Output regulation of multivariable nonlinear systems using extra outputs

Wang, Lei; Marconi, Lorenzo; Wen, Changyun; Su, Hongye

2018

Wang, L., Marconi, L., Wen, C., & Su, H. (2018). Output regulation of multivariable nonlinear systems using extra outputs. IFAC-PapersOnLine, 51(13), 13-18. doi: 10.1016/j.ifacol.2018.07.247

https://hdl.handle.net/10356/89927

https://doi.org/10.1016/j.ifacol.2018.07.247

Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 51-13 (2018) 13–18

Output Regulation of Multivariable Nonlinear Systems Using Extra Outputs

Lei Wang[*‡], Lorenzo Marconi[†], Changyun Wen[‡], Hongye Su[*]

[*] *State Key Laboratory of Industrial Control Technology, Institute of Cyber-Systems and Control, Zhejiang University, Hangzhou, China.*
[†] *CASY - DEI, University of Bologna, Italy*
[‡] *School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore.*

Abstract: We consider the output regulation problem for a class of nonlinear multivariable systems in the case, besides the regulated error, also additional measurements not necessarily zero in steady state are available. To offset the steady state value of the extra measurements and make the latter effective for stabilisation purposes, we propose a control structure in which those measurements are filtered by a post-processor specifically designed to block their steady state value. The specific scenario we have in mind is the one of nonlinear systems that are nonminimum-phase between the input and the regulated error and for which extra measurements are necessary, or simply desirable, to succeed in robust output feedback stabilisation.

Keywords: Multivariable nonlinear systems, output regulation, normal form, internal model

1. FRAMEWORK

1.1 Problem Statement

In this paper we deal with the problem of output regulation for a class of multivariable nonlinear systems of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f_0(w,z,e) \\ \dot{x}_{ij} &= x_{i,j+1}\,, \quad 1 \le i \le p\,, \quad 1 \le j \le r_i - 1 \\ \dot{x}_{i,r_i} &= q_i(w,z,x) + b_i(w,z,x)u \end{aligned} \tag{1}$$

in which $z \in \mathbb{R}^{n_0}$, $x = \text{col}\ (x_1, \ldots, x_p)$ with $x_i = \text{col}\ (x_{i1}, \ldots, x_{i,r_i})$, $i = 1, \ldots, p$, the control input u ranges in $\mathbb{R}^p$ and $e := \text{col}\ (x_{11}, \ldots, x_{p1}) \in \mathbb{R}^p$ denotes the regulated error. The variable $w \in \mathbb{R}^{n_w}$ models exogenous inputs, that might represent references/disturbances to be tracked/rejected or parametric uncertainties, and that are generated as solution of the autonomous system $\dot{w} = s(w)$ typically referred to as the exosystem. In this framework the problem of output regulation amounts to design a regulator such that the trajectories of the closed-loop system originating from a certain set of initial condition are bounded and the resulting regulated error converges to zero. The different solutions presented in literature, in fact, differ for the kind of assumptions done about the set of initial conditions (with global, semiglobal, or local results typically addressed) and the kind of measurements available for the design of the regulator. The majority of solutions presented so far, in particular, assume that only the regulated error e is available for feedback by thus limiting the class of systems that can be dealt with. While the knowledge of the error is an inescapable condition [1], there's no reason for disregarding other measurements that could be essential for enlarging the class of systems to deal with or simply to improve the attainable performances of the closed-loop system. For this reason, in the paper we consider the case in which the available measurements are of the form $y_m = \text{col}\ (e, y)$, in which $y = h(w,z) \in \mathbb{R}^q$, for some $q \ge 1$, denotes the extra measurements. We thus look for a regulator of the form

$$\begin{aligned} \dot{x}_c &= \varphi_c(x_c, y_m) \\ u &= \upsilon_c(x_c, y_m) \end{aligned} \tag{2}$$

capable to keep the closed loop trajectories originating from given compact sets bounded and to steer the regulated error to zero uniformly in the initial conditions.

The problem will be solved under a certain number of assumptions that are listed in the following. First, as usual in the literature of output regulation, it is assumed that $w \in \mathcal{W} \subset \mathbb{R}^{n_w}$, where $\mathcal{W}$ is a compact set, invariant for the dynamics of $\dot{w} = s(w)$.

Then we assume that the matrix $B(w,z,x) \in \mathbb{R}^{p\times p}$ defined as that matrix whose i-th row coincides with $b_i(w,z,x)$, is robustly invertible in the sense specified in the following.

Assumption 1. There exists a constant nonsingular matrix $\bar{B} \in \mathbb{R}^{p\times p}$ such that

$$B(w,z,x)\bar{B} + B^\top(w,z,x)\bar{B}^\top \ge \delta I \tag{3}$$

for some positive constant $\delta > 0$, for all $(w,z,x) \in \mathcal{W} \times \mathbb{R}^{n_0} \times \mathbb{R}^r$, $r := r_1 + \ldots + r_p$.

[⋆] This work is supported by the Science and Engineering Research Council Grant from the National Robotics Programme Singapore (SERC Grant No: 162 25 00036); H2020 European project AirBorne (ICT 780960); the Science Fund for Creative Research Groups of the National Natural Science Foundation of China (grant no. 61621002); Fundamental Research Funds for the Central Universities.

[1] For linear systems, it has been shown that robust regulation can be achieved only if the error, or extra measurement by which the error is "readable", is available for feedback (Francis & Wonham [1976]).

10.1016/j.ifacol.2018.07.247

It is worth remarking that the previous assumption limits the class of systems dealt with to be a subclass of systems that have a uniform vector relative degree from the input u to the regulated error e.

Furthermore, we assume that the regulator equations admit a solution. For the class of systems (1) the assumption in question can be formulated as follow.

Assumption 2. There exist functions $\pi_z(w) : \mathcal{W} \to \mathbb{R}^{n_0}$ and $c(w) : \mathcal{W} \to \mathbb{R}^p$ such that

$$\begin{aligned} L_{s(w)}\pi_z(w) &= f_0(w, \pi_z(w), 0) \\ 0 &= q_i(w, \pi_z(w), 0) + b_i(w, \pi_z(w), 0)c(w) \\ i &= 1, \dots, p\,. \end{aligned} \quad (4)$$

As it is well known the solutions $(\pi_z(w), c(w))$ of the previous equations represent the desired steady state of z and of u associated to the regulation objective $e(t) \equiv 0$. We can then identify the map that necessarily characterises the steady state value of the extra measurements y that is given by $\pi_y(w) := h(w, \pi_z(w))$. Namely, as long as the regulation objective is met, the extra measurements will asymptotically converge to $\pi_y(w(t))$ with $w(t)$ the trajectory of the exosystem. Unless very particular cases, we observe that such a steady state is not zero.

The next assumption, finally, restricts the possible maps $c(w)$ and $\pi_y(w)$ to fulfil a "regression-like" condition. This assumption is inherited from Byrnes & Isidori [2004].

Assumption 3. There exist two positive numbers d_u and d_y and two smooth functions $\phi_u : \mathbb{R}^{pd_u} \to \mathbb{R}^p$ and $\phi_y : \mathbb{R}^{qd_y} \to \mathbb{R}^q$ such that

$$L_s^{d_u} c(w) = \phi_u(c(w), L_s c(w), \dots, L_s^{d_u-1} c(w)) \quad (5)$$

$$L_s^{d_y} \pi_y(w) = \phi_y(\pi_y(w), L_s \pi_y(w), \dots, L_s^{d_y-1} \pi_y(w))\,. \quad (6)$$

It is worth remarking that most of the results presented in the paper can be given even without the previous assumption, by relying upon the general result of Marconi, Praly & Isidori [2007] in which immersion conditions (such as (5)-(6)) are removed in the design of internal model-based regulator. On the other hand, conditions (5)-(6), although limiting the class of systems, lead to constructive design solutions and simplify the forthcoming discussion.

We conclude the section by observing that the steady state maps previously introduced allow one to define an error system that will play a role in the following. In particular, by letting

$$\tilde{z} := z - \pi_z(w)\,, \quad \tilde{y} := y - \pi_y(w)\,, \quad \tilde{u} := u - c(w)$$

it is easy to obtain the error system

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}_0(w, \tilde{z}, e) \\ \dot{x}_{i,j} &= x_{i,j+1}\,, \quad 1 \le i \le p\,, \quad 1 \le j \le r_i - 1 \\ \dot{x}_{i,r_i} &= \tilde{q}_i(w, \tilde{z}, x) + b_i(w, z, x)\tilde{u} \\ \tilde{y} &= \tilde{h}(w, \tilde{z}) \\ e &= \text{col}\,(x_{11}, \dots, x_{p1}) \end{aligned} \quad (7)$$

where

$$\begin{aligned} \tilde{f}_0(w, \tilde{z}, e) &:= f_0(w, \tilde{z} + \pi_z(w), e) - f_0(w, \pi_z(w), 0) \\ \tilde{q}_i(w, \tilde{z}, x) &:= q_i(w, \tilde{z} + \pi_z(w), x) - q_i(w, \pi_z(w), 0) \\ &\quad + [b_i(w, \tilde{z} + \pi_z(w), x) - b_i(w, \pi_z(w), 0)]c(w) \\ \tilde{h}(w, \tilde{z}) &:= h(w, \tilde{z} + \pi_z(w)) - h(w, \pi_z(w)) \end{aligned}$$

1.2 Earlier contributions to the problem

The majority of the approaches presented in literature merely translates the problem of output regulation into the one of robust stabilisation to the origin of the error system (7) enriched with an internal model dynamics meant to asymptotically reproduce the term $c(w)$ matched with the control input (7). The robust stabilisation of the "extended" system is done by processing available information that, in most of the cases, is only the error e. This fact, in turn, practically limits the class of systems that can be successfully handled to one of minimum-phase systems, namely asking that $\dot{\tilde{z}} = \tilde{f}_0(w, \tilde{z}, e)$, viewed as a system with input e and state z has nice asymptotic properties that can be asymptotic (local exponential) stability of $\tilde{z} = 0$ (Isidori & Byrnes [1990], Marconi & Praly [2008]) or stronger properties such as input-to-state stability with respect to the input e (Huang [2007]). In this framework the internal model is mostly added following a "pre-processing" structure (following the terminology introduced in Astolfi, Isidori et. al. [2013]) in which the internal model dynamics act directly on the input of the plant and it is driven by the output of the stabiliser, and high-gain (error) feedback strategies, static or dynamic according to the relative degree of the plant, are used as stabilising action.

While in Marconi, Praly & Isidori [2007] it has been proved that an internal model-based regulator always exists, in practice assumptions are done on the ideal steady state input $c(w)$, asking that the latter fulfils appropriate immersion assumptions. Immersion into a linear system (Huang [2007]) and immersion into a system having canonical nonlinear observability form (Isidori, Marconi & Praly [2012]) are definitely the most common assumptions. The works quoted before (and indeed the majority of the approaches proposed so far) deal with the case of single-input single-error systems (namely $p = 1$). Attempts to extend high-gain design solutions of internal model-based regulators to *multivariable systems* have been done in Wang et. al. [2016, 2017] for a class of invertible nonlinear systems not necessarily possessing a vector relative degree.

High-gain solutions just using the regulated error for feedback have been shown to be effective also for a class of *non-minimum phase* single-input single-error ($p = 1$) nonlinear systems (see Marconi, Isidori & Serrani [2004]). In that approach it is assumed that the state $\tilde{z}$ of the zero dynamics is Uniformly Completely Observable (by using the terminology of Teel & Praly [1995]) from the regulated error e and the control input u. This condition and certain *nonlinear non-resonance conditions* properly defined in that paper were shown to be sufficient to obtain a high-gain internal model-based regulator under the assumption that the function $c(w)$ is immersed into a linear system. The approach proposed in Marconi, Isidori & Serrani [2004] borrows the design tool of Isidori [2000] for simple stabilisation problems of nonminimum-phase systems by output feedback.

If, besides the error e, other measurements y are available for feedback, the literature becomes much less rich. In Priscoli, Isidori & Marconi [2008] the problem at hand was addressed in case $p = 1$ under the (relatively strong) assumption that the extra measurements y have zero steady state value (namely $\tilde{h}(w, 0) = 0$ for all $w \in \mathcal{W}$). That assumption is indeed motivated by the fact that any robust regulator solving the problem at hand with a "pre-processing" structure must necessarily process measurements whose steady state value is zero (see Lemma 2 in Priscoli, Isidori & Marconi [2008]). To deal with the more general case in which y is not zero in steady state, the recent contribution Wang et. al. [2018] deals with the case in which $p = 1$ and the zero dynamics between the input u and the output $\tilde{y}$ is stabilisable by a linear feedback processing the error e. Under the same assumptions (5)-(6) but with $\phi_u(\cdot)$ and $\phi_y(\cdot)$ restricted to be *linear* functions, that paper shows how an internal model-based regulator can be designed. It must be remarked, however, that the formulated assumptions substantially limit the class of systems that can be dealt with and that an extension of the ideas presented in Wang et. al. [2018] to broaden the addressable class is hard to imagine. Considering the class of multivariable *linear systems* an interesting work that strongly inspired the actual paper is the one proposed in Antunes, Hespanha & Silvestre [2014] in which the case of systems having the same number of inputs and regulated errors but possibly extra measurements is considered in the context of multi-rate systems (namely systems in which sensors and actuators work at different rates). The main idea proposed in that paper is to pre-filter the extra measurements with filters that have zeros synchronised with the poles of the exosystems in order to "block" their steady value on the output, before effectively using the filtered extra output for stabilisation purposes.

In this paper we consider the problem of output regulation for the class of systems introduced in Subsection 1.1 in presence of "extra measurements" not necessarily vanishing in steady state by following the idea of adding "blocking zeros" introduced in Antunes, Hespanha & Silvestre [2014] for linear systems.

2. THE CASE OF LINEAR SYSTEMS

In preparation to the nonlinear analysis, in this section we briefly review the case of linear systems. Consider the output regulation problem for linear systems of the form

$$\dot{x} = Ax + Bu + Pw\,, \quad y_m = Cx + Qw \tag{8}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, measurement $y_m \in \mathbb{R}^{p+q}$, and exogenous variable $w \in \mathbb{R}^d$ governed by $\dot{w} = Sw$. The measurement y_m is comprised of two components $e = C_e x + Q_e w \in \mathbb{R}^p$, denoting the regulated error, and $y = C_y x + Q_y w \in \mathbb{R}^q$, the additional available measurements. In this framework, the problem is to design a *robust* regulator of the form

$$\dot{x}_r = A_r x_r + B_r y_m\,, \quad u = C_r x_r + D_r y_m \tag{9}$$

so that the resulting closed-loop system has bounded trajectories and the regulated error converges to zero. The adjective "robust" is here meant as in Francis & Wonham [1976]. Specifically, the list of all entries in the matrices of (8) is seen as a point $\mathcal{P}$ in a vector space of suitable dimension, and a regulator is said to be robust, at a given data point $\mathcal{P}_0$, if it solves the problem of output regulation for all data points $\mathcal{P}$ in a neighbourhood of $\mathcal{P}_0$.

In this context the regulator equations take the form

$$\begin{aligned} \Pi S &= A\Pi + B\Psi + P \\ 0 &= C_e \Pi + Q_e \end{aligned} \tag{10}$$

in the two unknowns (Π, Ψ). The following result in Francis & Wonham [1976] is well known.

Lemma 1. The problem of robust output regulation in the case of measurement feedback at a given data point $\mathcal{P}_0$ has a solution only if

- the triplet (A, B, C) is stabilisable and detectable at $\mathcal{P}_0$;
- the regulator equations (10) have solution (Π, Ψ) for all data points $\mathcal{P}$ in a neighbourhood of $\mathcal{P}_0$.$\triangleleft$

Moreover, it is a well known fact that the regulator equations (10) have a solution (Π, Ψ) for any (P, Q_e) *if and only if* the *non-resonance condition*

$$\operatorname{rank}\begin{pmatrix} A - \lambda I & B \\ C_e & 0 \end{pmatrix} = n + p\,, \ \text{for all } \lambda \in \sigma(S) \tag{11}$$

holds. Thus, by Lemma 1 and the previous fact, robust regulation necessarily asks that (11) is fulfilled. Furthermore, we observe that (11) implies that the solution (Π, Ψ) of (10) is *unique* (see Francis & Wonham [1976]) and that necessarily $m \geq p$.

The solution proposed in Francis & Wonham [1976] relies on a "post-processing" structure (see Isidori [2017]) in which an internal model

$$\dot{\eta} = (\Phi \otimes I_p)\eta + (G \otimes I_p)e\,, \tag{12}$$

where (Φ, G) is a controllable pair with the minimal polynomial of Φ coincident with the one of S, is added on the error. The cascade system, regarded as a system with input u and output (y_m, η), can be easily proved to be detectable (if (A, C) is such) and stabilisable (if (A, B) is such and (11) holds) when $w = 0$. A regulator of the form (9) solving the problem of robust output regulation at a given point $\mathcal{P}_0$ can be then completed with an output feedback stabiliser of the cascade (8), (12) (with $w = 0$) by the output (y_m, η) at $\mathcal{P}_0$.

An option to "post-processing" schemes is given by "pre-processing" structures in which an internal model of the form

$$\begin{aligned} \dot{\eta} &= \Phi_\eta \eta + G_\eta v \\ u &= \Gamma_\eta \eta + D_\eta v \end{aligned} \tag{13}$$

where $v \in \mathbb{R}^m$ is a residual input, $\Phi_\eta = (\Phi \otimes I_p)$ with Φ having the same expression as above, $G_\eta \in \mathbb{R}^{pd\times m}$ and $\Gamma_\eta \in \mathbb{R}^{m\times pd}$ with (G_η, Γ_η) chosen such that (Φ_η, G_η) is controllable and (Φ_η, Γ_η) is detectable, D_η is an arbitrary nonsingular matrix, is directly added on the input u. It can be proved that the cascade of the internal model with the system (8), viewed as a system with input v and output $y_m = (e, y)$, is stabilisable (if such is the pair (A, B)) and detectable (if such is the pair (A, C), and if (11) holds). The regulator can be then completed by designing an output feedback stabiliser for the cascade (13), (8) (with $w = 0$) at $\mathcal{P}_0$ of the form

$$\dot{\zeta} = A_c\zeta + B_{ce}e + B_{cy}y\,, \quad v = C_c\zeta + D_{ce}e + D_{cy}y\,. \tag{14}$$

With respect to the post-processing scheme, however, the stabiliser (14), besides stabilising the cascade at $\mathcal{P}_0$, is required to have necessarily a further crucial feature to obtain a *robust* regulator. This is specified in the following lemma.

Lemma 2. Let the non-resonance condition (11) hold. Then the regulator (13)-(14) solves the problem of robust output regulation at a given data point $\mathcal{P}_0$ only if

a) the closed-loop system (8), (13), (14) with $w = 0$ is asymptotically stable at $\mathcal{P}_0$;
b) For all Π_y the following set of equations

$$\begin{aligned} \Pi_\zeta S &= A_c \Pi_\zeta + B_{cy} \Pi_y \\ 0 &= C_c \Pi_\zeta + D_{cy} \Pi_y \end{aligned} \tag{15}$$

have a solution Π_ζ.◁

The second condition in the lemma, in particular, expresses the property that the steady state input y must be necessarily blocked on the output v of the stabiliser, namely that stabiliser, to be robust, necessarily possesses blocking zeros synchronised with the exosystem dynamics between the input y and the output v.

Notwithstanding pre-processing schemes are clearly more restrictive than post-processing solutions, they have been shown to be preferable in many contexts, such as regulation of multi-rate linear systems (can see Antunes, Hespanha & Silvestre [2014]) and robust regulation of nonlinear systems (see the references in the previous section). Moreover, it is interesting to observe that the constraint on the stabiliser given by (14) can be always fulfilled by adding a filter on the extra measurement y of the kind

$$\begin{aligned} \dot\xi &= (F_\xi \otimes I_q)\xi + (G_\xi \otimes I_q) y \\ y_{\rm f} &= y - (\Gamma_\xi \otimes I_q)\xi \end{aligned} \tag{16}$$

in which $(F_\xi, G_\xi) \in \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times 1}$ is a controllable pair with F_ξ Hurwitz, and $\Gamma_\xi \in \mathbb{R}^{1\times d}$ is such that $F_\xi + G_\xi \Gamma_\xi = \Phi$ where Φ is such that its minimal polynomial coincides with the one of S. For the resulting cascaded system the following holds.

Lemma 3. Suppose that the triplet (A, B, C) is stabilisable and detectable, and that (11) holds. Then the cascade of (8) and (16), viewed as a system with input u and output $(e, y_{\rm f})$, is stabilisable and detectable with $w = 0$. ◁

By this lemma and by the previous discussion about pre-processing solutions, we can then conclude that the cascade of systems (13), (8), (16), viewed as a system with input v and output $(e, y_{\rm f})$, is stabilisabile and detectable provided that the regulated plant is such by the input u and output y_m and (11) holds. A stabiliser of the form

$$\begin{aligned} \dot\zeta' &= A'_c \zeta' + B'_{ce} e + B'_{cy} y_{\rm f} \\ v &= C'_c \zeta' + D'_{ce} e + D'_{cy} y_{\rm f}\,. \end{aligned} \tag{17}$$

can be thus fixed to stabilise such a cascade at $\mathcal{P}_0$. By considering now the cascade of the systems (16) and (17) (which is a system with input (e, y) and output v) as the system (14), it turns out that the conditions of Lemma 2 are fullfilled.

Lemma 4. Let (17) be an output feedback stabiliser for the cascade of systems (13), (8), (16) at $\mathcal{P}_0$. Then, with system (14) defined as the cascade (16) and (17), the requirements (a) and (b) of Lemma 2 are fulfilled. ◁

The receipt to design pre-processing internal model is summarised in the following theorem.

Theorem 1. Let (A, B, C) stabilisable and detectable at $\mathcal{P}_0$ and let the non-resonance condition (11) holds. Then the system given by (8) extended with (13) and (16) is stabilisable and detectable. Let (17) be a stabiliser for the extended system with $w = 0$. Then the regulator (13), (16), (17) solves the problem of robust output regulation.

In the next section we show how the pre-processing design strategy illustrated above can be extended to the nonlinear framework delineated in Section 1.1.

3. MAIN RESULTS

Following now the linear pre-processing design paradigm, and by bearing in mind Assumption 3, we consider a pre-processing internal model of the form

$$\begin{aligned} \dot\eta &= A_{\rm u}\,\eta + B_{\rm u}\,\phi_u(\eta) + G_{\rm u}\,v \\ u &= C_{\rm u}\,\eta + D_u v \end{aligned} \tag{18}$$

with $\eta \in \mathbb{R}^{pd_u}$, $A_{\rm u} := A_1 \otimes I_p$, $B_{\rm u} := B_1 \otimes I_p$, $C_{\rm u} := C_1 \otimes I_p$, with $(A_1, B_1, C_1) \in \mathbb{R}^{d_u\times d_u} \times \mathbb{R}^{d_u\times 1} \times \mathbb{R}^{1\times d_u}$ a triplet in prime form, $G_{\rm u} := G_1 \otimes I_p$ with $G_1 \in \mathbb{R}^{pd_u\times p}$ and $D_u \in \mathbb{R}^{p\times p}$ to be properly designed, and a filter of the form

$$\begin{aligned} \dot\xi &= A_{\rm f}\,\xi + B_{\rm f}\,\phi_y(\xi) + G_{\rm f}\,y_{\rm f} \\ y_{\rm f} &= y - C_{\rm f}\,\xi \end{aligned} \tag{19}$$

with $\xi \in \mathbb{R}^{qd_y}$, $A_{\rm f} := A_2 \otimes I_q$, $B_{\rm f} := B_2 \otimes I_q$, $C_{\rm f} := C_2 \otimes I_q$, with $(A_2, B_2, C_2) \in \mathbb{R}^{d_y\times d_y} \times \mathbb{R}^{d_y\times 1} \times \mathbb{R}^{1\times d_y}$ a triplet in prime form and $G_{\rm f} := G_2 \otimes I_q$ with G_2 to be properly designed.

By letting

$$\pi_\eta(w) := \mathrm{col}\,\Big(\, c(w)\ \ L_{s(w)}c(w)\ \ldots\ L_{s(w)}^{d_u-1}c(w) \,\Big)$$

and $\tilde\eta := \eta - \pi_\eta(w)$, using Assumption 3, it is immediate to conclude that

$$\begin{aligned} \dot{\tilde\eta} &= A_{\rm u}\,\tilde\eta + B_{\rm u}\,\tilde\phi_\eta(w, \tilde\eta) + G_{\rm u}\,v \\ \tilde u &= C_{\rm u}\,\tilde\eta + D_{\rm u} v \end{aligned} \tag{20}$$

in which $\tilde\phi_u(w, \tilde\eta) := \phi_u(\tilde\eta + \pi_\eta(w)) - \phi_u(\pi_\eta(w))$.

Moreover, by letting

$$\pi_\xi(w) := \mathrm{col}\,\Big(\, \pi_y(w)\ \ L_{s(w)}\pi_y(w)\ \ldots\ L_{s(w)}^{d_y-1}\pi_y(w) \,\Big)$$

and $\tilde\xi := \xi - \pi_\xi(w)$, using again Assumption 3, it is immediate to conclude that

$$\begin{aligned} \dot{\tilde\xi} &= A_{\rm f}\,\tilde\xi + B_{\rm f}\,\tilde\phi_y(w, \tilde\xi) + G_{\rm f}\,\tilde y_{\rm f} \\ \tilde y_{\rm f} &= \tilde y - C_{\rm f}\,\tilde\xi \end{aligned} \tag{21}$$

in which $\tilde\phi_y(w, \tilde\xi) := \phi_y(\tilde\xi + \pi_\xi(w)) - \phi_y(\pi_\xi(w))$.

Following the linear paradigm, the starting point for the design of the regulator is a stabiliser for the cascade of the system (7) with the filter (21). We recall, in fact, by following the arguments of the previous section (see in particular

Lemma 3), that an output feedback stabiliser for the cascade in question always exists for linear systems under the natural assumption of stabilisability and detectability of the plant and under the non-resonance condition (11). By taking advantage of the normal form (7), we formulate the assumption in question as the existence of a stabiliser for the cascade of the zero dynamics of system (7) driving the filter (21), namely the system

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}_0(w, \tilde{z}, e) \\ \dot{\tilde{\xi}} &= A_{\rm f}\,\tilde{\xi} + B_{\rm f}\,\tilde{\phi}_y(w, \tilde{\xi}) + G_{\rm f}\,(\tilde{h}(w, \tilde{x}) - C_{\rm f}\,\tilde{\xi}) \end{aligned} \tag{22}$$

regarded as a system with input e and output $\tilde{y}_{\rm f} = \tilde{h}(w, \tilde{x}) - C_{\rm f}\,\tilde{\xi}$.

Assumption 4. The cascade system (22), viewed as a system with input e and output $\tilde{y}_{\rm f}$, is stabilisable by output feedback, namely there exists a stabiliser of the form

$$\begin{aligned} \dot{\chi} &= A_{\rm s}(\chi, e, \tilde{y}_{\rm f}) \qquad \chi \in \mathbb{R}^{\upsilon} \\ e &= C_{\rm s}(\chi, e, \tilde{y}_{\rm f}) \end{aligned} \tag{23}$$

$\upsilon \geq 0$, such that the equilibrium $(\tilde{z}, \tilde{\xi}, \chi) = 0$ of the closed-loop system (22)-(23) is asymptotically and locally exponentially stable with a domain of attraction an open set $\mathcal{A}_z \times \mathcal{A}_\xi \times \mathcal{A}_\chi \subseteq \mathbb{R}^{n_0} \times \mathbb{R}^{qd_y} \times \mathbb{R}^{\upsilon}$.

We emphasise that, by using standard backstepping results (see Teel & Praly [1995]), the previous assumption implies the existence of a dynamic output feedback stabiliser for the cascaded system given by the entire system (7) driving filter (21) (with an appropriate domain of attraction dependent on $\mathcal{A}$). Thus, by bearing in mind Lemma 3 for linear systems, the existence of the output feedback stabiliser (23), entails a stabilisability assumption by output feedback of system (7) and an appropriate nonlinear version of (11). The previous assumption is indeed sufficient to solve the problem of robust nonlinear output regulation in a semiglobal sense as specified by the next theorem.

Theorem 2. Consider the problem of output regulation for the class of systems (1) under the Assumptions 1-4 formulated above. Let $\mathcal{C}_z \subset \mathbb{R}^{n_0}$, $\mathcal{C}_x \subset \mathbb{R}^{rp}$, $\mathcal{C}_\eta \subset \mathbb{R}^{pd_u}$, $\mathcal{C}_\xi \subset \mathbb{R}^{qd_y}$ be arbitrary compact sets with $\mathcal{C}_z \subset \mathcal{A}_z$, $\mathcal{C}_\xi \subset \mathcal{A}_\xi$ and $\mathcal{C}_\chi \subset \mathcal{A}_\chi$. Then there exists a stabiliser of the form

$$\begin{aligned} \dot{\zeta} &= A'_{\rm s}(\zeta, e, y_{\rm f}) \qquad \zeta \in \mathbb{R}^{\iota} \\ v &= C'_{\rm s}(\zeta, e, y_{\rm f}) \end{aligned} \tag{24}$$

and a compact $\mathcal{C}_\zeta \subset \mathbb{R}^{\iota}$, such that the regulator given by (18), (19) and (24) solves the problem at hand for all initial conditions $(\tilde{z}(0), x(0), \tilde{\eta}(0), \tilde{\xi}(0), \chi(0), \zeta(0)) \in \mathcal{C}_z \times \mathcal{C}_x \times \mathcal{C}_\eta \times \mathcal{C}_\xi \times \mathcal{C}_\chi \times \mathcal{C}_\zeta$.

Proof. The key fact in proof of the result is that the $\tilde{y}_{\rm f}$ in (26) is indeed equal to $y_{\rm f}$ as a consequence of (6) in Assumption 3. This implies that the stabiliser (28) of the cascade of the system (27) is implementable with the available measurements. From this the regulator can be constructed by following pretty standard results in robust stabilisation of nonlinear system that are briefly summarised in the next part.

The design of the stabiliser (24), in particular, can be divided into two steps. In the first step a partial state feedback regulator, namely a regulator depending on the error e and its time derivatives (namely the vector x) and the extra measurements y, is designed able to solve the stabilisation to the origin of the error system. In the second step a regulator only processing the error and extra measurements is obtained by replacing the x with appropriate estimates provided by a "dirty" derivatives observer.

As for the first step, consider the change of variables

$$\begin{aligned} \tilde{x}_{i,1} &= x_{i,1} - C_{s,i}(\chi, e, \tilde{y}_f) \\ \tilde{x}_{i,j} &= \kappa^{1-j} x_{i,j}\,, \quad 1 \leq j \leq r_i - 1\,, 1 \leq i \leq p \\ \vartheta_i &= \tilde{x}_{i,r_i} + \alpha_{i,r_i-1}\tilde{x}_{i,r_i-1} + \ldots + \alpha_{i,2}\tilde{x}_{i,2} + \alpha_{i,1}\tilde{x}_{i,1} \end{aligned}$$

where $C_{s,i}(\chi, e, \tilde{y}_f)$ denotes the ith component of $C_s(\chi, e, \tilde{y}_f)$ for $1 \leq i \leq p$, κ is a design parameter to be determined, and all α_{ij}'s are chosen to be coefficients of Hurwitz polynomials. Let the compacts sets $\mathcal{C}_{\tilde{x}} \in \mathbb{R}^{(r-1)p}$ and $\mathcal{C}_\vartheta \in \mathbb{R}^p$ be such that

$$\begin{aligned} &(\tilde{z}(0), x(0), \tilde{\xi}(0), \tilde{\chi}(0)) \in \mathcal{C}_z \times \mathcal{C}_x \times \mathcal{C}_\xi \times \mathcal{C}_\chi \\ \Rightarrow &(\tilde{z}(0), \tilde{x}(0), \vartheta(0), \tilde{\xi}(0), \tilde{\chi}(0)) \in \mathcal{C}_z \times \mathcal{C}_{\tilde{x}} \times \mathcal{C}_\vartheta \times \mathcal{C}_\xi \times \mathcal{C}_\chi \end{aligned}$$

Moreover, to ease the notation, set $\tilde{x} = {\rm col}\,(\tilde{x}_1, \ldots, \tilde{x}_p)$ with $\tilde{x}_i = {\rm col}\,(\tilde{x}_{i1}, \ldots, \tilde{x}_{i,r_i-1})$ and $\vartheta = {\rm col}\,(\vartheta_1, \ldots, \vartheta_p)$. In these new coordinates the error system given by (7), (25), (26) and (28) can be easily seen to be described as

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}_0(w, \tilde{z}, C_{\rm s}(\chi, \tilde{y}_{\rm f}) + \tilde{e}) \\ \dot{\tilde{\xi}} &= A_{\rm f}\,\tilde{\xi} + B_{\rm f}\,\tilde{\phi}_y(w, \tilde{\xi}) + G_{\rm f}\,(\tilde{h}(w, \tilde{x}) - C_{\rm f}\,\tilde{\xi}) \\ \dot{\chi} &= A_{\rm s}(\chi, C_{\rm s}(\chi, \tilde{y}_{\rm f}) + \tilde{e}, \tilde{y}_{\rm f}) \\ \dot{\tilde{x}} &= \kappa\,H\,\tilde{x} + \varpi(w, \tilde{z}, \tilde{\xi}, \chi, \tilde{e}) + \kappa M \vartheta \\ \dot{\tilde{\eta}} &= A_{\rm u}\,\tilde{\eta} + B_{\rm u}\,\tilde{\phi}_\eta(w, \tilde{\eta}) + G_{\rm u}\,v \\ \dot{\vartheta} &= \tilde{Q}(w, \tilde{z}, \tilde{x}, \vartheta) + B(w, z, x)\,(D_u\,v + C_u\,\tilde{\eta}) \end{aligned} \tag{25}$$

where $\tilde{e} = {\rm col}\,(\tilde{x}_{11}, \ldots, \tilde{x}_{p1})$, $H = {\rm diag}(H_1, \ldots, H_p)$ with H_i appropriately defined Hurwitz matrices, $M = {\rm diag}(M_1, \ldots, M_p)$ with $M_i = {\rm col}\,(0, \ldots, 0, 1) \in \mathbb{R}^{r_i - 1}$ for $1 \leq i \leq p$, $\varpi(w, \tilde{z}, \tilde{\xi}, \chi, \tilde{e})$ and $\tilde{Q}(w, \tilde{z}, \tilde{x}, \vartheta)$ are some appropriate smooth functions, whose expressions are omitted for the sake of simplicity. It turns out that $\varpi(w, \tilde{z}, \tilde{\xi}, \chi, \tilde{e})$ is independent of κ and that $\varpi(w, 0, 0, 0, 0) = 0$ and $\tilde{Q}(w, 0, 0, 0) = 0$.

This system, viewed as a system with output ϑ and input v, has relative degree $(1, 1, \ldots, 1)$ and zero dynamics described by

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}_0(w, \tilde{z}, C_{\rm s}(\chi, \tilde{y}_{\rm f}) + \tilde{e}) \\ \dot{\tilde{\xi}} &= A_{\rm f}\,\tilde{\xi} + B_{\rm f}\,\tilde{\phi}_y(w, \tilde{\xi}) + G_{\rm f}\,(\tilde{h}(w, \tilde{x}) - C_{\rm f}\,\tilde{\xi}) \\ \dot{\chi} &= A_{\rm s}(\chi, C_{\rm s}(\chi, \tilde{y}_{\rm f}) + \tilde{e}, \tilde{y}_{\rm f}) \\ \dot{\tilde{x}} &= \kappa\,H\,\tilde{x} + \varpi(w, \tilde{z}, \tilde{\xi}, \chi, \tilde{e}) \\ \dot{\tilde{\eta}} &= (A_{\rm u} - G_{\rm u} D_{\rm u}^{-1} C_{\rm u})\,\tilde{\eta} + B_{\rm u}\,\tilde{\phi}_\eta(w, \tilde{\eta}) - \Delta(w, \tilde{z}, \tilde{x}) \end{aligned} \tag{26}$$

for some appropriate smooth function $\Delta(w, \tilde{z}, \tilde{x})$, satisfying $\Delta(w, 0, 0) = 0$.

In the previous zero dynamics the subsystem with state $(\tilde{z}, \tilde{\xi}, \chi)$ is, due to Assumption 4, asymptotically and locally exponentially stable at the origin when $\tilde{e} = 0$ with appropriate domain of attraction. Using the fact that the matrix H in the $\tilde{x}$ subsystem is Hurwitz, standard high gain arguments can be used to show that if κ is taken sufficiently large the interconnection of the $\tilde{x}$ subsystem with the$(\tilde{z}, \tilde{\xi}, \chi)$ subsystem is asymptotically and locally exponentially stable at the origin with domain of attraction that contains $\mathcal{C}_z \times \mathcal{C}_\xi \times \mathcal{C}_\chi \times \mathcal{C}_{\tilde{x}}$. Furthermore, by using Theorem 2 of Isidori, Marconi & Praly [2012], it immediately follows that there exists a $G_{\rm u}$ such that the $\tilde{\eta}$ dynamics in (26) with initial conditions in $\mathcal{C}_\eta$ is Input-to-State Stable with respect to the input $(\tilde{z}, \tilde{x})$ with linear gain function. Therefore, the origin of (26) is asymptotically and locally exponentially stable with a domain of attraction containing the compact sets of interest. With this being the case, according to the standard arguments about robust stabilisation of minimum-phase systems (see Isidori [1999]), and using Assumption 1, it is possible to prove that there exists a $\bar{\kappa}^\star > 0$ such that for all $\bar{\kappa} \geq \bar{\kappa}^\star$, the choice of v

$$v = -\bar{\kappa}\,\bar{B}\,\vartheta \tag{27}$$

succeeds in making the origin of the resulting closed-loop system (25)-(27) asymptotically and locally exponentially stable for any $(\tilde{z}(0), x(0), \tilde{\xi}(0), \chi(0), \tilde{\eta}(0)) \in \mathcal{C}_z \times \mathcal{C}_x \times \mathcal{C}_\xi \times \mathcal{C}_\chi \times \mathcal{C}_\eta$.

As for the second step, the partial state feedback stabiliser (27) is then replaced by

$$v = \mathrm{Sat}(-\bar{\kappa}\,B\,\hat{\vartheta}) \tag{28}$$

where $\mathrm{Sat}(\cdot)$ is a smooth vector valued saturation function with saturation level $L > 0$, and $\hat{\vartheta} = \mathrm{col}\,(\hat{\vartheta}_1, \ldots, \hat{\vartheta}_p)$ with $\hat{\vartheta}_i = \alpha_{i1}\tilde{x}_{i1} + \sum_{i=2}^{r_1-1}\alpha_{ij}\kappa^{1-j}\hat{x}_{ij} + \kappa^{1-r_i}\hat{x}_{i,r_i}$, in which $\hat{x}_{i,j}$'s are states of the dirty-derivative high-gain observer

$$\begin{aligned} \dot{\hat{x}}_{ij} &= \hat{x}_{i,j+1} + \ell^j a_{ij}(x_{i1} - \hat{x}_{i1})\,, \\ &\qquad 1 \leq i \leq p\,, 1 \leq j \leq r_i - 1 \\ \dot{x}_{i,r_i} &= \ell^{r_i} a_{i,r_i}(x_{i1} - \hat{x}_{i1}) \end{aligned} \tag{29}$$

Standard arguments (originally presented in Teel & Praly [1995]) can be then used to show that by choosing parameters a_{ij}'s, the saturation level L and the high-gain parameter ℓ appropriately, the resulting pure output feedback regulator preserves the same properties of partial state feedback regulator.

4. CONCLUSIONS

The paper presented preliminary results about the design of internal model-based regulators for a class of nonlinear multivariable systems in which, in addition to regulated errors, additional measurements not necessarily having zero steady state are also available. Strongly motivated by a linear analysis that has shown how, for linear systems, robust regulators based on "pre-processing structures" necessarily have zeros blocking the effect of the extra output in steady state, we proposed a design solution for nonlinear systems in which the extra outputs are filtered with a controller semiglobally stabilising the cascade of the internal model, the plant and the additional filter. The presented framework is the starting point of many future research activities that aim to improve the presented result in several direction. Among the others, the design of constructive design methodologies for stabilising the cascade of the plant and the filter starting from the output feedback stabiliser of the regulated plant is definitely one the research direction under investigation.

REFERENCES

D. Antunes, J. P. Hespanha and C. Silvestre, "Output regulation for non-square linear multi-rate systems," *International Journal of Robust and Nonlinear Control*, vol.24, pp.968-990, 2014.

L. Marconi, and L. Praly, "Uniform Practical Nonlinear Output Regulation," *IEEE Trans. Autom. Contr.*, vol. 53, no. 5, pp.1184-1202, 2008.

A. Isidori. "A tool for semi-global stabilization of uncertain non-minimum-phase nonlinear systems via output feedback," *IEEE Transactions on Automatic Control*, vol.45, no.10, pp.1817-1827, 2000.

L. Marconi, A. Isidori and A. Serrani, "Non-resonance conditions for uniform observability in the problem of nonlinear output regulation", *System & Control Letter*, vol. 53, pp. 281-298, 2004.

L. Wang, C. Wen, L. Marconi and H. Su, "Output Regulation for a Class of Nonlinear Systems not Detectable by Regulated Output," European Control Conference 2018.

L. Wang, A. Isidori, H. Su and L. Marconi. "Nonlinear output regulation for invertible nonlinear MIMO systems," *Internal Journal of Robust and Nonlinear Control*, vol. 26, no. 11, pp. 2401-2417, 2016.

L. Wang, A. Isidori, Z. Liu, and H. Su. "Robust output regulation for invertible nonlinear MIMO systems," *Automatica*, vol.82, pp.278-286, 2017.

B.A. Francis and W. M. Wonham, "The internal model principle of control theory", *Automatica*, vol.12, pp.457-465, 1976.

A. Isidori, L. Marconi and L. Praly. "Robust design of nonlinear internal models without adaptation," *Automatica*, vol.48, pp.2409-2419, 2012.

C.I. Byrnes, A. Isidori. Nonlinear internal models for output regulation. IEEE Transactions on Automatic Control, 49, 2244C2247, 2004.

A. Isidori, C.I. Byrnes. "Output regulation of nonlinear systems," *IEEE Trans. Automatic Control* vol. 25, pp.131-140, 1990.

D. Astolfi, A. Isidori, L. Marconi and L. Praly. Nonlinear output regulation by post-processing internal model for multi-input multi-output systems. *Proceedings of the 9th IFAC Symposium on Nonlinear Control Systems*, 2013.

A. Teel, L. Praly, "Tools for semiglobal stabilization by partial state and output feedback," *SIAM Journal on Control and Optimization*, vol.33, no.5, pp. 1443-1488, 1995.

L. Marconi, L. Praly, A. Isidori. "Output stabilization via nonlinear Luenberger observers," *SIAM Journal on Control and Optimization*, vol.45, no.6, pp.2277-2298, 2007.

F.D. Priscoli, A. Isidori, L. Marconi. "Output regulation with redundant measurements," *SICE Journal of Control, Measurement, and System Integration*, vol.1, no.2, pp. 92-101, 2008.

A. Isidori. *Nonlinear Control Systems II.* New York:Springer,1999.

A. Isidori. *Lectures on Feedback Design for Multivariable Systems.* Springer Verlag, 2017.

J. Huang. *Nonlinear output regulation theory and applications.* SIAM Series: Advances in Design and Control, Cambridge University Press, 2007.