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Optimality Properties of Speed Optimization for a Vessel Operating with Time Window Constraint

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Abstract

We consider speed optimization for a vessel that has to arrive at every port along its voyage within a time window at each port. The objective of the problem is to minimize the vessel's bunker fuel cost given that fuel consumption rate is a convex function of speed. The intent of this paper is to establish the optimality properties for this problem and show that a solution with such properties (which we refer to as a *good solution*) is unique and optimal. The optimality properties established in this paper facilitate the proof of exactness for existing and future algorithms, as one needs only to show that the solution provided by an algorithm satisfies the definition of a good solution. As an illustration, we show how we can apply our results to prove the exactness of an existing algorithm in literature. Our work contributes to the understanding of the problem's optimality structure, which will provide intuition for development of algorithms for this problem.

Keywords: Bunker fuel; speed optimization; optimality properties.

Introduction

International shipping transports over 90% of the world trade. Over the last ten years, prices of bunker fuel have had an almost threefold increase. The high prices is a bane to shipping companies since bunker fuel accounts for the single most important cost in a vessel's operating cost, often over 50% at today's prices. Moreover, the sluggish trade conditions after the 2008 financial crises have made the situation worse with low cargo volumes and depressed freight rates. It is estimated that bunker fuel consumption per unit time for most vessels is proportional to the third power of the sailing speed (see, e.g., Christiansen et al. 2007). The underlying sensitivity of fuel consumption to speed has made speed optimization critical to bunker fuel cost. In this paper, we consider the problem of speed optimization for a vessel with the objective of minimizing bunker fuel cost in its voyage. In particular, the optimization problem is subject to the constraint of time windows. Regardless of the type of carriers (e.g., tankers, dry bulk carriers or container ships), vessels always follow a time window assigned to them, which defines the arrival and departure time for the vessel at a particular port. The use of time window is to facilitate the berth scheduling process so as to ease port congestions.

We first review related literature on speed optimization in minimizing bunker fuel cost. Among the earlier works, Ronen (1982) develop a set of optimization models for the different voyages leg, namely income generating leg, positioning (empty) leg and mixed leg. With the recent surge in bunker fuel price, research on optimizing vessel's speed has become much more active. Ronen (2011) develop a cost model to analyze the relationship between bunker fuel price and sailing speed, and proposes a procedure to determine the optimal sailing speed that minimizes the annual operating cost of a container service route. Some of the recent works consider the problem in different operational settings. For example, Meng and Wang (2011) determine the optimal sailing speed and fleet deployment plan for a long-haul liner service route; Wang and Meng (2012) find the optimal sailing speed on individual legs of a liner service route with considerations of transshipment and container routing. As carbon emission depends on the amount of fuel consumed, Psaraftis and Kontovas (2013) survey the models for

sea transport with an emphasis on the role of speed in influencing emission. Interested readers may refer to Wang et al. (2013) for a review of related studies and optimization methods for minimizing bunker fuel consumption. For a recent review of more general ship routing and scheduling problems, refer to Christiansen et al. (2013).

Within this stream of work, our paper is most related to the works on speed optimization that consider the time window constraint. Fagerholt et al. (2010) propose an algorithm to solve the problem, which is non-linear because the fuel consumption rate is a cubic function of speed. The algorithm discretizes the vessel arrival times and the problem is then solved as a shortest path problem on a directed acyclic graph. Norstad et al. (2011) studies the routing and scheduling problem for tramp shipping in which the vessel speed optimization on each sailing leg is a sub-problem. Two algorithms are proposed for the sub-problem. One is by discretizing vessel's arrival time (that is based on Fagerholt et al. 2010); the other is by Recursive Smoothing Algorithm (RSA) in Norstad et al. (2011), which is based on the idea that speed should be constant and as low as possible since fuel consumption function is convex over the range of feasible speeds. Our work makes use of the same concept of smoothing speeds over the voyage and thus it can be regarded as a generalization of the concepts in RSA. Hvattum et al. (2013) present the analytical proof that the RSA is exact.

Our work differs from the aforementioned papers in that our intent is not to develop an efficient algorithm for the problem. Rather, we establish the optimality properties for this problem and show that a solution with such properties (which we defined as a good solution) is unique and optimal. Our work contributes to this research stream as it improves the understanding of the problem's optimality structure. The optimality properties established in this paper facilitate the proof of exactness for existing and future algorithms, as an algorithm can be proven to be exact by just showing its solution satisfies the definition of the good solution. Besides, the results obtained in this paper will provide intuition for future development of efficient algorithms, since one can simply focus on finding a good solution to ensure optimality.

In this paper, we also consider the case in which the bunker fuel consumption rate (as

a function of speed) is convex. Thus our results are not restricted to a strictly convex bunker consumption function. Even though numerous studies, e.g., Wang and Meng (2012), have found that the consumption rate is approximately a cubic function of speed, these studies are restricted to the normal operating speeds. Recently, some container shipping companies have embarked on "ultra slow steaming", in which vessels sail at untypical low speeds to achieve drastic cuts in bunker fuel consumption. For example, more than 100 of Maersk Line's vessels have utilized slow steaming since 2007, and bunker consumptions on major routes were cut by as much as 30% over two years; Maersk are expected to further slow down their vessels to 12-16 knots (Husdal 2010). Despite this trend, to our knowledge, there is no research carried out yet to establish the fuel consumption profile at such low speeds, although the common belief is that consumption rate is much less sensitive to speed at this speed range. By assuming a convex function (rather than only strictly convex), our results are developed based on a reasonable bound for the fuel consumption profile.

The rest of the paper is organized as follows. In the next section, we describe the problem and present the notations and assumptions. Then we establish the properties of the good solution and prove that the good solution is optimal if there exists a feasible solution. Next, we illustrate the use of the optimality properties in proving that the RSA in Norstad et al. (2011) is exact. Specifically, we show that the RSA finds the good solution, i.e., the good solution exists. Finally, we provide some concluding remarks and future research opportunities.

Problem Description

The notations used are listed as follows.

P_i	Port i along the voyage.
v_{max}	Maximum sailing speed of vessel.
v_i	Average sailing speed from port P_i to P_{i+1} .
t_i	Arrival time at P_i .

$[t_i^l, t_i^u]$	Time window for arrival time t_i .
T_i	Sailing time from P_i to P_{i+1} .
$D_{[t_1, t_2]}$	Total sailing distance from time t_1 to t_2 .
D_{i_1, i_2}	Total sailing distance from port P_{i_1} to P_{i_2} .
D_i	Distance between P_i and P_{i+1} .
S_n	Schedule of arrival times at all $n+1$ ports along the voyage, i.e., $[t_0 < t_1 < \dots < t_n]$.
C_{S_n}	Minimum bunker cost for schedule S_n . $C = C(T_0, T_1, T_2, \dots, T_{n-1})$.
b_i^n	b_i at n -th step after the OptimizeSpeed(s, e) is called in RSA.
$t_i^{S_n}$	t_i of output plan S_n .
$v_{il}^{S_n}(v_{ir}^{S_n})$	Average speed between P_{i-1} and P_i (P_i and P_{i+1}) in S_n

Note that in the proof for Theorem 4 (for RSA), S_n denotes the output plan at the n -th state after the function OptimizeSpeed(s, e) is called.

Problem: Consider a vessel (with full speed v_{max}) starts sailing from port P_0 through ports $P_1, P_2, \dots$ and finally complete the voyage at port P_n . Assume t_i is in a closed time interval $[t_i^l, t_i^u]$ (where $t_i^l \leq t_i^u, t_0^l = t_0^u$; for $i < j$, $t_i^l < t_j^u$), which is the time window within which the vessel has to arrive at P_i . $C = C(T_0, T_1, T_2, \dots, T_{n-1})$ represents the cost function of bunker fuel. The objective is to determine the sailing schedule that minimizes the bunker fuel cost for the ship's propulsion.

We consider the vessel speed $v(t)$ as a continuous function of time t . Suppose that the bunker cost incurred per unit time $\frac{dC}{dt}$ is related to vessel speed v by $\frac{dC}{dt} = f(v(t))$, where $f(v)$ is a convex and continuous function on $[0, v_{max}]$. We assume that no bunker is consumed at $v=0$, i.e., $f(0) = 0$. Inherently, we consider only the bunker consumed for the ship's propulsion; we do not take into account the (much smaller rate of) bunker consumed when the ship is stationary with its engine idling. Furthermore, we assume that the duration of stopping at each port is fixed. As such, without loss of generality, we

can simply ignore these time periods in the problem. Furthermore, since $T_i = t_{i+1} - t_i$ and $D_{[t_1, t_2]} = \int_{t_1}^{t_2} v dt$, the total bunker fuel cost for the voyage can be expressed as $\int_{t_0}^{t_n} f \circ v(t) dt = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f \circ v(t) dt$.

Based on previous research (see Christiansen et al. 2007), the bunker cost incurred per unit time is a convex function of speed. Since $f(v)$ is convex and $f(0) = 0$, by the definition of convexity, $f(v)/v$ is monotone increasing; if $f(v)$ is strictly convex on $[v_1, v_2]$, $f(v)/v$ is strictly monotone increasing. As stated in many other works, the relationship between sailing speed and bunker consumption at normal operational speed range can be expressed as a cubic function of speed. At lower speeds, we assume that the relationship is convex (i.e., it need not be strictly convex). We consider that the increment of bunker cost $\frac{df(v)}{dv}$ is continuous, which implies that $f(v)$ is convex on $[0, v_{max}]$.

Next, we develop the model and simplify it into a convex programming problem using Jensen's inequality. We show that the optimal solution exists for the problem and we propose a generalized solution which we refer to as a *good solution*. The purpose of doing so is to allow us to prove later that under the condition $f(v)$ is convex, the good solution is an optimal solution and any algorithm that finds the good solution is an exact algorithm.

Jensen's Inequality: Suppose t_i and t_{i+1} are fixed. The cost function in each time unit $\frac{dC}{dt} = f(v)$ is convex, where sailing speed $0 \leq v(t) \leq v_{max}$. If $v(t)$ is a constant which equals to $\int_{t_i}^{t_{i+1}} v(t) dt / (t_{i+1} - t_i)$, then total bunker cost $\int_{t_i}^{t_{i+1}} f \circ v dt$ is minimum. Suppose $f(v)$ is strictly convex, $\int_{t_i}^{t_{i+1}} f \circ v dt$ is minimum if and only if $v(t) = \int_{t_i}^{t_{i+1}} v(t) dt / (t_{i+1} - t_i)$.

Proof. Proof is in Appendix. □

Jensen's inequality shows that given a fixed schedule S_n , sailing at a constant speed between each pair of adjacent ports results in the minimum cost C_{S_n} .

Proposition 1: Suppose $f(v)/v$ is monotone increasing. For any schedule S_n with $t_n \neq t_n^u$, there exists a schedule S'_n with $t'_n = t_n^u$ such that $C_{S'_n} \leq C_{S_n}$. Suppose $f(v)/v$ is

strictly monotone increasing, then $C_{S'_n} < C_{S_n}$.

Proof. Obviously, for the schedule S_n , a vessel can spend more time on the last segment of the voyage. There is a schedule S'_n where the arrival time $t'_i = t_i$ for all $i < n$ and $t'_n = t_n^u$. The distance between the last two ports is D_{n-1} . It is easy to see that $v_{n-1} > v'_{n-1}$ in the last leg. We have $C_{S_n} - C'_{S_n} = f(v_{n-1})D_{n-1}/v_{n-1} - f(v'_{n-1})D_{n-1}/v'_{n-1}$. Since $f(v)/v$ is (strictly) monotone increasing and $v_{n-1} > v'_{n-1}$, we get $C_{S_n} \geq C_{S'_n}$ ($C_{S_n} > C_{S'_n}$). $\square$

At this point, we can conclude that by Jensen's inequality and Proposition 1, sailing at constant speed between each pair of successive ports and $t_n = t_n^u$ are sufficient conditions to achieve the minimum total bunker fuel cost for a fixed schedule S_n .

Properties of Good Solution

Theorem 1: Suppose there is at least one feasible solution for the problem, then there exists a minimum cost for the problem.

Proof. The vessel sails at a constant speed $v_i = D_i/T_i$. The total bunker fuel cost can be expressed as:

$$C = C(T_0, T_1, T_2, \dots, T_{n-1}) = \sum_{i=0}^{n-1} f(D_i/T_i)T_i \quad (1)$$

where $t_j^l - t_0 \leq \sum_{i=0}^{j-1} T_i \leq t_j^u - t_0$ for $j = 1, 2, 3, \dots, n$ and $T_i > 0$ for $i = 0, 1, \dots, n-1$. Since f is continuous, $C = \sum_{i=0}^{n-1} f(D_i/T_i)T_i$ is also continuous.

$$S := \{T_0, T_1, \dots, T_{n-1} \mid t_j^l - t_0 \leq \sum_{i=0}^{j-1} T_i \leq t_j^u - t_0 \text{ for } j = 1, 2, 3, \dots, n, \\ T_i \geq \frac{D_i}{v_{max}} \text{ for } i = 0, 1, \dots, n-1.\} \quad (2)$$

S is closed and bounded, i.e., S is compact. By continuity of $C(T_0, T_1, T_2, \dots, T_{n-1})$, if S is not empty, then there exists a point $X_0 = (x_0, x_1, x_2, \dots, x_{n-1})$ in S , s.t. $C(X_0)$ is minimum over S . By Jensen's inequality and Proposition 1, $C(X_0)$ is the minimum bunker fuel cost for the problem. $\square$

Note that the problem to be discussed in the rest of this section have at least one feasible solution, i.e. the problem has an optimal solution. In the paper's last section, we will discuss the implication if the feasible solution does not exist due to the violation of the constraint $v_i \leq v_{max}$.

By Jensen's inequality, we proved that sailing at constant speeds between successive ports is optimal. The insights provided by the next lemma is related to that of Jensen's inequality, but it considers speeds over two successive voyage segments. It shows that for the same total time duration in any two successive voyage segments (i.e., from P_i to P_{i+1} and then to P_{i+2}), the cost decreases if the constant speeds in each segment (i.e., v_i and v_{i+1}) gets closer in values.

Lemma 1: *Given S_n , $v_i > v_{i+1}$ ($v_i < v_{i+1}$) and $t_{i+1} < t_{i+1}^u$ ($t_{i+1} > t_{i+1}^l$). For S'_n , if $t'_j = t_j$, for $j \neq i+1$, are fixed, then $C_{S'_n}$ is a function of t'_{i+1} ; if t'_{i+1} satisfies $v_i > v'_i = \frac{D_i}{t'_{i+1}-t_i} \geq v'_{i+1} = \frac{D_{i+1}}{t_{i+2}-t'_{i+1}} > v_{i+1}$ ($v_i < v'_i = \frac{D_i}{t'_{i+1}-t_i} \leq v'_{i+1} = \frac{D_{i+1}}{t_{i+2}-t'_{i+1}} < v_{i+1}$), then $C_{S'_n}$ is a monotone decreasing (increasing) function of t'_{i+1} .*

Figure 1: If $v_i > v_{i+1}$ (S_n), v'_i and v'_{i+1} get closer in value as t'_{i+1} increases (S'_n)

In Lemma 1, the only difference between S_n and S'_n is in the arrival time at P_{i+1} , i.e., $t_{i+1} \neq t'_{i+1}$. Lemma 1 implies that when $v_i > v_{i+1}$ ($v_i < v_{i+1}$), there exists slack for t_{i+1} to increase (decrease) since $t_{i+1} < t_{i+1}^u$ ($t_{i+1} > t_{i+1}^l$), i.e., to bring v_i and v_{i+1} closer in value. In S'_n , this slack is reduced as t'_{i+1} increases (decreases) (i.e., v'_i gets closer to v'_{i+1}) and results in a lower $C_{S'_n}$. This is illustrated in Figure 1. Thus, when f is strictly convex, $C_{S'_n}$ is strictly monotone decreasing (increasing) function of t'_{i+1} .

Proof. It is sufficient to show

$$T_i f(v_i) + T_{i+1} f(v_{i+1}) \geq T'_i f(v'_i) + T'_{i+1} f(v'_{i+1}) \quad (3)$$

where $v_i > v'_i \geq v'_{i+1} > v_{i+1}$ and where $T_i v_i + T_{i+1} v_{i+1} = T'_i v'_i + T'_{i+1} v'_{i+1} = D_i + D_{i+1}$. Figure 1 shows the timelines for S_n and S'_n , where $v_i > v'_i \geq v'_{i+1} > v_{i+1}$ and $t'_{i+1} > t_{i+1}$.

Since $T_i + T_{i+1} = T'_i + T'_{i+1}$,

$$\begin{aligned} T_i f(v_i) + T_{i+1} f(v_{i+1}) &\geq T'_i f(v'_i) + T'_{i+1} f(v'_{i+1}) \\ \iff \frac{T_i}{T_i + T_{i+1}} f(v_i) + \frac{T_{i+1}}{T_i + T_{i+1}} f(v_{i+1}) &\geq \frac{T'_i}{T'_i + T'_{i+1}} f(v'_i) + \frac{T'_{i+1}}{T'_i + T'_{i+1}} f(v'_{i+1}). \end{aligned}$$

Let $\alpha_1 := \frac{T_i}{T_i + T_{i+1}}$, $\alpha_2 := \frac{T_{i+1}}{T_i + T_{i+1}}$, $\alpha_3 := \frac{T'_i}{T'_i + T'_{i+1}}$ and $\alpha_4 := \frac{T'_{i+1}}{T'_i + T'_{i+1}}$. It is easy to find α_5 and α_6 s.t. $\alpha_5 v_i + \alpha_6 v'_{i+1} = \alpha_1 v_i + \alpha_2 v_{i+1}$ and $\alpha_5 + \alpha_6 = 1$ (where $0 \leq \alpha_5 < \alpha_1$ and $\alpha_6 > 0$). Because f is convex, $\frac{\alpha_1 - \alpha_5}{\alpha_6} + \frac{\alpha_2}{\alpha_6} = 1$ and $v'_{i+1} = \frac{\alpha_1 - \alpha_5}{\alpha_6} v_i + \frac{\alpha_2}{\alpha_6} v_{i+1}$. So we can then find the following:

$$\begin{aligned} \frac{\alpha_1 - \alpha_5}{\alpha_6} f(v_i) + \frac{\alpha_2}{\alpha_6} f(v_{i+1}) &\geq f(v'_{i+1}) \\ \iff (\alpha_1 - \alpha_5) f(v_i) + \alpha_2 f(v_{i+1}) &\geq \alpha_6 f(v'_{i+1}) \\ \iff \alpha_1 f(v_i) + \alpha_2 f(v_{i+1}) - \alpha_5 f(v_i) - \alpha_6 f(v'_{i+1}) &\geq 0 \\ \iff \alpha_1 f(v_i) + \alpha_2 f(v_{i+1}) &\geq \alpha_5 f(v_i) + \alpha_6 f(v'_{i+1}). \end{aligned} \tag{4}$$

Similarly,

$$\alpha_3 f(v'_i) + \alpha_4 f(v'_{i+1}) \leq \alpha_5 f(v_i) + \alpha_6 f(v'_{i+1}). \tag{5}$$

By (4) and (5), we obtain (3). Hence $C_{S'_n}$ is a decreasing function of t'_{i+1} . By a similar proof, $v_i < v_{i+1}$ implies $C_{S'_n}$ is increasing function of t'_{i+1} . When f is strictly convex, since $\alpha_1 - \alpha_5 > 0$ and $\alpha_2 > 0$, the strict inequality holds. $\square$

Corollary 1: Assume f is strictly convex and C_{S_n} is minimum. If $v_i > v_{i+1}$, then $t_{i+1} = t_{i+1}^u$. If $v_i < v_{i+1}$, then $t_{i+1} = t_{i+1}^l$.

Proof. We prove the above by contradiction. Suppose $v_i > v_{i+1}$ and $t_{i+1} < t_{i+1}^u$. Because f is strictly convex, by Lemma 1, C_{S_n} is a strictly monotone decreasing function of t_{i+1} . It contradicts the assumption C_{S_n} is minimum. Hence if $v_i > v_{i+1}$, then $t_{i+1} = t_{i+1}^u$. By a similar proof, if $v_i < v_{i+1}$, then $t_{i+1} = t_{i+1}^l$. $\square$

Corollary 1 is equivalent to the following: given that f is strictly convex and C_{S_n} is minimum, if $t_{i+1} \in (t_{i+1}^l, t_{i+1}^u)$ then $v_i = v_{i+1}$; if $t_{i+1} = t_{i+1}^l$ then $v_i \leq v_{i+1}$, and if $t_{i+1} = t_{i+1}^u$ then $v_i \geq v_{i+1}$. We can now formally define the good solution.

Good Solution: S_n is a good solution if and only if $t_0 < t_1 < \dots < t_n = t_n^u$ and for any $0 \leq i \leq n-2$, $t_{i+1} \in [t_{i+1}^l, t_{i+1}^u]$, if $t_{i+1} \in (t_{i+1}^l, t_{i+1}^u)$ then $v_i = v_{i+1}$; if $t_{i+1} = t_{i+1}^l$ then $v_i \leq v_{i+1}$; if $t_{i+1} = t_{i+1}^u$ then $v_i \geq v_{i+1}$. Specifically, while f is strictly convex, Corollary 1 and Proposition 1 imply that the optimal solution S_n must be a good solution.

Consider the range of t'_{i+1} in Lemma 1 that t'_{i+1} should be in $[t_{i+1}^l, t_{i+1}^u]$ and it satisfies $v_i > v'_i = \frac{D_i}{t'_{i+1} - t_i} \geq v'_{i+1} = \frac{D_{i+1}}{t_{i+2} - t'_{i+1}} > v_{i+1}$ ($v_i < v'_i = \frac{D_i}{t'_{i+1} - t_i} \leq v'_{i+1} = \frac{D_{i+1}}{t_{i+2} - t'_{i+1}} < v_{i+1}$). By generalizing this result to $m+1$ ports, we have following lemma.

Lemma 2: Given S_n with $v_{i-1} > v_i = v_{i+1} = \dots = v_{i+m}$ ($v_{i-1} < v_i = v_{i+1} = \dots = v_{i+m}$) and $t_j < t'_j$ ($t_j > t'_j$), where $t'_j \in [t_j^l, t_j^u]$, for $j = i, i+1, \dots, i+m$. If the arrival times t_k are fixed for $k \neq i, i+1, \dots, i+m$, then C_{S_n} is a monotone decreasing (increasing) function of arrival time t_i .

Proof. The proof is similar to Lemma 1. In this lemma, since $v_i = v_{i+1} = \dots = v_{i+m}$, we can consider these consecutive legs as a single leg. As such, it becomes the same as Lemma 1. To be more specific, one can conceive that when t_i increases, $v_i = v_{i+1} = \dots = v_{i+m}$ increases and so are the times $t_{i+1}, t_{i+2}, \dots, t_{i+m}$. Concurrently, the value of C_{S_n} keeps decreasing till one of the t_j reaches the value of t'_j or $v_{i-1} = v_i = v_{i+1} = \dots = v_{i+m}$. $\square$

Corollary 2: Given S_n with $v_{i-1} > v_i = v_{i+1} = \dots = v_{i+m}$ ($v_{i-1} < v_i = v_{i+1} = \dots = v_{i+m}$) and $t_j < t'_j$ ($t_j > t'_j$), where $t'_j \in [t_j^l, t_j^u]$, for $j = i, i+1, \dots, i+m$. If the arrival times t_k are fixed for $k \neq i, i+1, \dots, i+m$, then there exists a new S_n with C_{S_n} decreases and the property: $\{v_{i-1} = v_i = v_{i+1} = \dots = v_{i+m} \text{ or there exists an } l \text{ (} i \leq l \leq i+m \text{) such that } t_l = t'_l \text{ holds}\}$.

Proof. Denote $\{t_i | v_{i-1}(t_i) > v_i(t_i) = v_{i+1}(t_i) = \dots = v_{i+m}(t_i) \text{ and } t_j(t_i) < t'_j \text{ for } j = i, i+1, \dots, i+m\}$ ($\{t_i | v_{i-1}(t_i) < v_i(t_i) = v_{i+1}(t_i) = \dots = v_{i+m}(t_i) \text{ and } t_j(t_i) > t'_j$

for $j = i, i+1, \dots, i+m$) as X ; $\{t_i | v_{i-1}(t_i) \geq v_i(t_i) = v_{i+1}(t_i) = \dots = v_{i+m}(t_i) \text{ and } t_j(t_i) \leq t'_j \text{ for } j = i, i+1, \dots, i+m\}$ ($\{t_i | v_{i-1}(t_i) \leq v_i(t_i) = v_{i+1}(t_i) = \dots = v_{i+m}(t_i) \text{ and } t_j(t_i) \geq t'_j \text{ for } j = i, i+1, \dots, i+m\}$) as Y . Obviously, $X \subset Y$.

The speed $v_{i-1}(t_i)$ is continuous and strictly monotone decreasing. In addition, $v_i(t_i)$, $v_{i+1}(t_i)$, $\dots$, $v_{i+m}(t_i)$ and $t_j(t_i)$ are continuous and strictly monotone increasing. Hence we know that the set Y equals to the addition of X and a limit point x_0 . On the limit point x_0 , $v_{i-1}(x_0) = v_i(x_0) = v_{i+1}(x_0) = \dots = v_{i+m}(x_0)$ or there exists an l ($i \leq l \leq i+m$) such that $t_l(x_0) = t'_l(x_0)$ holds. Clearly, all the speeds still satisfy the constraint $v_i \leq v_{max}$ when t_i is in Y . It implies feasibility and the continuity of $C_{S_n}(t_i)$ on Y .

By Lemma 2, if $x < y$ (or $x > y$) are in X , then $C_{S_n}(x) \geq C_{S_n}(y)$. By the continuity of $C_{S_n}(t_i)$ on Y , the inequality $C_{S_n}(x) \geq C_{S_n}(y)$ still holds on the limit point x_0 (as y goes to x_0). Therefore at point x_0 , there exists a new S_n with a decreased C_{S_n} and has the property: $\{v_{i-1} = v_i = v_{i+1} = \dots = v_{i+m} \text{ or there exists an } l \text{ (} i \leq l \leq i+m \text{) such that } t_l = t'_l \text{ holds}\}$. $\square$

Note that for S_n , $t_j \leq t'_j$ ($t_j \geq t'_j$) for $j = i, i+1, \dots, i+m$ hold.

Theorem 2: *If the good solution exists, then the good solution is unique.*

Proof. Consider two *good* solutions a and b . Suppose there exists an i s.t. $t_{ia} \neq t_{ib}$ (note that t_{ia} is t_i of solution a), otherwise a and b are the same. By the well-ordering principle there exists a minimum i s.t. $t_{ia} \neq t_{ib}$. Without loss of generality, we assume $t_{ia} < t_{ib}$ and therefore $v_{i-1}^a > v_{i-1}^b$ and $t_{ia} < t_i^u$, $t_{ib} > t_i^l$. By the definition of the good solution, we have $v_{ib} \leq v_{i-1}^b < v_{i-1}^a \leq v_{ia}$. Furthermore, since $t_{ia} < t_{ib}$ and $v_{ib} < v_{ia}$, we have $t_{i+1}^a < t_{i+1}^b$. By induction we can obtain $t_n^a < t_n^b$, which contradicts the definition of the good solution $t_n = t_n^u$. This shows that the good solution is unique. $\square$

Up till this point, we have proven that if f is strictly convex and a feasible solution exists, then the good solution exists and must be optimal. By Theorem 2, the good solution is unique, so that if f is strictly convex, the optimal solution is unique.

As stated earlier, the bunker cost function has not been proven to be strictly convex at low speeds. Thus, to accommodate this, we need to consider the more general case in which f is convex. For this case, we have two questions. Does the good solution exist? If the good solution exists, is it optimal? Fortunately, the answers to both questions are “yes”. Theorem 3 shows that if the good solution and a feasible solution exist, then the good solution must be optimal.

For proving the next theorem, we introduce the concept of sub-dividing the voyage. Suppose there is an n -leg voyage with time windows $[t_0^l, t_0^u]$, $[t_1^l, t_1^u]$, ... $[t_n^l, t_n^u]$ and a schedule S_n . When we sub-divide S_n at any port P_k ($0 < k < n + 1$), it results in two sub-voyages and their associated schedules, where t_k is the arrival time at P_k .

- The first sub-voyage is a k -leg voyage, which comprises the first $k + 1$ ports with time windows $[t_0^l, t_0^u]$, $[t_1^l, t_1^u]$, ... $[t_k^l, t_k^u]$. Its schedule is the same as the first $k + 1$ ports in S_n .
- The second sub-voyage is an $(n - k)$ -leg voyage, which includes the last $n - k + 1$ ports with time windows $[t_k^l, t_k^u]$, $[t_{k+1}^l, t_{k+1}^u]$, ... $[t_n^l, t_n^u]$. Its schedule coincides with the last $n - k + 1$ ports in S_n .

Consider an optimal solution S_n for the n -leg voyage. We let $\{S_k, S_{n-k}\}$ denotes the sub-voyages of S_n . That is, if we sub-divide S_n at any port P_k ($0 < k < n + 1$), we obtain $\{S_k, S_{n-k}\}$. Note that the number of legs in S_k (or S_{n-k}) is less than n and S_k (or S_{n-k}) is optimal in the sub-voyage (with speeds satisfying the constraint of v_{max}). Consider the good solution S'_n of the voyage. Similarly, if we sub-divide S'_n at any port P_k ($0 < k < n + 1$), the number of legs in S'_k (or S'_{n-k}) is less than n and S'_k (or S'_{n-k}) is a good solution in the sub-voyage.

Theorem 3: *If a voyage has an optimal solution and a good solution, the good solution must be an optimal solution.*

Proof. We prove that the theorem holds for any integer $n \geq 1$ (i.e., any n -leg voyage) by the strong form of mathematical induction. We need to prove the following: (a) For

$n = 1$, the theorem holds. (b) For $n > 1$ and for any integer m ($1 \leq m < n$), any m -leg voyage that have an optimal solution and a good solution, the good solution is optimal. Then for any n -leg voyage that have both optimal and good solution, the good solution must be optimal. For a 1-leg voyage, if the optimal solution exists, then by Jensen's inequality and Proposition 1, the good solution is an optimal solution. (Note that the speed is less than v_{max} by Proposition 1). Thus we have proven that (a) is true. What remains is to show that (b) is true. For a fixed n -leg voyage that has an optimal solution and a good solution, we let S'_n denotes the good solution. We assume that S_n is an optimal solution for the n -leg voyage. The purpose here is to show S'_n is optimal (i.e. $C_{S'_n} = C_{S_n}$ and that all the speeds in S'_n satisfy the v_{max} constraint).

We construct an optimal solution $\tilde{S}_n$ s.t. there exists a k ($1 \leq k \leq n$) and $\tilde{t}_k = t'_k$ that sub-divides the n -leg voyage with schedule $\tilde{S}_n$ (and S'_n) into two sub-voyages $\{\tilde{S}_k, \tilde{S}_{n-k}\}$ (and $\{S'_k, S'_{n-k}\}$). Since $\tilde{t}_k = t'_k$, the related sub-voyages are the same. Because the number of legs in each sub-voyage are less than n , then by (b), we obtain S'_k and S'_{n-k} as optimal solutions (with $C_{S'_k} = C_{\tilde{S}_k}$ and $C_{S'_{n-k}} = C_{\tilde{S}_{n-k}}$). Hence we have $C_{S'_n} = C_{S'_k} + C_{S'_{n-k}} = C_{\tilde{S}_k} + C_{\tilde{S}_{n-k}} = C_{\tilde{S}_n}$ and with all the speeds in S'_n satisfying the v_{max} constraint. To complete the proof, we need to find out the optimal solution $\tilde{S}_n$ s.t. there exists a k s.t. $1 \leq k \leq n$ with $\tilde{t}_k = t'_k$. We need to consider the following two cases.

Case 1: Suppose there exists a k ($1 \leq k \leq n$) s.t. the arrival time at port P_k is $t_k = t'_k$, then by the above results, S'_n is an optimal solution.

Case 2: Suppose there does not exist such a k , we claim that there is another optimal solution $\tilde{S}_n$ with $\tilde{t}_k = t'_k$.

Figure 2: Applying Corollary 2 to construct an optimal solution with a same arrival time as good solution, such that mathematical induction can be applied

Without loss of generality, assume that the arrival time t'_1 in S'_n is later than the arrival time t_1 in S_n (i.e. $t_1 < t'_1$) and $t_n = t_n^u$ (Proposition 1). Clearly, $t_1 < t'_1$ implies $v_0 > v'_0$. Since S'_n is a good solution and $t_1 < t'_1$, by the definition of good solution, we have $v'_0 \geq v'_1$. By assumption, there exists a smallest i such that $v_i < v'_i$ as shown in Figure 2. (Otherwise there is a contradiction to $t_n = t_n^u$.)

Clearly $t_j < t'_j$ for all $j \leq i$, so that by the definition of the good solution, we have $v'_0 \geq v'_1 \geq \dots \geq v'_i > v_i$ and $v_{i-1} \geq v'_{i-1} \geq v'_i > v_i$. By Corollary 2, there is a solution $\bar{S}_n$ with $C_{\bar{S}_n} \leq C_{S'_n}$ such that $\bar{v}_{i-1} = \bar{v}_i$ or $\bar{t}_i = t'_i$. Obviously, $\bar{S}_n$ is an optimal solution. Now we consider the new optimal solution $\bar{S}_n$ as S_n (i.e., $\bar{S}_n$ replaces S_n).

The following three sub-cases need to be considered:

- **Case 2.1:** If $t_i = t'_i$, by Case 1, S'_n is an optimal solution.
- **Case 2.2:** If $v_{i-1} = v_i$ and $v_{i-1} \geq v'_{i-1}$, then $v_j \geq v'_j$ for all $j \leq i$.
- **Case 2.3:** If $v_{i-1} = v_i$ and $v_{i-1} < v'_{i-1}$, then $v_{i-2} \geq v'_{i-2} \geq v'_{i-1} > v_{i-1} = v_i$ and $t_i < t'_i, t_{i-1} < t'_{i-1}$. By Corollary 2, there is an optimal solution $\bar{S}_n$ such that $\bar{v}_{i-2} = \bar{v}_{i-1} = \bar{v}_i$ or $\bar{t}_j = t'_j$ ($j = i$ or $i - 1$). Let this new optimal solution be S_n . If $t_j = t'_j$, by Case 1, S'_n is an optimal solution. If $v_{i-2} = v_{i-1} = v_i$ and $v_{i-2} \geq v'_{i-2}$, then $v_j \geq v'_j$ for all $j \leq i$. If $v_{i-2} = v_{i-1} = v_i$ and $v_{i-2} < v'_{i-2}$, Corollary 2 applies. Since i is finite and if we view the above as a search process, the process will eventually be terminated by one of the following conditions: (i) S'_n is an optimal solution; (ii) $v_j \geq v'_j$ for all $j \leq i$. It may appear that $v_0 < v'_0$ is a terminating condition. However this condition is infeasible because for S_n , $t_1 \leq t'_1 \Rightarrow v_0 \not\leq v'_0$.

For condition (ii) ($v_j \geq v'_j$ for all $j \leq i$), we note that during the above search process, $t_j \leq t'_j$ for $j \leq i$ always hold in S_n . Obviously, if for any $1 \leq j \leq i$, we have $t_j = t'_j$, then by Case 1, S'_n is an optimal solution. Suppose that $v_j \geq v'_j$ for all $j \leq i$ and $t_1 < t'_1 \Rightarrow v_j \geq v'_j$ for all $j \leq i$ and $t_s < t'_s$ for all $s \leq i + 1$, then there exists a smallest $i_2 > i$ such that $v_{i_2} < v'_{i_2}$. (Otherwise it contradicts $t_n = t_n^u$.)

In summary, after the above process, there are two possible results.

- S'_n is an optimal solution;
- There exists a smallest $i_2 > i$ such that $v_{i_2} < v'_{i_2}$.

The first result is what we intend to prove. The second is the same as the assumption for Case 2, except that i_2 replaces i ($i_2 > i$). Hence we have a similar result where S'_n is an optimal solution or there exists $i_3 > i_2 > i$ such that $v_{i_3} < v'_{i_3}$. Since n is finite and $i < i_2 < i_3 < \dots < n + 1$, eventually we obtain S'_n as an optimal solution. $\square$

Example

In this section, we demonstrate the use of our results by proving that the RSA in Norstad et al. (2011) finds the good solution. By Theorem 1, 2 and 3, it then follows that the RSA is an exact algorithm. For completeness, we reproduced the RSA as follows; see Norstad et al. (2011) for details.

	OptimizeSpeed(s, e):
<hr/>	
1:	$\beta \leftarrow 0$
2:	$p \leftarrow 0$
3:	$v^R \leftarrow (\sum_{i=s}^{e-1} D_i) / (t_e - t_s)$
4:	for $i \leftarrow s + 1$ to e do
5:	$v_i \leftarrow v^R$
6:	$t_i \leftarrow t_{i-1} + d_i / v_i$
7:	$b_i \leftarrow \max\{0, t_i - t_i^u, t_i^l - t_i\}$
8:	if $b_i > \beta$ then
9:	$\beta \leftarrow b_i$
10:	$p \leftarrow i$
11:	end if
12:	end for
13:	if $\beta > 0$ and $t_p > t_p^u$ then
14:	$t_p \leftarrow t_p^u$
15:	OptimizeSpeed(s, p)

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16:   OptimizeSpeed( $p, e$ )
17: end if
18: if  $\beta > 0$  and  $t_p < t_p^l$  then
19:    $t_p \leftarrow t_p^l$ 
20:   OptimizeSpeed( $s, p$ )
21:   OptimizeSpeed( $p, e$ )
22: end if

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The function $\text{OptimizeSpeed}(s, e)$ optimizes the speeds in a voyage between P_s (starting port) and P_e (ending port). If sailing at a constant speed $v^R \leftarrow (\sum_{i=s}^{e-1} D_i)/(t_e - t_s)$ between P_s and P_e does not violate any time window constraint, then the speed is the solution. If sailing at speed $v^R \leftarrow (\sum_{i=s}^{e-1} D_i)/(t_e - t_s)$ between P_s and P_e does violate the time window $[t_p^l, t_p^u]$ at Port p , then the arrival time t_p that violates the time window is adjusted to be as close as possible to the endpoint t_p^l or t_p^u . $\text{OptimizeSpeed}(s, p)$ and $\text{OptimizeSpeed}(p, e)$ are recurred till no time window is violated and a solution is then obtained.

We prove that for any fixed voyage with a finite number of legs N and time windows $[t_i^l, t_i^u]$ (where $t_i^l \leq t_i^u$ and $t_0^l = t_0^u$; for $i < j$, $t_i^l < t_j^u$ and $i, j = 0, 1, \dots, N$), the good solution can be found.

In RSA, we consider changing of time t_p (by the notation of RSA) as a step and the $N + 1$ fixed arrival times $t_0 < t_1 < t_2 < \dots < t_N$ as a state. So a step means a change from one state to another. Before we prove that the good solution exists, we need to show that the RSA ends at a state that satisfies the time windows.

Lemma 3: *The RSA ends at a state which satisfies the time windows and at any state $t_0 < t_1 < t_2 < \dots < t_N = t_N^u$.*

At the initial state the arrival times t_0 and $t_N = t_N^u$ are fixed. We find every step in RSA by fixing the arrival time t_p at $P_p(t_p = t_p^l \text{ or } t_p^u)$, then the rest of the arrival times are determined by the fixed t_p . We let $t_{p_i}(t_{p_i} = t_{p_i}^l \text{ or } t_{p_i}^u)$ denotes the arrival time fixed at the i -th step and let $\tilde{t}_{p_i}$ denotes the arrival time that is replaced by t_{p_i} . Given that

$\beta = \text{Max}\{0, \tilde{t}_{p_i} - t_{p_i}^u, t_{p_i}^l - \tilde{t}_{p_i}\}$, β is the maximum of b_i in $\text{OptimizeSpeed}(s, e)$. (Note that $\tilde{t}_{p_i}$ may differ at the different steps.)

Proof. Using the above notations, we claim that for any i, j , if $p_i < p_j$, then $t_{p_i} < t_{p_j}$. This statement ensures $t_0 < t_1 < t_2 < \dots < t_N$, so that RSA continues to search for the solution. Since the statement holds for the initial state and only one arrival time t_p can be fixed when $\text{OptimizeSpeed}(s, e)$ is called, we just need show that $t_s < t_p < t_e$ (where t_s and t_e are t_0 or t_N^u respectively, or the $t_{p_i}(s)$).

We first show that $t_p < t_e$: (i) if $t_p = t_p^u$, then $t_p < \tilde{t}_p < t_e$; (ii) if $t_p = t_p^l$, then there are two cases: (1) $t_e = t_e^u$; by the assumption $p < e$, $t_p^l < t_e^u$ and hence $t_p = t_p^l < t_e^u = t_e$; (2) $t_e = t_e^l$, where we have to consider the step to fix t_e . At the step, $\beta = t_e^l - \tilde{t}_e$ and $\beta \geq \text{Max}\{0, \tilde{t}_p - t_p^u, t_p^l - \tilde{t}_p\}$. Hence we get $t_e^l - \tilde{t}_e \geq t_p^l - \tilde{t}_p$. By step 6 of $\text{OptimizeSpeed}(s, e)$, it implies $\tilde{t}_e > \tilde{t}_p$. Therefore $t_e = t_e^l \geq t_p^l + \tilde{t}_e - \tilde{t}_p > t_p^l = t_p$.

Figure 3: $t_p < t_e$

In a similar way, we can prove that $t_s < t_p$: (i) if $t_p = t_p^l$, then $t_p > \tilde{t}_p > t_s$; (ii) if $t_p = t_p^u$, then there are two cases: (1) $t_s = t_s^l$; by the assumption $s < p$ and thus $t_s = t_s^l < t_p^u = t_p$; (2) $t_s = t_s^u$, where we consider the step to fix t_s . At this step, $\beta = \tilde{t}_s - t_s^u$ and $\beta \geq \text{Max}\{0, \tilde{t}_p - t_p^u, t_p^l - \tilde{t}_p\}$. Hence we have $\tilde{t}_s - t_s^u \geq \tilde{t}_p - t_p^u$. By step 6 of $\text{OptimizeSpeed}(s, e)$, $\tilde{t}_s < \tilde{t}_p$. Therefore $t_s = t_s^u \leq t_p^u + \tilde{t}_s - \tilde{t}_p < t_p^u = t_p$.

Figure 4: $t_s < t_p$

Hence we find that $t_s < t_p < t_e$. It implies that $t_0 < t_1 < t_2 < \dots < t_N = t_N^u$ at any state. Since N is finite, we can only fix a finite $t_p(s)$. To fix a new t_p , we only need to

try less than $N + 1$ pairs of (s, e) . Thus the RSA has to finally stop and the final state satisfies the time windows. $\square$

Lemma 3 suggests that the RSA continues to search until the constraint of the time windows are satisfied. Next we show that the final state is a good solution.

Theorem 4: *The RSA finds the good solution (i.e., the good solution exists).*

Proof. We consider the following property. Property (A): For any $0 \leq i \leq N - 2$ and $t_{i+1} \in [t_{i+1}^l, t_{i+1}^u]$, if $t_{i+1} \in (t_{i+1}^l, t_{i+1}^u)$, then $v_i = v_{i+1}$; if $t_{i+1} = t_{i+1}^l$, then $v_i \leq v_{i+1}$ and if $t_{i+1} = t_{i+1}^u$, then $v_i \geq v_{i+1}$. A state $t_0 < t_1 < t_2 < \dots < t_N = t_N^u$ with Property (A) means that it is a good solution. It is not always true that $t_{i+1} \in [t_{i+1}^l, t_{i+1}^u]$ at any state of RSA. So we consider another property. Property (B): For any $0 \leq i \leq N - 2$, if $t_{i+1} \in (t_{i+1}^l, t_{i+1}^u)$, then $v_i = v_{i+1}$; if $t_{i+1} = t_{i+1}^l$, then $v_i \leq v_{i+1}$; if $t_{i+1} = t_{i+1}^u$, then $v_i \geq v_{i+1}$; if $t_{i+1} \notin [t_{i+1}^l, t_{i+1}^u]$, then $v_i = v_{i+1}$. Obviously, the initial state of sailing at a constant speed satisfies (B). We prove Theorem 4 by showing that every state of the RSA satisfies (B), so that the final state satisfies (B). Because Lemma 3 ensures that RSA eventually stops at a state that satisfies $t_{i+1} \in [t_{i+1}^l, t_{i+1}^u]$, it means that the final state is good solution.

We show that (B) holds for every step of RSA. Suppose (B) holds at a port P_{i+1} ($0 \leq i \leq N - 2$). That is, (1) $t_{i+1} \in (t_{i+1}^l, t_{i+1}^u)$ and $v_i = v_{i+1}$; or (2) $t_{i+1} = t_{i+1}^l$ and $v_i \leq v_{i+1}$; or (3) $t_{i+1} = t_{i+1}^u$ and $v_i \geq v_{i+1}$; or (4) $t_{i+1} \notin [t_{i+1}^l, t_{i+1}^u]$ and $v_i = v_{i+1}$. If (B) holds at a state, it can then follows that (B) holds at every port at the state.

Any step that changes t_p will only affect the arrival times at ports $P_{s+1}, \dots, P_p, \dots, P_{e-1}$. However, since the speeds between P_s and P_p or between P_p and P_e satisfy $v_i = v_{i+1}$, so (B) holds at these ports. We need to show that (B) still holds at P_s , P_p and P_e . By the RSA, (B) holds at P_p if either $t_p = t_p^u$ or $t_p = t_p^l$. Hence we only need to show that (B) holds at P_s and P_e . We find that P_s and P_e is either P_0 or P_N respectively or have been chosen as P_p at a particular step. Since at P_0 and P_N , t_0 is fixed and $t_N = t_N^u$, it is sufficient to prove that (B) holds at P_p at any state.

Before the function $\text{OptimizeSpeed}(s, e)$ is called, $v_{i-1} = v_i$ for port P_i ($s < i < e$), so that (B) holds (at P_i). Assuming P_p is chosen in the first step of $\text{OptimizeSpeed}(s, e)$, we only need to show that (B) holds at P_p for the states after $\text{OptimizeSpeed}(s, e)$ is called.

To relate the notations used in this paper with those used for the RSA, we denote S_n as the output plan at the n -th state (after the $\text{OptimizeSpeed}(s, e)$ is called), b_i^n as the b_i at n -th step (after the $\text{OptimizeSpeed}(s, e)$ is called), $t_i^{S_n}$ as t_i of S_n , $v_{il}^{S_n}(v_{ir}^{S_n})$ as the average speed between P_{i-1} and P_i (P_i and P_{i+1}) in S_n .

At the first step, there is an i_1 s.t. $b_{i_1}^1 = \text{Max} \{b_i^1 \mid i = s \dots e\} > 0$. (Otherwise it becomes the trivial case where $\text{OptimizeSpeed}(s, e)$ does not affect any arrival time.) Since $v_{i_1 l}^{S_0} = v_{i_1 r}^{S_0}$ is the average speed between P_s and P_e in S_0 , we define $v^{S_0} := v_{i_1 l}^{S_0}$.

Without loss of generality, we assume $t_{i_1}^{S_0} < t_{i_1}^l$ (Figure 5).

Figure 5: Arrival time before time window

We let $t_{i_1}^{S_1} = t_{i_1}^l$, then we have $v_{i_1 l}^{S_1} < v^{S_0} < v_{i_1 r}^{S_1}$ (Figure 6).

Figure 6: Arrival time set at lower bound of time window

We claim that for any n , it follows that $v_{i_1 l}^{S_n} \leq v^{S_0} \leq v_{i_1 r}^{S_n}$ and $t_{i_1}^{S_n} = t_{i_1}^l$, which we can prove by induction. For $n = 1$, $v_{i_1 l}^{S_1} \leq v^{S_0} \leq v_{i_1 r}^{S_1}$ and $t_{i_1}^{S_1} = t_{i_1}^l$. Suppose $1 \leq m \leq n - 1$, we have $v_{i_1 l}^{S_m} \leq v^{S_0} \leq v_{i_1 r}^{S_m}$ and $t_{i_1}^{S_m} = t_{i_1}^l$. We need to show that for n , $v_{i_1 l}^{S_n} \leq v^{S_0} \leq v_{i_1 r}^{S_n}$ and $t_{i_1}^{S_n} = t_{i_1}^l$. Assume that the function $\text{OptimizeSpeed}(a, b)$ is called for n . If both arguments a and b of the function are not i_1 , then $v_{i_1 l}^{S_{n-1}}$, $v_{i_1 r}^{S_{n-1}}$ and $t_{i_1}^{S_{n-1}}$ will not be changed in the n -th step (after $\text{OptimizeSpeed}(s, e)$ is called). It then follows that $v_{i_1 l}^{S_n} \leq v^{S_0} \leq v_{i_1 r}^{S_n}$ and $t_{i_1}^{S_n} = t_{i_1}^l$. Note that in each step, the function $\text{OptimizeSpeed}(a, b)$ should only affect arrival time(s) $t_{a+1}, \dots, t_{b-1}$.

Case 1: Suppose b is i_1 and there exists an i_n s.t. $b_{i_n}^n = \text{Max}\{b_i^n \mid i = a \dots i_1\} > 0$ (Figure 7). Only arrival times $t_{a+1}^{S_{n-1}}, t_{a+2}^{S_{n-1}}, \dots, t_{i_1-1}^{S_{n-1}}$ will be changed. So $v_{i_1r}^{S_n} = v_{i_1r}^{S_{n-1}} \geq v^{S_0}$ and $t_{i_1}^{S_n} = t_{i_1}^l$. So we need to show that $v_{i_1l}^{S_n} \leq v^{S_0}$. We present two cases for the value of $t_{i_n}^{S_n}$. Note that $b_{i_n}^1 = \text{Max}\{0, t_{i_n}^l - t_{i_n}^{S_0}, t_{i_n}^{S_0} - t_{i_n}^u\}$ and $s \leq a < i_n < i_1 < e$.

Figure 7: $b = i_1$

Case 1.1: If $t_{i_n}^{S_n} = t_{i_n}^u$, then by steps 13 and 14 of `OptimizeSpeed(s, e)`, $t_{i_n}^{S_{n-1}} > t_{i_n}^u = t_{i_n}^{S_n}$. Since $t_{i_1}^{S_n} = t_{i_1}^{S_{n-1}} = t_{i_1}^l$, we have $t_{i_1}^{S_n} - t_{i_n}^{S_n} > t_{i_1}^{S_{n-1}} - t_{i_n}^{S_{n-1}}$. By Lemma 3, this leads to $t_{i_n}^{S_{n-1}} < t_{i_1}^{S_{n-1}}$, i.e., $t_{i_1}^{S_{n-1}} - t_{i_n}^{S_{n-1}} > 0$. Hence $v_{i_1l}^{S_n} = \frac{D_{i_n, i_1}}{t_{i_1}^{S_n} - t_{i_n}^{S_n}} < \frac{D_{i_n, i_1}}{t_{i_1}^{S_{n-1}} - t_{i_n}^{S_{n-1}}} = v_{i_1l}^{S_{n-1}}$. By mathematical induction, $v_{i_1l}^{S_{n-1}} \leq v^{S_0}$ and thus $v_{i_1l}^{S_n} < v^{S_0}$.

Case 1.2: If $t_{i_n}^{S_n} = t_{i_n}^l$, then by the definition of $b_{i_1}^1$, we have $t_{i_1}^l - t_{i_1}^{S_0} \geq \text{Max}\{0, t_{i_n}^l - t_{i_n}^{S_0}, t_{i_n}^{S_0} - t_{i_n}^u\} \geq t_{i_n}^l - t_{i_n}^{S_0}$. It implies that $t_{i_1}^l - t_{i_n}^l \geq t_{i_1}^{S_0} - t_{i_n}^{S_0}$. By Lemma 3, $t_{i_n}^{S_0} < t_{i_1}^{S_0}$ and $t_{i_n}^{S_n} = t_{i_n}^l < t_{i_1}^{S_n} = t_{i_1}^l$, and therefore $v_{i_1l}^{S_n} = \frac{D_{i_n, i_1}}{t_{i_1}^{S_n} - t_{i_n}^{S_n}} \leq \frac{D_{i_n, i_1}}{t_{i_1}^{S_0} - t_{i_n}^{S_0}} = v^{S_0}$.

Case 2: Suppose a is i_1 and there exists an i_n s.t. $b_{i_n}^n = \text{Max}\{b_i^n \mid i = i_1 \dots b\} > 0$ (Figure 8). Only arrival times $t_{i_1+1}^{S_{n-1}}, t_{i_1+2}^{S_{n-1}}, \dots, t_{b-1}^{S_{n-1}}$ will be changed. So $v_{i_1l}^{S_n} = v_{i_1l}^{S_{n-1}} \leq v^{S_0}$ and $t_{i_1}^{S_n} = t_{i_1}^l$. We need to show that $v_{i_1r}^{S_n} \geq v^{S_0}$. Note that $t_{i_n}^{S_n} = t_{i_n}^l$ or $t_{i_n}^u$ and $s < i_1 < i_n < b \leq e$.

Figure 8: $a = i_1$

Case 2.1: If $t_{i_n}^{S_n} = t_{i_n}^u$, then by steps 13 and 14 of $\text{OptimizeSpeed}(s, e)$, we have $t_{i_n}^{S_{n-1}} > t_{i_n}^u = t_{i_n}^{S_n}$. By Lemma 3, $t_{i_1}^l = t_{i_1}^{S_{n-1}} < t_{i_n}^{S_{n-1}}$ and $t_{i_1}^l = t_{i_1}^{S_n} < t_{i_n}^{S_n}$. Therefore $t_{i_n}^{S_{n-1}} - t_{i_1}^l > t_{i_n}^{S_n} - t_{i_1}^l > 0$ and so $0 < v_{i_1 r}^{S_{n-1}} = \frac{D_{i_1, i_n}}{t_{i_n}^{S_{n-1}} - t_{i_1}^l} < \frac{D_{i_1, i_n}}{t_{i_n}^{S_n} - t_{i_1}^l} = v_{i_1 r}^{S_n}$. By mathematical induction, we have $v^{S_0} \leq v_{i_1 r}^{S_{n-1}}$ and so $v^{S_0} \leq v_{i_1 r}^{S_{n-1}} < v_{i_1 r}^{S_n}$.

Case 2.2: If $t_{i_n}^{S_n} = t_{i_n}^l$, then by the definition of $b_{i_1}^1$, we have $t_{i_1}^l - t_{i_1}^{S_0} \geq \text{Max}\{0, t_{i_n}^l - t_{i_n}^{S_0}, t_{i_n}^{S_0} - t_{i_n}^u\} \geq t_{i_n}^l - t_{i_n}^{S_0}$ and thus $t_{i_n}^{S_0} - t_{i_1}^{S_0} \geq t_{i_n}^l - t_{i_1}^l$. By Lemma 3, $t_{i_1}^l = t_{i_1}^{S_n} < t_{i_n}^{S_n} = t_{i_n}^l$ and $v^{S_0} = \frac{D_{i_1, i_n}}{t_{i_n}^{S_0} - t_{i_1}^{S_0}} \leq \frac{D_{i_1, i_n}}{t_{i_n}^l - t_{i_1}^l} = v_{i_1 r}^{S_n}$.

Therefore for any n , $v_{i_1 l}^{S_n} \leq v^{S_0} \leq v_{i_1 r}^{S_n}$ and $t_{i_1}^{S_n} = t_{i_1}^l$. (For the case $t_{i_1}^{S_0} > t_{i_1}^u$, we have similar results, i.e., $v_{i_1 l}^{S_n} \geq v^{S_0} \geq v_{i_1 r}^{S_n}$ and $t_{i_1}^{S_n} = t_{i_1}^u$.) Thus after any step n , (B) holds for P_{i_1} . $\square$

Theorem 4 implies that the good solution always exists, so we can restate Theorem 3 as follows:

Theorem 3.1: *If a voyage has at least one feasible solution (i.e., an optimal solution exists), the good solution must be an optimal solution.*

Proof. The proof follows from Theorem 3 and Theorem 4. $\square$

Up till this point, we have considered the problem to have at least one feasible solution (Theorem 1). Specifically, we have supposed that if all the speeds $v_i \leq v_{max}$ hold, then by Theorem 3, an optimal solution exists and the good solution must be optimal. However, by the definition of the good solution, $v_i \leq v_{max}$ may not hold, i.e., the good solution may not be feasible. To address this issue, we state the following theorem.

Theorem 5: *If the good solution is not feasible (i.e. there exists at least one v_i in the good solution s.t. $v_i > v_{max}$), then there is no feasible solution for the problem.*

Proof. We prove the theorem by contraposition. Suppose there exists an i , s.t. $v_i \leq v_{max}$

does not hold, the good solution is not feasible and thus not optimal. By the contraposition of Theorem 3.1, the optimal solution does not exist, i.e. no feasible solution exists for the problem. $\square$

Based on the above results, the good solution is always optimal without the constraint $v_i \leq v_{max}$, which we discuss as follows. With $v_i \leq v_{max}$, the domain of T_i are compact (as stated in the proof for Theorem 1). If the constraint is removed, then $T_i > 0$ will replace the constraint $T_i \geq \frac{D_i}{v_{max}}$. Hence the domain may no longer be compact. We can consider the unconstrained problem as follows. We first find the good solution since it always exists (Theorem 4). Then we select a v_{max} s.t. all the speeds in the good solution satisfy $v_i \leq v_{max}$. That is, we consider the problem with the constraint $v_i \leq v_{max}$. Since the good solution is a feasible solution, it must be optimal. Therefore, for any $v > v_{max}$ (where $v_i \leq v_{max} < v$), the good solution is optimal for the problem with constraint $v_i \leq v$. This means that the good solution must be optimal for the unconstrained problem.

Conclusion

In this paper, we present the results for the speed optimization problem of a vessel. First, we show that the necessary conditions for the optimal solution is to sail at constant speed between each pair of successive ports and that the arrival time at the last port must coincide with the upper bound of the port's time window. Then, if at least one feasible solution exists, we show that there exists an optimal solution for the problem (Theorem 1). We present the properties of a good solution and prove that if there is a good solution for the problem, it is unique (Theorem 2) and must be optimal when a feasible solution exists (Theorem 3). The results imply that any algorithm that leads to a good solution is an exact optimization method. We demonstrate how the results can be applied to prove the exactness of an existing algorithm.

We believe the insights provided in this paper will contribute to development of solution methods for this problem and possibly its future extensions. Since fuel consumption

of most other transport modes is also convex with speed, it is worthwhile to find how the results can be applied in other transportation services (e.g., scheduled bus service). It would also be valuable if one could explore how the results can be extended to or employed in richer scheduling problems for a fleet of vessels, e.g., liner shipping with requirements for service frequency and port-to-port transit times.

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Appendix

We provide the proof of the Jensen’s inequality.

Jensen’s Inequality: Suppose t_i and t_{i+1} are fixed. The cost function in unit time $\frac{dC}{dt} = f(v)$ is convex, where sailing speed $0 \leq v(t) \leq v_{max}$. When $v(t)$ is a constant $\int_{t_i}^{t_{i+1}} v(t)dt/(t_{i+1}-t_i)$, total cost $\int_{t_i}^{t_{i+1}} f \circ v dt$ is minimum. Suppose $f(v)$ is strictly convex, $\int_{t_i}^{t_{i+1}} f \circ v dt$ is minimum if and only if $v(t) = \int_{t_i}^{t_{i+1}} v(t)dt/(t_{i+1}-t_i)$.

By Jensen’s inequality, if $\mu(\Omega) = c$, $c\varphi(\frac{\int_{\Omega} f d\mu}{c}) \leq \int_{\Omega} f d\mu$; see, e.g., Rudin (1986). We can apply Jensen’s inequality with the condition $\mu(\Omega) = 1$ to imply this consequence.

Proof. The sailing time is $T = t_{i+1} - t_i > 0$. Consider the time interval $[t_i, t_{i+1}]$ as the measure of the space Ω . Since we use the stand-time t as the measure on Ω , then we have $t(\Omega) = T$. We intend to apply Jensen’s inequality to show that $Tf(\frac{\int_{\Omega} v(t)dt}{T}) \leq \int_{\Omega} f(v(t))dt$. To achieve this, we need to construct a new measure μ on Ω such that $\int_{\Omega} d\mu = 1$. Assume $\mu(E) = \int_E \frac{1}{T} dt$, for all measurable set $E \subseteq \Omega$. We can

then obtain:

$$\int_{\Omega} f \circ v d\mu = \int_{\Omega} (f \circ v) \frac{1}{T} dt, \quad \int_{\Omega} v d\mu = \int_{\Omega} v \frac{1}{T} dt, \quad \int_{\Omega} d\mu = \int_{\Omega} \frac{1}{T} dt = \frac{T}{T} = 1. \quad (6)$$

Since f is convex, $0 \leq v(t) \leq v_{max}$, $\int_{\Omega} v d\mu = \int_{\Omega} v \frac{1}{T} dt < +\infty$ and μ is a positive measure s.t. $\mu(\Omega) = 1$, By Jensen's inequality, we can then show:

$$\begin{aligned} f\left(\int_{\Omega} v d\mu\right) &\leq \int_{\Omega} f \circ v d\mu \implies f\left(\int_{\Omega} v \frac{1}{T} dt\right) \leq \int_{\Omega} (f \circ v) \frac{1}{T} dt \\ \implies T f\left(\frac{\int_{\Omega} v dt}{T}\right) &\leq \int_{\Omega} (f \circ v) dt. \end{aligned} \quad (7)$$

As $\int_{\Omega} v dt = D_i$, when $v = \frac{D_i}{T}$, the cost $\int_{t_i}^{t_{i+1}} f \circ v dt = T f\left(\frac{D_i}{T}\right)$ is minimum.

From the above we show that $T f\left(\int_{\Omega} v \frac{1}{T} dt\right) \leq \int_{\Omega} (f \circ v) dt$ (t is a measure for time space Ω), when $v(t) = \int_{t_i}^{t_{i+1}} v(t) dt / T$ the equality holds. Next we show that when f is strictly convex the equality holds if and only if $v(t) = \int_{t_i}^{t_{i+1}} v(t) dt / T$.

First we define set $S_1 = \{t | v(t) \geq \int_{t_i}^{t_{i+1}} v(t) dt / T\}$ and $S_2 = \{t | v(t) < \int_{t_i}^{t_{i+1}} v(t) dt / T\}$, and $t(S_1) + t(S_2) = T$. Suppose $v(t) \neq \int_{t_i}^{t_{i+1}} v(t) dt / T$ and by continuity of $v(t)$, both $t(S_1)$ and $t(S_2)$ are greater than 0 and less than T . By applying the above results on the measure space S_1 and S_2 respectively, by the definition of strict convexity and $\frac{\int_{S_1} v(t) dt}{t(S_1)} > \frac{\int_{S_2} v(t) dt}{t(S_2)}$, we can find that:

$$\begin{aligned} \int_{\Omega} (f \circ v) dt &= \int_{S_1} (f \circ v) dt + \int_{S_2} (f \circ v) dt \\ &\geq t(S_1) f\left(\frac{\int_{S_1} v(t) dt}{t(S_1)}\right) + t(S_2) f\left(\frac{\int_{S_2} v(t) dt}{t(S_2)}\right) \\ &> T f\left(\frac{\int_{S_1} v(t) dt}{t(S_1)} \frac{t(S_1)}{T} + \frac{\int_{S_2} v(t) dt}{t(S_2)} \frac{t(S_2)}{T}\right) = T f\left(\frac{\int_{S_1} v(t) dt}{T}\right). \end{aligned} \quad (8)$$

Since $\int_{\Omega} (f \circ v) dt > T f\left(\frac{\int_{S_1} v(t) dt}{T}\right)$, the above proves that the equality in (7) holds if and only if $v(t) = \int_{t_i}^{t_{i+1}} v(t) dt / T$. $\square$