frac{\partial^2 \mathbf{u}_{A1}}{\partial t^2} = (\lambda_A + 2\mu_A) \nabla(\nabla \cdot \mathbf{u}_{A1}) - \mu_A \nabla \times (\nabla \times \mathbf{u}_{A1}) \quad \text{in } \Omega_A, \quad (16)$$

$$\rho_B \frac{\partial^2 \mathbf{u}_{B1}}{\partial t^2} = (\lambda_B + 2\mu_B) \nabla(\nabla \cdot \mathbf{u}_{B1}) - \mu_B \nabla \times (\nabla \times \mathbf{u}_{B1}) \quad \text{in } \Omega_B, \quad (17)$$

$$\mathbf{s}_{A1} \cdot \mathbf{n}_A + \mathbf{s}_{B1} \cdot \mathbf{n}_B^{\text{inn}} = 0, \quad \mathbf{u}_{A1} = \mathbf{u}_{B1} \quad \text{on } \Gamma_{A/B}, \quad (18)$$

$$\mathbf{u}_{B1} \rightarrow 0, \quad \mathbf{s}_{B1} \rightarrow 0 \quad \text{at infinity.} \quad (19)$$

In the second-order perturbation equations,

$$\rho_A \frac{\partial^2 \mathbf{u}_{A2}}{\partial t^2} = (\lambda_A + 2\mu_A) \nabla(\nabla \cdot \mathbf{u}_{A2}) - \mu_A \nabla \times (\nabla \times \mathbf{u}_{A2}) + \bar{\mathbf{f}}^{(1)} \quad \text{in } \Omega_A, \quad (20)$$

$$\rho_B \frac{\partial^2 \mathbf{u}_{B2}}{\partial t^2} = (\lambda_B + 2\mu_B) \nabla(\nabla \cdot \mathbf{u}_{B2}) - \mu_B \nabla \times (\nabla \times \mathbf{u}_{B2}) \quad \text{in } \Omega_B, \quad (21)$$

$$\mathbf{s}_{A2} \cdot \mathbf{n}_A + \mathbf{s}_{B2} \cdot \mathbf{n}_B^{\text{inn}} = -\bar{\mathbf{s}}^{(1)} \cdot \mathbf{n}_A, \quad \mathbf{u}_{A2} = \mathbf{u}_{B2} \quad \text{on } \Gamma_{A/B}, \quad (22)$$

$$\mathbf{u}_{B2} \rightarrow 0, \quad \mathbf{s}_{B2} \rightarrow 0 \quad \text{at infinity.} \quad (23)$$

It should be noted that $\bar{\mathbf{f}}^{(1)}$ and $\bar{\mathbf{s}}^{(1)}$ are functions of the primary displacement field. Equations (16)–(19), the homogeneous first-order perturbation equations, describe the primary wave propagation in the open waveguide. Once the solution is obtained, the terms $\bar{\mathbf{f}}^{(1)}$ and $\bar{\mathbf{s}}^{(1)}$ can be calculated by substituting the primary solution into Eqs. (12) and (13). Equations (20)–(23), the inhomogeneous second-order perturbation equations, are the description of the second harmonic wave in the open waveguide, which can be interpreted physically as an excited solution by an external body force $\bar{\mathbf{f}}^{(1)}$ applied in Ω_A and a traction $\bar{\mathbf{s}}^{(1)}$ on $\Gamma_{A/B}$. Therefore, the original nonlinear problem reduces to seeking two separate linear solutions.

For the primary wave field, analytical solutions can only be extracted for regular waveguides (such as plates, rods, and pipes) through the transfer matrix method,²⁴ global matrix method,²⁵ and dispersion equations.²⁶ In order to solve the wave propagation in open waveguides with arbitrary cross-sectional geometries, numerical methods are usually required. Three main methods have been developed to extract wave modes in open waveguides, the SAFE method coupled with absorbing layers (SAFE-AL),^{27,28} the SAFE method incorporating PMLs (SAFE-PML),^{29–31} and combining the SAFE method and the boundary element method (SAFE-BEM).³²

It should be noted that complex reciprocity relation and modal expansion analysis are useful to derive the secondary field. However, their application to open waveguides

remains intricate due to the fact that leaky modes cannot form the modal basis in open waveguides.¹⁸ As indicated in Sec. I, this difficulty can be addressed by introducing PMLs truncating the infinite surrounding medium within a restricted space, under which the trapped modes, leaky modes, and PML modes form the modal basis, and any arbitrary wave field can be expressed by superposition of the modal basis.²³ Accordingly, an equivalent open waveguide with PMLs is indeed necessary to be built in replacement of the infinite open waveguide not only for extracting the primary wave field via SAFE-PML method, but also for the appropriate application of modal expansion in open waveguides.

Figure 1(b) presents an equivalent open waveguide with a PML around the surrounding medium. Ω_B now consists of the surrounding medium and the PML. The main function of the PML is to absorb the leaky wave in order to simulate the infinite surrounding medium, as well as to transform the radiation modes existing in the infinite surrounding medium into PML modes that resonate in the PML. The damping property of the PML comes from a mathematical coordinate mapping from real space to the complex plane by introducing a continuous, complex-valued stretching function.³³ Therefore the PML has the same material properties as the surrounding medium making its impedance perfectly matched with the surrounding material. As the complex coordinate mapping only occurs in the PML, the governing equations [Eqs. (16), (17), (20), and (21)] and the interface boundary conditions [Eqs. (18) and (22)] in the infinite open waveguide are still available in the finite open waveguide. The governing equations in the PML appear a bit complex due to complex coordinate mapping. However, such coordinate mapping in the PML will not complicate the problem, as only interface and boundary conditions will be utilized during the following mathematical derivation. Therefore, it is not necessary to explicitly write the governing equations of the PML here. The boundary conditions at infinity have to be converted to boundary conditions on Γ_B ,

$$\mathbf{u}_{B1} = 0, \quad \mathbf{s}_{B1} = 0 \quad \text{on } \Gamma_B, \quad (24)$$

$$\mathbf{u}_{B2} = 0, \quad \mathbf{s}_{B2} = 0 \quad \text{on } \Gamma_B. \quad (25)$$

In the PML coupled open waveguide as shown in Fig. 1(b), the solution of the primary wave can be obtained numerically through SAFE-PML method. In order to solve the secondary field, the real reciprocity relation and modal expansion method are used. The real reciprocity relation can be written as³⁴

$$\nabla \cdot (\mathbf{v}_1 \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_1) = \mathbf{v}_2 \cdot \mathbf{f}_1 - \mathbf{v}_1 \cdot \mathbf{f}_2, \quad (26)$$

where subscripts “1” and “2” represent two possible solutions in the open waveguide; $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocity tensors; $\mathbf{s}_1$ and $\mathbf{s}_2$ demonstrate the stress tensors and $\mathbf{f}_1$ and $\mathbf{f}_2$ represent the volume forces.

Solution “1” is considered to be the n th mode of the open waveguide without external force fields. Solution “2” is assumed to be a solution having the external forces applied by the primary wave shown in Eqs. (20) and (22), which

means that solution “2” is actually the secondary field. Solution “1” and solution “2” can be written as

$$\mathbf{v}_1 = \mathbf{v}_n e^{ik_n z}, \quad \mathbf{s}_1 = \mathbf{s}_n e^{ik_n z}, \quad \mathbf{f}_1 = 0, \quad (27)$$

$$\mathbf{v}_2 = \mathbf{v}_2(x, y, z), \quad \mathbf{s}_2 = \mathbf{s}_2(x, y, z), \quad \mathbf{f}_2 = \bar{\mathbf{f}}^{(1)}. \quad (28)$$

Substituting the solutions into Eq. (26), the equation can be expressed as

$$\begin{aligned} & \nabla_{\perp} \cdot (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \\ & + \frac{\partial}{\partial z} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_z \\ & = -\mathbf{v}_n e^{ik_n z} \cdot \bar{\mathbf{f}}^{(1)}, \end{aligned} \quad (29)$$

where $\nabla_{\perp} = (\partial/\partial x)(\cdot) \cdot \mathbf{n}_x + (\partial/\partial y)(\cdot) \cdot \mathbf{n}_y$, $\mathbf{n}_z$ is the unit vector in the wave propagation direction. Integrating the formula over the cross section as shown in Fig. 1(b), the formula becomes

$$\begin{aligned} & \int_{\Omega} \nabla_{\perp} \cdot (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) d\Omega \\ & + \int_{\Omega} \frac{\partial}{\partial z} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_z d\Omega \\ & = - \int_{\Omega} \mathbf{v}_n e^{ik_n z} \cdot \bar{\mathbf{f}}^{(1)} d\Omega, \end{aligned} \quad (30)$$

where $\Omega = \Omega_A + \Omega_B$.

The secondary field can be written as a linear combination of the waveguide modes

$$\mathbf{v}_2(x, y, z) = \sum_{m=1}^M A_m(z) \mathbf{v}_m(x, y), \quad (31)$$

$$\mathbf{s}_2(x, y, z) = \sum_{m=1}^M A_m(z) \mathbf{s}_m(x, y), \quad (32)$$

where M is the number of modes. It should be noted that the mode subscripts m and n can be positive and negative values to represent the positive-going and negative-going waves in the z direction. Here, only positive-going modes are considered for the secondary field. In addition, it has been demonstrated that the PML modes mainly oscillate in the PML region and only contribute to long-term diffraction, geometrical decay of the field,²³ which is of little interest in the context of NDT, so that the sums can only run over all trapped and leaky modes.

Substituting Eqs. (31) and (32) into Eq. (30) and using the Gauss divergence theorem to calculate the first term on the left-hand side of Eq. (30) gives

$$\begin{aligned} & \int_{\Omega} \nabla_{\perp} \cdot (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) d\Omega \\ & = \int_{\Gamma_{A/B}} \mathbf{v}_n e^{ik_n z} \cdot (-\bar{\mathbf{s}}^{(1)}) \cdot \mathbf{n}_A d\Gamma. \end{aligned} \quad (33)$$

Additional steps for the derivation of Eq. (33) are given in Appendix A.

For the second term on the left-hand side of Eq. (30), it can be casted into

$$\begin{aligned} & \int_{\Omega} \frac{\partial}{\partial z} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_z d\Omega \\ & = -4Q_{n,m} \frac{\partial}{\partial z} \left(e^{ik_n z} \sum_{m=1}^M A_m(z) \right), \end{aligned} \quad (34)$$

where $Q_{n,m} = -(1/4) \int_{\Omega} (\mathbf{v}_n \cdot \mathbf{s}_m - \mathbf{v}_m \cdot \mathbf{s}_n) \cdot \mathbf{n}_z d\Omega$, is the general orthogonality relation in open waveguides,²³ which is similar to Auld’s real orthogonality relation.³⁴

As the external volume force $\bar{\mathbf{f}}^{(1)}$ only exists in Ω_A , the term on the right-hand side of Eq. (30) can be expressed as

$$- \int_{\Omega} \mathbf{v}_n e^{ik_n z} \cdot \bar{\mathbf{f}}^{(1)} d\Omega = - \int_{\Omega_A} \mathbf{v}_n e^{ik_n z} \cdot \bar{\mathbf{f}}^{(1)} d\Omega. \quad (35)$$

Hence, Eq. (30) can be rewritten as

$$\begin{aligned} & 4Q_{n,m} \left(\frac{d}{dz} + ik_n \right) \sum_{m=1}^M A_m(z) \\ & = \int_{\Omega_A} \mathbf{v}_n \cdot \bar{\mathbf{f}}^{(1)} d\Omega + \int_{\Gamma_{A/B}} \mathbf{v}_n \cdot (-\bar{\mathbf{s}}^{(1)}) \cdot \mathbf{n}_A d\Gamma. \end{aligned} \quad (36)$$

The general orthogonality relation is given by²³

$$Q_{n,m} = 0, \quad \text{if } n \neq -m. \quad (37)$$

That means the positive-going m th mode is orthogonal to all modes except the negative-going m th mode in the general orthogonality relation. Therefore, Eq. (36) can be transformed into an ordinary differential equation whose solution (A_m) is the amplitude of the secondary mode

$$4Q_{-m,m} \left(\frac{d}{dz} + ik_{-m} \right) A_m(z) = e^{i2k_l z} (\bar{f}_{\text{vol}}^1 + \bar{f}_{\text{surf}}^1), \quad (38)$$

where $\bar{f}_{\text{vol}}^1 = \int_{\Omega_A} \mathbf{v}_{-m} \cdot \hat{\mathbf{f}}^{(1)} d\Omega$ and $\bar{f}_{\text{surf}}^1 = \int_{\Gamma_{A/B}} \mathbf{v}_{-m} \cdot (-\hat{\mathbf{s}}^{(1)}) \cdot \mathbf{n}_A d\Omega$, having $\bar{\mathbf{f}}^{(1)} = e^{i2k_l z} \hat{\mathbf{f}}^{(1)}$ and $\bar{\mathbf{s}}^{(1)} = e^{i2k_l z} \hat{\mathbf{s}}^{(1)}$; k_l represents the wavenumber of the primary wave, which is assumed to be the l th mode of the open waveguide. Solving the equation by placing a relationship between positive-going and negative-going modes, $k_{-m} = -k_m$, into Eq. (38), and following an initial condition, $A_m(0) = 0$, the amplitude of the secondary mode can be obtained,

$$A_m(z) = \frac{i(\bar{f}_{\text{vol}}^1 + \bar{f}_{\text{surf}}^1)}{4Q_{-m,m}(2k_l - k_m)} e^{ik_m z} (1 - e^{i(2k_l - k_m)z}). \quad (39)$$

It can be seen that, calculations of $\bar{f}_{\text{vol}}^1$, $\bar{f}_{\text{surf}}^1$, and $Q_{-m,m}$ require for the modal properties of the negative-going modes, which is not convenient in the numerical computations. For propagating modes, the fields of the modes must satisfy the following relations:³⁴

$$\mathbf{v}_{-m}(x, y) = -\mathbf{v}_m^*(x, y), \quad \mathbf{s}_{-m}(x, y) = \mathbf{s}_m^*(x, y), \quad (40)$$

where $\mathbf{v}_m^*(x, y)$ and $\mathbf{s}_m^*(x, y)$ represent the complex conjugations. Substituting these relationship into definitions of $\bar{f}_{\text{vol}}^1$, $\bar{f}_{\text{surf}}^1$, and $Q_{-m,m}$ leads to the following equations:

$$\bar{f}_{\text{vol}}^1 = - \int_{\Omega_A} \mathbf{v}_m^* \cdot \hat{\mathbf{f}}^{(1)} d\Omega = -f_{\text{vol}}^1, \quad (41)$$

$$\bar{f}_{\text{surf}}^1 = - \int_{\Gamma_{A/B}} \mathbf{v}_m^* \cdot (-\hat{\mathbf{s}}^{(1)}) \cdot \mathbf{n}_A d\Gamma = -f_{\text{surf}}^1, \quad (42)$$

$$Q_{-m,m} = - \frac{1}{4} \int_{\Omega} \{ -\mathbf{v}_m^* \cdot \mathbf{s}_m - \mathbf{v}_m \cdot \mathbf{s}_m^* \} \cdot \mathbf{n}_z d\Omega = -P_{mm}, \quad (43)$$

where f_{vol}^1 , f_{surf}^1 , and P_{mm} represent the external power flux due to the volume source and surface source and the power flux of the m th mode in the wave propagation direction, respectively, which have the same expression as those of closed waveguides in the previous studies.^{4,8} It is worth noting that $Q_{-m,m}$ is calculated on the whole cross section, including the PML region. Therefore, the amplitude of the second harmonic wave can be rewritten as

$$A_m(z) = \frac{i(f_{\text{vol}}^1 + f_{\text{surf}}^1)}{4P_{mm}(2k_l - k_m)} e^{ik_m z} (1 - e^{i(2k_l - k_m)z}). \quad (44)$$

Although this formula appears similar in form to that from closed waveguides in the previous studies,^{4,8} the wavenumbers of the primary and secondary modes can be complex when taking energy leakage into account.

As a side remark, the same results can be obtained for immersed waveguides, which are immersed in a fluid only by replacing the governing equations of the surrounding medium with acoustic wave equations and following the same mathematical procedures presented above.

It is known that, a linearly cumulative second harmonic wave in closed waveguides can be selected as the dominant contribution to the secondary wave field only if it satisfies the internal resonance conditions: (1) phase velocity matching and (2) non-zero power flux. Under these conditions, the other secondary modes are negligible due to minimal contribution to the expansion. Therefore the selection of the primary mode and dominant secondary mode (mode pair) is of significance to maximize the nonlinear response of the waveguide. This situation in open waveguides is complicated due to energy leakage, which usually makes the amplitudes of both the primary and secondary modes intricate and unpredictable. However, it is also possible to select the dominant term in the modal expansion by considering two general cases similar to closed waveguides. In the first case, if there is no energy transfer from primary wave to second harmonic wave, the amplitude of the secondary mode must be zero, leading to no contribution to the expansion. In the second case, if phase velocity mismatching occurs between the primary wave and second harmonic wave, the amplitude of the secondary mode is bound in value and oscillates with propagation distance, whose contribution is usually minimal being of little interest for application purposes. Consequently, two conditions are proposed in open waveguides, which seem

exactly the same as that in closed waveguides: (1) phase velocity matching, $\text{Re}(2k_l) = \text{Re}(k_m)$; (2) non-zero power flux, $f_{\text{vol}}^1 + f_{\text{surf}}^1 \neq 0$. As long as these two criteria are met, the second harmonic wave is dominant although its amplitude with propagation distance may not be linearly cumulative and has to be calculated numerically.

III. NUMERICAL METHODS

A. Nonlinear SAFE method

The SAFE method has been widely used to study the guided waves of waveguides with arbitrary cross-sectional geometries such as railways,³⁵ square rods,³⁶ grooved plates,³⁷ and composite bends.³⁸ In the method, a FE representation of the cross section of the waveguide is used together with a harmonic wave propagation being assumed along the propagation direction, which transforms the three-dimensional problem to a two-dimensional problem, written as

$$\mathbf{u}(x, y, z, t) = \mathbf{U}(x, y) e^{i(kz - \omega t)}, \quad (45)$$

where z is the direction of the wave propagation and $\mathbf{U}(x, y)$ is the displacement in the cross section; k is the wavenumber; $\omega = 2\pi f$ is the circular frequency, and f is the frequency.

The SAFE method has been extended to embedded³⁰ and immersed waveguides³¹ by coupling the PML (SAFE-PML), in which a finite PML is introduced to absorb the leaky waves simulating an infinite surrounding material via mapping real coordinates into complex coordinates, which is defined as

$$\begin{cases} \tilde{x}(x) = \int_0^x \gamma_1(\xi) d\xi \\ \tilde{y}(y) = \int_0^y \gamma_2(\xi) d\xi \\ \tilde{z}(z) = z, \end{cases} \quad (46)$$

where $\gamma_1(x)$ and $\gamma_2(y)$ denote the non-zero, continuous, complex-valued coordinate stretching functions, also named PML functions, which satisfy

$$\gamma_1(x) = 1 \text{ outside the PML,} \quad (47)$$

$$\text{Im}\{\gamma_1(x)\} > 0 \text{ inside the PML,}$$

$$\gamma_2(y) = 1 \text{ outside the PML,} \quad (48)$$

$$\text{Im}\{\gamma_2(y)\} > 0 \text{ inside the PML.}$$

In this paper, the following parabolic PML functions are adopted:³⁰

$$\gamma_1(x) = \begin{cases} 1 & \text{outside the PML} \\ 1 + \hat{\gamma}_1 \left(\frac{|x| - d_x}{h_x} \right)^2 & \text{inside the PML,} \end{cases} \quad (49)$$

$$\gamma_2(y) = \begin{cases} 1 & \text{outside the PML} \\ 1 + \hat{\gamma}_2 \left(\frac{|y| - d_y}{h_y} \right)^2 & \text{inside the PML,} \end{cases} \quad (50)$$

where $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are complex with the positive imaginary part, which quantify the PML absorption and will be given explicitly in each case study.

In addition, a user-defined parameter is used as a filtering condition in post-processing to identify and remove the PML modes,

$$\frac{|\bar{P}_{\Omega_A}|}{|\bar{P}_{\Omega}|} > \eta, \quad (51)$$

where $\bar{P}$ represents the average power flux along the wave propagation direction, defined as

$$\bar{P} = \frac{\int_S |P| ds}{\int_S ds}, \quad (52)$$

in which S denote the cross-sectional area of interest.

According to the mathematical derivation, two conditions have to be satisfied before the calculation of the amplitude of the second harmonic wave. Accordingly, the computational procedure can be divided into two stages corresponding to the two conditions.

In the first stage, the dispersion curves, including phase velocity and attenuation dispersion curves of a given open waveguide, can be calculated using the SAFE-PML method. Then, the phase velocity matching condition is utilized to identify the primary wave and second harmonic wave by plotting the phase velocity dispersion curves of the primary wave, which then overlap the dispersion curves of the second harmonic wave at its half frequency, where all the intersection points between the two dispersion curves are selected as the potential mode pair, which satisfy the phase velocity matching condition. In addition, modal properties of the primary and secondary mode, like wavenumber, attenuations, and mode shapes, can also be extracted from the SAFE-PML method, which will be used in next stage.

The second stage is to calculate the amplitude of the second harmonic wave with the wave propagation distance. In this stage, according to Eqs. (41)–(43), the power flux transferred from the primary wave and that in the wave propagation direction can be obtained via numerical integration, during which mode shapes obtained in the first stage are used to compute the velocity vector and the first P-K stress tensor. If the power flux transferred from the primary wave is different from zero, the amplitude of the second harmonic wave is not zero and can be calculated numerically by substituting the power flux and wavenumbers of the primary and second harmonic waves into Eq. (44).

B. Time domain FE model

The nonlinear SAFE method can provide modal properties of nonlinear guided waves in open waveguides, including the excitation frequency and the amplitude of the second harmonic wave with propagation distance. Based on the prediction from the nonlinear SAFE method, time domain FE model is developed to simulate the propagation of the

nonlinear guided wave, and the second harmonic wave is collected to cross validate the prediction of the nonlinear SAFE method.

Embedded waveguide models consist of a nonlinear elastic core structure and an elastic surrounding material, which is usually set to be large enough to avoid boundary reflections. In the core structure, the Murnaghan model is used to describe the material behavior, which is equivalent to the Landau-Lifshitz nonlinear hyperelastic constitutive model in the nonlinear SAFE method, written as

$$w = \frac{1}{2} \lambda (\text{tr}(\mathbf{E}))^2 + \mu \text{tr}(\mathbf{E}^2) + \frac{1}{3} (l + 2m) (\text{tr}(\mathbf{E}))^3 - m \text{tr}(\mathbf{E}) \left((\text{tr}(\mathbf{E}))^2 - \text{tr}(\mathbf{E}^2) \right) + n \det(\mathbf{E}), \quad (53)$$

where l , m , and n are Murnaghan constants; $\text{tr}()$ and $\det()$ denote the trace and determinant of the tensor, respectively. The relations between the third-order elastic constants in Eq. (3) and l , m , and n are given by

$$l = B + C, \quad m = \frac{1}{2} A + B, \quad n = A. \quad (54)$$

In the surrounding material, a linear elastic material model is used together with a continuum boundary condition being applied between the core structure and the surrounding material.

Immersed waveguide models are termed fluid–solid interaction systems due to the coupling between the core structure and the surrounding fluid. The Murnaghan model is also used to simulate the solid core structure, while an acoustic model is applied to the surrounding fluid. In order to simulate the coupling, acoustic-structure boundary conditions have to be implemented between the core structure and the surrounding fluid.

In this paper, both the nonlinear SAFE method and the time domain FE model were implemented in a commercial FE software package.³⁹ In Sec. IV, two case studies were carried out using the simulation tools discussed above: an aluminum plate attached to an elastomer and an aluminum plate with water loaded on one side.

IV. CASE STUDIES

A. An aluminum plate attached to a semi-infinite half-space of an elastomer material

1. Nonlinear SAFE method computations

The first example consists of an aluminum plate with thickness of 4 mm attached to a semi-infinite half-space of an elastomer material. The schematic of the SAFE-PML model is shown in Fig. 2(a). A narrow strip of the structure with width of 1 mm was imposed with periodic boundary conditions to represent continuity of displacements and stresses between the two edges. The thickness of the plate was 4 mm and the elastomer was set to be 1 mm considering the fact that PMLs work better when close to the core structures. A 1 mm thick layer of the PML was attached to the elastomer, which was thick enough to absorb all of the leaky

FIG. 2. Nonlinear SAFE method computations with (a) the schematic of the SAFE-PML model for the aluminum plate attached to a semi-infinite half-space of an elastomer material, (b) phase velocity and (c) attenuation dispersion curves for the embedded plate, and (d) variation of the modal amplitude of the secondary mode with propagation distance.

waves. In the simulation, the PML function was chosen as $\hat{\gamma}_2 = 3 + 12i$ and it was solved from 50 kHz to 1000 kHz with $\eta = 1.3$ being used to identify and filter the PML modes during post-processing. Continuity of displacements and stresses was imposed between the plate and the elastomer. In addition, stress-free conditions were applied at the outer surface of the structure. The whole geometry was meshed by 1186 triangular elements with side length of 0.11 mm, which were automatically generated by the software and the number of degrees of freedom was 14970. All of the material properties used in the simulation are listed in Table I.

Figure 2(b) presents the phase velocity dispersion curves for the aluminum plate attached to an elastomer in 50–500 kHz, including the primary modes in solid lines and the second harmonic modes (at half frequency) in dashed lines. A few mode pairs can be found at the intersections of the solid lines and the dashed lines that satisfy the phase velocity matching condition, among which, only one mode pair marked with a triangle can satisfy the additional condition: non-zero power flux. The mode pair consists of a shear horizontal (SH) mode and a Lamb mode, (SH0,s0), at the excitation frequency of 427.55 kHz, where the uppercase letter and the lowercase letter represent the primary and secondary modes, respectively.

Figure 2(c) shows the attenuation dispersion curves for the embedded plate in 50–1000 kHz. The attenuations of the primary wave and second harmonic wave can be obtained and are indicated with two arrows in Fig. 2(c), showing that the attenuation of the SH0 mode is predicted to be 71 dB/m and that of the s0 mode is calculated as 279 dB/m. It is clear that both the primary wave and second harmonic wave are strongly attenuative.

The distance variation of the modal amplitude of the secondary mode for the mode pair (SH0,s0) is shown in Fig. 2(d). It can be seen that the amplitude of the s0 mode increases and then decreases rapidly to zero in a very short distance. This phenomenon can be understood from the viewpoint of energy transfer. In the beginning, even though both the primary and secondary modes are leaky, the energy transferred from the SH0 mode to the s0 mode is larger than the energy that leaks away from the s0 mode, thus, the amplitude of the s0 mode increases progressively. With the wave propagating, the leakage of the SH0 mode leads to the decay of its amplitude, which greatly reduces the energy transfer to the second harmonic wave, as it is proportional to the square of the amplitude. Once the energy transfer is less than the energy leakage of the s0 mode up to a certain

TABLE I. Mechanical properties of materials used in the models.

Material	ρ (kg/m ³)	C_L (m/s)	C_S (m/s)	A (GPa)	B (GPa)	C (GPa)	α_L and α_T (dB/ λ)
Aluminum	2780	6347.3	3116.4	-351.2	-149.4	-102.8	—
Elastomer	1100	953.5	522.2	—	—	—	2.73
Water	1000	1500	—	—	—	—	—

distance, the amplitude of the s0 mode starts to decrease and finally goes to zero as shown in Fig. 2(d).

2. Time domain finite-element method (FEM) simulations

Time domain FE simulations are carried out in this section for the mode pair (SH0,s0) at excitation frequency of 427.55 kHz in a 4 mm thick plate. The schematic of the embedded plate model used for the simulation is shown in Fig. 3. A one-element wide three-dimensional model with periodic boundary condition is used in FE simulations to simulate the propagation of the SH wave.⁴⁰ It is worth mentioning that the SH wave could also be simulated by an axisymmetric model with a very large radius and out-of-plane displacements.⁴¹ The length of the plate was assumed to be 300 mm and the thickness of the elastomer was set to be 46 mm, long enough to delay boundary reflections. Both the plate and the elastomer were set to be 0.1 mm (one element) in width with periodic conditions being applied to simulate infinity in the x direction and reduce computational cost. Continuity boundary condition was imposed between the plate and the elastomer to force the displacement and the stress to be continuous, and stress-free boundary conditions were applied to other boundaries. A ten-cycle Hann windowed tone burst with central frequency of 427.55 kHz was applied in a manner of prescribed displacement boundary condition in the x direction at the left end of the plate to generate the SH0 mode, and monitors were placed at the center of the plate to receive displacements from 10 mm to 250 mm along the z direction with a step of 10 mm. The displacements in the x and z directions were recorded for the analysis of the primary mode SH0 and the secondary mode s0, respectively, as they are dominant in these directions. In the simulation, tetrahedron elements were used with a maximum element size of 0.45 mm, and the maximum time step was set to be 0.05 μ s. All of the material parameters used in the FE simulation were taken from Table I.

Figure 4(a) plots time domain signals obtained in the x direction for the primary mode at $z = 50$ mm. It is apparent

FIG. 3. Schematic of the embedded plate in the FE simulation.

that a pure SH0 mode has been excited in the plate because of the mode-shape-matching excitation. In Fig. 4(b), displacements obtained in the z direction in which the secondary mode is dominant are displayed, which seems to be a bit complex. It is interesting to find out that the complex wave pattern consists of the s0 mode and the edge resonance, which is induced by the s0 mode propagation around the secondary frequency. Edge resonance is known as an unusual type of vibration that only exists around specific frequencies in which the motion is confined to a narrow region around the edge, and its sustained vibration results in a huge duration wave pattern.⁴² (A simulation case to demonstrate the generation of the edge resonance induced by exciting the s0 mode around the secondary frequency is not shown here for conciseness.) It can be understood in physics, with the primary wave (SH0) propagation, the non-linear response of the plate continuously transfers a small part of the energy from the primary to the secondary mode, forming the second harmonic wave s0, whose propagation further leads to the edge resonance. The propagation of the s0 mode together with the subsequent edge resonance results in the complex wave pattern. For the complex wave pattern, it is not easy to identify the s0 mode. However, considering the fact that all the energy of the s0 is transferred from the SH0 mode, fast Fourier transform (FFT) was therefore applied on the windowed signal, whose interval has the same width of the SH0 mode, even though a small part of the s0 mode may be outside of the window due to the group velocity mismatch and the small dispersion properties of the s0 mode.

Figures 4(c) and 4(d) show frequency spectra calculated by converting the time domain windowed signals in Figs. 4(a) and 4(b) into frequency domain via FFT, respectively. As indicated by Fig. 4(c), the distinct peak existing at the primary frequency represents the primary mode. In Fig. 4(d), the peak at the second harmonic frequency reveals that most of the energy concentrates on the secondary wave motion, meaning that the vibration in the z direction actually comes from the nonlinear response of the material nonlinearity and finite elasticity. The amplitude of the s0 mode can be collected from the frequency spectrum at the second harmonic frequency.

The normalized amplitude variation of the s0 mode (normalized by the maximum value over the propagation distance) with propagation distance from FEM simulation is plotted in comparison with that from the nonlinear SAFE method shown in Fig. 5. Very good agreement can be observed between the two results, which not only validates the predictions from the nonlinear SAFE method, but also shows a complicated behavior of the nonlinear guided wave in embedded plates.

FIG. 4. Time domain signals received at $z = 50$ mm in (a) the x direction and (b) the z direction; frequency spectra by using fast Fourier transform (FFT) to the signals received in (c) the x direction and (d) the z direction.

B. An aluminum plate with water loaded on one side

1. Nonlinear SAFE method computations

The second case includes a 4 mm thick aluminum plate with water loaded on one side. Figure 6(a) presents the schematic of the SAFE-PML model. Similarly with the previous case, a narrow strip of the structure with periodic boundary conditions was used to represent continuity of displacements and stresses between the two edges. The thickness of the plate was 4 mm and the water layer was set to be 1 mm. In order to absorb the leaky waves in the water layer, a 1.5 mm thick layer of the PML was attached to the water. $\hat{\gamma}_2 = 3 + 12i$ was chosen as the PML function together with $\eta = 1.5$ being used for post-processing in the simulation calculated from 20 kHz to 1000 kHz. Water loading interface conditions³¹ were imposed between the plate and the water, and stress free conditions were imposed on the other boundaries of the model. 814 triangular elements were automatically generated with the number of degrees of freedom being 7479. All of the material properties used were taken from Table I.

Phase velocity dispersion curves for the aluminum plate with water loaded on one side in solid curves overlapping the second harmonic modes at its half frequency in dashed curves are shown in Fig. 6(b) in the frequency range of

20–500 kHz. The mode pair (SH0,s0) at the excitation frequency of 425 kHz was also selected for studies in this case, as the primary and secondary modes in this mode pair can satisfy the conditions of phase velocity matching and the non-zero energy transfer, respectively.

FIG. 5. Comparison on the normalized modal amplitude of the secondary mode in the embedded plate between the FEM simulation and the nonlinear SAFE method computation.

FIG. 6. Nonlinear SAFE method computations with (a) the schematic of the SAFE-PML model for the aluminum plate with water loaded on one side, (b) phase velocity and (c) attenuation dispersion curves for the one-sidedly immersed plate, and (d) variation of the modal amplitude of the secondary mode with propagation distance.

Figure 6(c) shows the attenuation dispersion curves for the immersed plate in 20–1000 kHz. As water cannot support shear stress, the SH0 mode is non-leaky (in fact, all SH modes are non-leaky). The s0 mode is dominant in the y direction, which is strongly coupled with the water layer leading to energy leakage. Therefore, the s0 mode is a leaky wave with a strong attenuation of 413 dB/m at the second harmonic frequency. In this case, the primary mode is non-leaky, but the secondary mode is strongly leaky.

The variation of the modal amplitude of the s0 mode with propagation distance is calculated with the nonlinear SAFE method for the mode pair (SH0,s0) and shown in Fig. 6(d). An interesting phenomenon can be observed in which the amplitude of the s0 mode increases and then keeps constant with the propagation distance. Such a phenomenon can also be understood from the viewpoint of the energy transfer. As the primary wave is non-leaky, the energy transfer from the SH0 mode to the s0 mode can be considered as constant. For the s0 mode, the energy transfer (input energy) is prevalent and larger than the leakage (energy consuming) at the beginning propagation distance, so that the amplitude of the s0 mode increases cumulatively. At a certain distance, the energy transfer becomes nearly equal to the leakage, leading

to a balanced activity of the input energy and the energy consumed by the s0 mode. The amplitude of the s0 mode therefore becomes constant with the propagation distance.

Such a phenomenon can also be mathematically derived. As the primary mode is non-leaky, the wavenumber of the primary wave is assumed to be a positive number without an imaginary part as $k_l = a_1$. To present a leaky secondary mode, having the same phase velocity of the primary mode, the wavenumber of the second harmonic wave has to be assumed as a complex number: $k_m = 2a_1 + bi$, where the imaginary part is positive, demonstrating the attenuation of the second harmonic wave. Substituting the wavenumbers into Eq. (44), the amplitude of the secondary mode can be obtained as

$$A(z) = \frac{(f_{\text{vol}}^1 + f_{\text{surf}}^1)}{4P_{mm}b} (1 - e^{-bz}) e^{2a_1 z}. \quad (55)$$

When $z \rightarrow \infty$, $e^{-bz} \rightarrow 0$. The absolute value of the modal amplitude goes to

$$|A| = \frac{|f_{\text{vol}}^1 + f_{\text{surf}}^1|}{4P_{mm}b}, \quad (56)$$

which is a constant whose value is proportional to the power flux transferred from the primary mode and inversely

FIG. 7. Schematic of the immersed plate in the FE modeling.

proportional to the power flux and the attenuation of the secondary mode.

2. Time domain FEM simulations

The mode pair (SH0,s0) at 425 kHz in a 4 mm thick plate with water loaded on one side is selected for the FE simulation. The schematic of the three-dimensional model used for the simulation is shown in Fig. 7. The length of the plate was assumed to be 600 mm. The thickness of the water was set to be 96 mm, which is thick enough to avoid boundary reflections within the interested time period. Similarly to the previous case study, both the plate and the water were set

to be 0.2 mm in width with periodic conditions being applied to represent infinity in the x direction. In order to model the coupling between the plate and the water, an acoustic structure boundary condition was imposed between the plate and the water, with displacements and stresses in the normal direction of the solid–fluid interface being continuous. Apart from the interface, stress-free boundary conditions were applied to other boundaries. Prescribed displacement boundary condition as an input signal were applied at the left end of the plate to excite the SH0 mode, using a ten-cycle Hann windowed tone burst with central frequency of 425 kHz. Monitors were placed at the free surface of the plate to record the dominant displacement for the primary mode in the x direction and the secondary mode in the y direction from 10 mm to 450 mm along the z direction with a step of 10 mm. In the simulation, tetrahedron elements were used with a maximum element size of 0.3 mm, and the maximum time step was set to be $0.05 \mu\text{s}$. The material parameters in the FE simulation were taken from Table I.

Figures 8(a) and 8(b) present time domain signals obtained in the x and y directions at $z = 70 \text{ mm}$, respectively. The SH0 and s0 modes can be identified in the time domain, and it is interesting to find out that the s0 mode remains in the same location as the primary mode in the time trace even

FIG. 8. Time domain signals received at $z = 70 \text{ mm}$ in (a) the x direction and (b) the y direction, frequency spectra by using FFT to the signals received in (c) the x direction and (d) the y direction.

FIG. 9. Comparison on the normalized modal amplitude of the secondary mode in the immersed plate between the FEM simulation and the nonlinear SAFE method computation.

if it has a slightly slower group velocity (predicted to be 2380 m/s). This is mainly due to the strong attenuation. It is known that the s_0 mode is generated by the propagating SH_0 mode and has a number of wave packets overlapping with each other^{43,44} because of group velocity mismatch. As the s_0 is strongly attenuated, the first-generated wave packet will first leak into the water leaving the newly generated wave packet within the SH_0 packet. Therefore, the s_0 mode appears to move with the same speed as the SH_0 mode. In addition, it is also interesting to see some edge resonance behind the s_0 mode in the time domain signal, although its amplitude is negligible.

FFT was applied to the time domain signals in Figs. 8(a) and 8(b) to obtain the frequency spectra shown in Figs. 8(c)

and 8(d), respectively. The amplitudes of the SH_0 and s_0 modes can be determined and collected at the primary frequency and second harmonic frequency, respectively.

The normalized amplitude variation of the s_0 mode (normalized with maximum value) with propagation distance was extracted from FEM simulation and compared with the nonlinear SAFE computation, as shown in Fig. 9. It can be seen that the FEM simulation captures the feature of the constant amplitude of the second harmonic wave, agreeing very well with the prediction from the nonlinear SAFE computation.

It can be seen from the two case studies that the nonlinear ultrasonic guided wave properties in open waveguides are quite different from those in closed waveguides. It is interesting to find out that the secondary modal amplitude keeps constant during wave propagation in the one-sidedly immersed aluminum plate for the mode pair (SH_0, s_0) at a particular frequency. It should be noted that such a feature can be extended to any immersed waveguides if the primary wave is non-leaky, as demonstrated mathematically in Sec. IV B 1. This could bring potential applications for underwater inspections. In Sec. V, experiments are carried out to verify the finds from the simulations.

V. EXPERIMENTAL VALIDATIONS

A. Experimental setup

Experiments were designed to verify the constant amplitude of the second harmonic wave for the mode pair (SH_0, s_0) at $fd = 1700 \text{ kHz mm}$ in a plate with water loaded on one side. The schematic of the experimental setup is shown in Fig. 10. A 3 mm thick aluminum plate (Al 1100) was placed in a water tank with water loaded on the bottom surface. Magnetostrictive transducers (MSTs) were used for the generation of the SH_0 mode, which consists of a thin

FIG. 10. (Color online) Schematic of experimental setup.

FIG. 11. (a) In-plane displacement signals and (b) out-of-plane displacement signals received by lasers at $z = 210$ mm, frequency spectra of (c) the in-plane displacement signals and (d) the out-of-plane displacement signals.

iron-cobalt foil rigidly bounded to the aluminum plate with cyanoacrylate and a meandering electric coil placed upon the foil, and a permanent magnet with its generated magnetic field was aligned with the direction of the current in the coil. The spacing of the coil elements was 2.74 mm corresponding to the half wavelength of the primary wave.

A high voltage ten-cycle Hann windowed tone burst signal at the frequency of 566.7 kHz was generated by a high power gated amplifier (RITEC RAM-5000 SNAP, RITEC Inc., Warwick, RI, USA) and fed into a 600 kHz low pass filter to filtrate nonlinear signals from the electronics. The purified signal was then sent to the MST via an impedance-matching network. Ultrasonic waves were measured using two laser vibrometers (Polytec OFV 5000, Polytec South-East Asia Pte. Ltd., Singapore) to pick up both in-plane displacements (u , in the x direction) and out-of-plane displacements (v , in the y direction) along the wave propagation direction (z direction) from 60 mm to 450 mm with a step of 30 mm. An oscilloscope was used to store the time trace of the signal with 3000 averages to improve the signal-to-noise ratio, and the data were transferred to computers for post-processing using FFT to extract the amplitudes of the primary wave and second harmonic wave. Each set of measurements was repeated ten times.

B. Calibration and experimental results

It is known that the system nonlinearity always exists and varies with different transducer setups. However, the material nonlinearity needs to be dominant in the measurement for a successful experiment. Therefore, a calibration procedure is required before making measurements. During the calibration, the lasers were fixed at the propagation distance of 210 mm to receive the signals. The output level of the gated amplifier increased from 20% to 90% with a step of 10%. For each output level, the lasers recorded both the in-plane and out-of-plane displacements.

Figures 11(a) and 11(b) plot the in-plane displacements, measuring the SH0 mode, and out-of-plane displacements, demonstrating the s0 mode at output level of 90%. The SH0 mode was identified according to the group velocity (3100 m/s) in the time domain. Despite the SH0 mode, the strongly dispersive SH1 mode was also generated in the plate due to the similarity of their mode shapes. As demonstrated in Sec. IV B 2, the secondary mode (s0 mode) should arrive at the same time as the primary mode, even it has a slightly slower group velocity. Figures 11(c) and 11(d) demonstrate the frequency spectra for the Hann windowed SH0

FIG. 12. Normalized nonlinearity parameter as a function of amplifier output level.

and s0 signals in semi-log coordinates, respectively. The amplitude of the primary wave (A_1) was collected at the primary frequency, although there was a small component at the second harmonic frequency due to the nonlinearity in the system. The amplitude of the second harmonic wave (A_2) was picked up from Fig. 11(d) at the second harmonic frequency.

Figure 12 plots the normalized modal amplitude ratio (nonlinearity parameter), A_2/A_1^2 , as a function of the excitation output, which gives an indication of the measurement system nonlinearity. It can be seen that in the modal amplitude ratio there exist two stages with varying output level of the gated amplifier: stage 1, a rapid decrease of A_2/A_1^2 with increasing the output level; stage 2, A_2/A_1^2 remains almost constant with varying input and output. It should be noted that the amplitude of the second harmonic is expected to increase in proportion to the square of the primary mode. Therefore, the existence of stage 2 means that the instrumentation nonlinearity, although present to a certain extent, is small so that its amplification by higher input voltage does

not have significant influence on the measurement of the material nonlinearity.⁴⁵ Based on the calibration, a 90% output level in the measurements can ensure that the contribution of the material nonlinearity is dominant in the measurements.

In order to assess the variation of the s0 mode with the propagation distance, the amplitudes of the SH0 and s0 modes were collected and plotted as a function of the propagation distance. However, it is known that geometric divergence of the wave front in plates leads to a decay of the wave amplitude, introducing difficulties in estimating the variation of the s0 mode. A simplified compensation for the geometric divergence is therefore used to correct the wave amplitudes. As waves in the plate follow the two-dimensional divergence case, the amplitude of the primary wave decays approximately with a factor of $1/\sqrt{z}$,⁴⁶ where z is the propagation distance. Thus, the amplitude of the primary wave is compensated by $\sqrt{z}$, meaning that $\hat{A}_1 = A_1\sqrt{z}$, where $\hat{A}_1$ is the compensated amplitude and A_1 is the measurement from the experiments. According to the mathematical derivation, the amplitude of the second harmonic wave is proportional to the energy transfer from the primary wave. Therefore, the amplitude of the second harmonic wave is compensated by z , as the energy transfer is proportional to the square of the amplitude of the primary wave. It also should be noted that the decay of the secondary mode itself was neglected in this case. The compensated amplitudes of the SH0 and s0 modes, as well as the standard deviation in the measurements at each propagation distance, are shown in Fig. 13. It can be seen that, as the SH0 mode is non-leaky, there is almost no decay in the amplitude of this mode. For the secondary mode, even though the amplitude of this mode shows a relatively big variation and they cannot capture the sharp increase in the beginning propagation distances as predicted in the theory and the simulations, the current results can give an intuitive indication of the constant amplitude with propagation distance in the far field.

VI. DISCUSSIONS

It is known that, in closed waveguides, the relative nonlinearity parameter, defined as the amplitude of the

FIG. 13. The compensated amplitude of (a) the primary mode and (b) the secondary mode with propagation distance.

FIG. 14. The amplitude of the secondary wave changes with proportional variation of the third-order elastic constants.

secondary mode over the square of the amplitude of the primary mode, can increase linearly with respect to the propagation distance, and its slope is strongly related to the higher-order elastic constants of the material. Thus, the slope of the nonlinearity parameter can be used to characterize the material state of closed waveguides. Using essentially the same principle, the constant amplitude of the secondary mode in immersed waveguides can also provide useful information. According to Eq. (56), the constant value of the secondary modal amplitude is proportional to the power flux transferred from the primary mode, which is affected by the change of the higher-order elastic constants in the waveguide according to Eqs. (41) and (42). On the other hand, the power flux and the attenuation of the secondary mode are not affected by the higher-order elastic constants. Therefore, a relationship could be established between the amplitude of the second harmonic wave and the material states, which could be represented by the higher-order elastic constants. As shown in Fig. 14, the constant modal amplitude of the secondary mode varies proportionally with the third-order elastic constants. It should be noted that the amplitude of the secondary mode is usually very weak, especially in leaky cases, and proper methods have to be proposed to enhance the secondary mode to obtain a reasonable signal-to-noise ratio in the measurements.

VII. CONCLUSIONS

In this paper, the properties of nonlinear ultrasonic guided waves in open waveguides have been studied. A mathematical framework has been established based on the reciprocity relation and modal expansion by introducing PMLs truncating the infinite open waveguide into a finite one, in which the amplitude of the secondary mode has been derived and two conditions have been proposed to select the dominant mode in the secondary field. Numerical models have been developed, including the nonlinear SAFE method to predict the properties of nonlinear ultrasonic guided waves and FE models to simulate the wave propagation and cross validate the predictions from the nonlinear SAFE method. Two examples, an aluminum plate attached

to a semi-infinite half-space of an elastomer and an aluminum plate with water loaded on one side have been studied to demonstrate the proposed methods, as well as show modal properties of the secondary mode in open waveguides. Even though only embedded/immersed plates were studied in this paper, the methods developed are applicable to any open waveguides with arbitrary cross sections. It is interesting to find out that the amplitude of the secondary mode keeps constant during the wave propagation when the primary wave is non-leaky, e.g., SH wave in immersed plates. Experiments were carried out to validate the feature of the constant amplitude of the second harmonic wave in immersed plates, which may provide potential applications of the nonlinear ultrasonic guided waves for underwater inspections.

ACKNOWLEDGMENTS

This work was supported by MOE AcRF Tier 1, RG99/17. We would also like to acknowledge the Zheda Jingyi Electromechanical Technology Engineering Co., Ltd. for providing iron-cobalt foil for MSTs.

APPENDIX: DERIVATIONS OF EQ. (33)

For the first term on the left-hand side of Eq. (30), the Gauss divergence theorem is used to transform the surface integral into line integrals,

$$\begin{aligned} & \int_{\Omega} \nabla_{\perp} \cdot (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) d\Omega \\ &= \int_{\Gamma_{A/B}} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_A d\Gamma \\ &+ \int_{\Gamma_{A/B}} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_B^{\text{inn}} d\Gamma \\ &+ \int_{\Gamma_B} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_B^{\text{out}} d\Gamma. \end{aligned} \quad (\text{A1})$$

As the wave field vanishes on Γ_B [Eqs. (24) and (25)], the third term on the right-hand side of Eq. (A1) goes to zero, so that Eq. (A1) can be rewritten as

$$\begin{aligned} & \int_{\Omega} \nabla_{\perp} \cdot (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) d\Omega \\ &= \int_{\Gamma_{A/B}} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_A d\Gamma \\ &+ \int_{\Gamma_{A/B}} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_B^{\text{inn}} d\Gamma \\ &= \int_{\Gamma_{A/B}} \mathbf{v}_n e^{ik_n z} \cdot (\mathbf{s}_2|_A \cdot \mathbf{n}_A + \mathbf{s}_2|_B \cdot \mathbf{n}_B^{\text{inn}}) d\Gamma \\ &+ \int_{\Gamma_{A/B}} -\mathbf{v}_2 e^{ik_n z} \cdot (\mathbf{s}_n|_A \cdot \mathbf{n}_A + \mathbf{s}_n|_B \cdot \mathbf{n}_B^{\text{inn}}) d\Gamma. \end{aligned} \quad (\text{A2})$$

According to the interface boundary conditions on $\Gamma_{A/B}$, the following equations are satisfied:

$$\mathbf{s}_2|_A \cdot \mathbf{n}_A + \mathbf{s}_2|_B \cdot \mathbf{n}_B^{\text{inn}} = -\bar{\mathbf{s}}^{(1)} \cdot \mathbf{n}_A, \quad (\text{A3})$$

$$\mathbf{s}_n|_A \cdot \mathbf{n}_A + \mathbf{s}_n|_B \cdot \mathbf{n}_B^{\text{inn}} = 0. \quad (\text{A4})$$

Substituting Eqs. (A3) and (A4) into Eq. (A2), this equation can be simplified as

$$\begin{aligned} & \int_{\Omega} \nabla_{\perp} \cdot (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) d\Omega \\ &= \int_{\Gamma_{A/B}} \mathbf{v}_n e^{ik_n z} \cdot (-\bar{\mathbf{s}}^{(1)}) \cdot \mathbf{n}_A d\Gamma. \end{aligned} \quad (\text{A5})$$

- ¹M. Deng, "Cumulative second-harmonic generation accompanying nonlinear shear horizontal mode propagation in a solid plate," *J. Appl. Phys.* **84**(7), 3500–3505 (1998).
- ²M. Deng, "Cumulative second-harmonic generation of Lamb-mode propagation in a solid plate," *J. Appl. Phys.* **85**(6), 3051–3058 (1999).
- ³M. Deng, "Analysis of second-harmonic generation of Lamb modes using a modal analysis approach," *J. Appl. Phys.* **94**(6), 4152–4159 (2003).
- ⁴W. J. N. de Lima and M. F. Hamilton, "Finite-amplitude waves in isotropic elastic plates," *J. Sound Vib.* **265**(4), 819–839 (2003).
- ⁵M. F. Müller, J. Y. Kim, J. Qu, and L. J. Jacobs, "Characteristics of second harmonic generation of Lamb waves in nonlinear elastic plates," *J. Acoust. Soc. Am.* **127**(4), 2141–2152 (2010).
- ⁶A. Srivastava and F. L. di Scalea, "On the existence of antisymmetric or symmetric Lamb waves at nonlinear higher harmonics," *J. Sound Vib.* **323**(3), 932–943 (2009).
- ⁷Y. Liu, V. K. Chhillara, and C. J. Lissenden, "On selection of primary modes for generation of strong internally resonant second harmonics in plate," *J. Sound Vib.* **332**(19), 4517–4528 (2013).
- ⁸W. J. N. de Lima and M. F. Hamilton, "Finite amplitude waves in isotropic elastic waveguides with arbitrary constant cross-sectional area," *Wave Motion* **41**(1), 1–11 (2005).
- ⁹A. Srivastava and F. L. di Scalea, "On the existence of longitudinal or flexural waves in rods at nonlinear higher harmonics," *J. Sound Vib.* **329**(9), 1499–1506 (2010).
- ¹⁰V. K. Chhillara and C. J. Lissenden, "Analysis of second harmonic guided waves in pipes using a large-radius asymptotic approximation for axis-symmetric longitudinal modes," *Ultrasonics* **53**(4), 862–869 (2013).
- ¹¹C. Nucera and F. Lanza di Scalea, "Nonlinear semianalytical finite-element algorithm for the analysis of internal resonance conditions in complex waveguides," *J. Eng. Mech.* **140**(3), 502–522 (2014).
- ¹²C. Nucera and F. Lanza di Scalea, "Nondestructive measurement of neutral temperature in continuous welded rails by nonlinear ultrasonic guided waves," *J. Acoust. Soc. Am.* **136**(5), 2561–2574 (2014).
- ¹³A. Pau, "Derivation of wave mode orthogonality from reciprocity in direct notation," *ASME J. Nondestr. Eval.* **1**(2), 024501 (2018).
- ¹⁴D. Marcuse, *Theory of Dielectric Optical Waveguides* (Academic, New York, 1974).
- ¹⁵R. E. Collin, *Field Theory of Guided Waves* (Wiley–IEEE, New York, 1991).
- ¹⁶K. L. Nguyen, F. Treyssède, and C. Hazard, "Numerical modeling of three-dimensional open elastic waveguides combining semi-analytical finite element and perfectly matched layer methods," *J. Sound Vib.* **344**, 158–178 (2015).
- ¹⁷V. Shevchenko, "The expansion of the fields of open waveguides in proper and improper modes," *Radiophys. Quant. Electron.* **14**(8), 972–977 (1971).
- ¹⁸M. Gallezot, F. Treyssède, and L. Laguerre, "Contribution of leaky modes in the modal analysis of unbounded problems with perfectly matched layers," *J. Acoust. Soc. Am.* **141**(1), EL16–EL21 (2017).
- ¹⁹R. Sammut and A. W. Snyder, "Leaky modes on a dielectric waveguide: Orthogonality and excitation," *Appl. Optics* **15**(4), 1040–1044 (1976).
- ²⁰A. W. Snyder, "Leaky-ray theory of optical waveguides of circular cross section," *Appl. Phys.* **4**(4), 273–298 (1974).
- ²¹S. L. Lee, Y. Chung, L. A. Coldren, and N. Dagli, "On leaky mode approximations for modal expansion in multilayer open waveguides," *IEEE J. Quantum Elect.* **31**(10), 1790–1802 (1995).
- ²²T. Vogt, M. Lowe, and P. Cawley, "The scattering of guided waves in partly embedded cylindrical structures," *J. Acoust. Soc. Am.* **113**(3), 1258–1272 (2003).
- ²³M. Gallezot, F. Treyssède, and L. Laguerre, "A modal approach based on perfectly matched layers for the forced response of elastic open waveguides," *J. Comput. Phys.* **356**, 391–409 (2018).
- ²⁴A. H. Nayfeh and P. B. Nagy, "General study of axisymmetric waves in layered anisotropic fibers and their composites," *J. Acoust. Soc. Am.* **99**(2), 931–941 (1996).
- ²⁵B. N. Pavlakovic, "Leaky guided ultrasonic waves in NDT," Ph.D. thesis, Imperial College, London, 1998.
- ²⁶R. N. Thurston, "Elastic waves in rods and clad rods," *J. Acoust. Soc. Am.* **64**(1), 1–37 (1978).
- ²⁷M. Castaings and M. Lowe, "Finite element model for waves guided along solid systems of arbitrary section coupled to infinite solid media," *J. Acoust. Soc. Am.* **123**(2), 696–708 (2008).
- ²⁸Z. Fan, M. J. S. Lowe, M. Castaings, and C. Bacon, "Torsional waves propagation along a waveguide of arbitrary cross section immersed in a perfect fluid," *J. Acoust. Soc. Am.* **124**(4), 2002–2010 (2008).
- ²⁹F. Treyssède, K. L. Nguyen, A. S. Bonnet-BenDhia, and C. Hazard, "Finite element computation of trapped and leaky elastic waves in open stratified waveguides," *Wave Motion* **51**(7), 1093–1107 (2014).
- ³⁰P. Zuo, X. Yu, and Z. Fan, "Numerical modeling of embedded solid waveguides using SAFE-PML approach using a commercially available finite element package," *NDT&E Int.* **90**, 11–23 (2017).
- ³¹P. Zuo and Z. Fan, "SAFE-PML approach for modal study of waveguides with arbitrary cross sections immersed in inviscid fluid," *J. Sound Vib.* **406**, 181–196 (2017).
- ³²M. Mazzotti, I. Bartoli, A. Marzani, and E. Viola, "A coupled SAFE-2.5D BEM approach for the dispersion analysis of damped leaky guided waves in embedded waveguides of arbitrary cross-section," *Ultrasonics* **53**(7), 1227–1241 (2013).
- ³³J. P. Béranger, "A perfectly matched layer for the absorption of electromagnetic waves," *J. Comput. Phys.* **114**(2), 185–200 (1994).
- ³⁴B. A. Auld, *Acoustic Fields and Waves in Solids*, Vol. 2 (Krieger, Malabar, FL, 1990).
- ³⁵T. Hayashi, W. J. Song, and J. L. Rose, "Guided wave dispersion curves for a bar with an arbitrary cross-section, a rod and rail example," *Ultrasonics* **41**(3), 175–183 (2003).
- ³⁶T. Hayashi, C. Tamayama, and M. Murase, "Wave structure analysis of guided waves in a bar with an arbitrary cross-section," *Ultrasonics* **44**(1), 17–24 (2006).
- ³⁷M. V. Predoi, M. Castaings, B. Hosten, and C. Bacon, "Wave propagation along transversely periodic structures," *J. Acoust. Soc. Am.* **121**(4), 1935–1944 (2007).
- ³⁸X. Yu, M. Ratsasapp, P. Rajagopal, and Z. Fan, "Anisotropic effects on ultrasonic guided waves propagation in composite bends," *Ultrasonics* **72**, 95–105 (2016).
- ³⁹COMSOL, *User's Guide and Introduction*, COMSOL Multiphysics, available at <http://www.comsol.com/> (Last viewed 20 October 2017).
- ⁴⁰P. Belanger, "High order shear horizontal modes for minimum remnant thickness," *Ultrasonics* **54**(4), 1078–1087 (2014).
- ⁴¹A. Pau and D. V. Achilopoulos, "Interaction of shear and Rayleigh-Lamb waves with notches and voids in plate waveguides," *Materials* **10**(7), 841 (2017).
- ⁴²B. A. Auld and E. M. Tsao, "A variational analysis of edge resonance in a semi-infinite plate," *IEEE Trans. Sonics Ultrason.* **24**, 317–326 (1977).
- ⁴³M. Deng, Y. Xiang, and L. Liu, "Time-domain analysis and experimental examination of cumulative second-harmonic generation by primary Lamb wave propagation," *J. Appl. Phys.* **109**(11), 113525 (2011).
- ⁴⁴P. Zuo, Y. Zhou, and Z. Fan, "Numerical studies of nonlinear ultrasonic guided waves in uniform waveguides with arbitrary cross sections," *AIP Adv.* **6**(7), 075207 (2016).
- ⁴⁵C. Bermes, J. Y. Kim, J. Qu, and L. J. Jacobs, "Nonlinear Lamb waves for the detection of material nonlinearity," *Mech. Syst. Signal Process.* **22**(3), 638–646 (2008).
- ⁴⁶J. B. Keller, "Geometrical theory of diffraction," *J. Opt. Soc. Am.* **52**(2), 116–130 (1962).