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Three-body interactions in generic fractional quantum Hall systems and impact of Galilean invariance breaking

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We derive full analytic expressions of three-body interactions from Landau-level (LL) mixing in fractional quantum Hall systems with Schrieffer-Wolff transformation. The formalism can be applied to any LL, and to very general systems without rotational or Galilean invariance. We illustrate how three-body pseudopotentials can be readily computed from the analytical expressions for a wide variety of different systems, and show that for realistic systems, softening the bare Coulomb interactions (e.g., finite thickness or screening) can significantly suppress three-body interactions. More interestingly, for experimental systems without Galilean invariance (which is common for real materials), there is strong evidence that higher orders in band dispersion can drive the Moore-Read state from anti-Pfaffian to Pfaffian phase. Our analysis points to the importance of the realistic band structure details to the non-Abelian topological phases, and the analytical expressions we derived can also be very useful for high fidelity numerical computations.

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Introduction. The fractional quantum Hall (FQH) system is a class of two-dimensional strongly correlated electron systems in which the kinetic energy is almost completely quenched by strong magnetic field and low temperature [1,2]. A prototypical example of strongly correlated topological phases, FQH states are the breeding grounds for exotic and potentially highly useful physical phenomena, especially with non-Abelian quasiparticle excitations [3–6]. In the FQH regime, electrons partially fill the topmost Landau level (LL) and stay there most of the time due to the large energy gaps between LLs. More interestingly, while fundamentally only two-body Coulomb interaction can exist between electrons, there can be effective three or more body interactions due to the quenching of the kinetic energies and virtual excitations into higher LLs [7]. Such effective many-body interactions are of utmost importance to some of the most interesting, non-Abelian topological phases in FQH systems [8–10], with rich physical implications and potential practical applications including the design of topological quantum computers.

A number of detailed perturbative and numerical calculations for the effects of LL mixing from realistic interactions have been reported in literature [7,11–17], almost all of which focuses on the three-body interactions relevant to the simplest of all non-Abelian FQH states: the Moore-Read (MR) state [8]. In practice, the perturbative calculations are computationally involved, giving individual three-body pseudopotentials (PPs) which only lead to a partial understanding of the full interaction. It is useful to have full analytic expressions of the three-body interactions from the LL mixing of realistic experimental systems, from which different physical contributions can be transparently interpreted. For numerical analysis, such expressions also allow us to perform computations without using truncated PPs, as such truncations can lead to approximations with unpredictable errors when combined with finite-size effects [12,18–22].

From experimental perspectives, there still remain many challenges in the study of the Moore-Read states, even though the plateau of the Hall conductivity at filling factor $\mu = 5/2$ has been unambiguously observed [23,24]. The main difficulty is because the MR state is quite fragile, with small incompressibility gap requiring very high quality samples [25–28]; and there exists ambiguity between the Pfaffian (Pf) and anti-Pfaffian (APf) phases in realistic systems, which belong to two distinct non-Abelian universality classes at the same filling factor [29,30]. The APf is the particle-hole conjugate of Pf, with the same energy for any two-body interactions. Three-body interactions from LL mixing break particle-hole symmetry and could be decisive on which phase is favored. Recent propositions of the particle-hole Pfaffian (PH-Pf) also lead to more possibilities [31,32]. Given the sensitivity of the nature of the MR states to various experimental details, additional theoretical tools with better insights can go a long way in tackling the experimental difficulties in realizing robust MR states with unambiguous non-Abelian statistics [29,30,33].

In this Rapid Communication, we describe a first quantized formalism in deriving full analytic expressions of effective three-body interactions from LL mixing using the Schrieffer-Wolff (SW) transformation [34]. The formalism can be applied to anisotropic systems [35–40], or systems where LLs are not equally spaced (no Galilean invariance [41]). The analytical expressions show clear and illuminating contributions from realistic bare interactions, LL spectrum, form factors within LL, form factors from virtual excitations into higher LLs, and various metrics in the system. We study some general features of the three-body interactions. In particular, we show that small breaking of the Galilean invariance can significantly drive the interaction from favoring APf to Pf phase. Given that in realistic materials Galilean invariance is naturally broken with higher orders in band dispersion, this

has deep implications on the importance of the detailed band structures for non-Abelian phases at half-filled LLs.

Shrieffer-Wolff transformation. As a form of degenerate perturbative calculation, the (SW transformation for quantum many-body systems is a standard and powerful tool for a number of applications [34]. We start with the full Hamiltonian of the quantum Hall system as follows:

$$H = \sum_i h(a_i^\dagger a_i) + \int d^2q V_{\vec{q}} \sum_{i \neq j} e^{iq_a r_i^a} e^{-iq_a r_j^a} \quad (1)$$

$$= H_0 + H_1 + H_2, \quad (2)$$

where $H_0 = \sum_i h(a_i^\dagger a_i)$ is the kinetic energy with $a_i, a_i^\dagger$ as ladder operators defining the LLs, and $h(x)$ can be nonlinear for systems without Galilean invariance; i, j are particle indices. In Eq. (1) the interaction part comes from the fundamental two-body Coulomb interaction between electrons. For the special case of two-dimensional systems with zero thickness, $V_{\vec{q}} = 1/q$ with $q = |\vec{q}|$, the Fourier component of the Coulomb interaction. We focus on the perturbative limit that the energy scale of H_0 dominates. The complication comes from the fact that while H_0 is diagonal in LLs, it does not commute with the interaction part of H : the Coulomb interaction will mix LLs. We thus separate the interaction energy into two parts (H_1, H_2) in Eq. (2), with $[H_0, H_1] = 0$. Explicitly we have

$$H_2 = \int d^2q V_{\vec{q}} e^{-q^2/2} \sum_{i \neq j} f_q(R_{ij}) \sum' c_{1234}^q \hat{V}_{ij}^{1234}, \quad (3)$$

$$\hat{V}_{ij}^{1234} = (a_i^\dagger)^{n_1} (a_i)^{n_3} (a_j^\dagger)^{n_2} (a_j)^{n_4}, \quad (4)$$

$$f_q(R_{ij}) = e^{iq_x R_{ij}^x + iq_y R_{ij}^y}, \quad (5)$$

$$c_{1234}^q = \frac{(i\mathbf{q})^{n_1} (-i\mathbf{q})^{n_2} (i\mathbf{q}^*)^{n_3} (-i\mathbf{q}^*)^{n_4}}{n_1! n_2! n_3! n_4!}, \quad (6)$$

where $\mathbf{q} = (q_x + iq_y)/\sqrt{2}$, $\sum'$ sums over all terms that do not commute with H_0 , and $R_{ij}^{x,y} = R_i^{x,y} - R_j^{x,y}$ are guiding center coordinates commuting with $a_i, a_i^\dagger$. H_1 is identical to H_2 with the only difference that the summation goes over all terms commuting with H_0 . We would like to construct an effective Hamiltonian $H_{\text{eff}} = e^S H e^{-S}$ that is diagonal in LLs, related to the original Hamiltonian via an SW transformation with the antiunitary operator S . We organize S and the transformation as follows:

$$S = \sum_{n=1}^{\infty} S_n, \quad H_{\text{eff}} = H + [S, H] + \frac{1}{2}[S, [S, H]] + \dots \quad (7)$$

with $S_n \sim (\Delta)^{-n}$, which is the inverse power of the typical energy scale in H_0 . If LLs are equally spaced, $\Delta^{-1} \sim \kappa = e^2/(\hbar\omega_c \epsilon l_B)$, the small parameter commonly used in literature [7,11–17]. Our goal is to keep H_{eff} diagonal in LL order by order. To $\mathcal{O}(1)$ we have $H_{\text{eff}} = H + [S_1, H_0] + \mathcal{O}(\Delta^{-1})$. We

thus need the following condition:

$$H_2 + [S_1, H_0] = 0. \quad (8)$$

This can be solved exactly, with the following explicit expression that can be easily checked by plugging into Eq. (8):

$$S_1 = \int d^2q V_q e^{-q^2/2} \sum_{i \neq j} f_q(R_{ij}) \sum' c_{1234}^q \hat{V}_{ij}^{1234} \hat{G}_{ij}^{1234}, \quad (9)$$

$$\hat{G}_{ij}^{1234} = (h_i^{n_1 n_3} + h_j^{n_2 n_4} - h_i^{00} - h_j^{00})^{-1}, \quad (10)$$

where we have $h_i^{mn} = h(a_i^\dagger a_i - n + m)$, and $[\hat{G}_{ij}^{1234}, H_0] = 0$. So far we obtained $H_{\text{eff}} = H_0 + H_1$ in the limit of $\Delta \rightarrow \infty$. To go beyond the lowest order, we note that at $\mathcal{O}(\Delta^{-1})$ we have

$$[S_1, H_1 + H_2] + \frac{1}{2}[S_1, [S_1, H_0]] = [S_1, H_1 + \frac{1}{2}H_2], \quad (11)$$

where we have used Eq. (8), and Eq. (11) again contains a part that commutes with H_0 . Just like we used S_1 to cancel the part that does not commute with H_0 at $\mathcal{O}(1)$, we now assume that we can use S_2 to cancel the part that does not commute with H_0 at $\mathcal{O}(\Delta^{-1})$. We thus only need to calculate the LL conserving part of Eq. (11), given the explicit expression in Eq. (9). This part contains corrections to both two-body and three-body interactions from LL mixing at the order of Δ^{-1} .

Effective three-body interactions. To extract from Eq. (11) the part that commutes with H_0 , one should note that given $[S_1, H_0] \neq 0$ and $[H_1, H_0] = 0$, we only need to compute $[S_1, H_2]$. With some straightforward algebra [42], both the two-body and three-body interactions can be explicitly evaluated. In this work, we only focus on the three-body interactions, which in the lowest Landau level (LLL) is given as follows:

$$V_{3\text{bdy}}^{(0)}(\vec{q}_1, \vec{q}_2) = V_{\vec{q}_1} V_{\vec{q}_2} \mathcal{J}_0\left(\frac{q_1^2}{2}, \frac{q_2^2}{2}\right) \mathcal{F}_0(\vec{q}_1, \vec{q}_2), \quad (12)$$

$$\mathcal{J}_\alpha(x, y) = e^{-x-y} L_\alpha(x) L_\alpha(y), \quad (13)$$

$$\mathcal{F}_0(\vec{q}_1, \vec{q}_2) = - \sum_{n=1}^{\infty} \frac{(-P_{12})^n}{\Delta_0^n n!} \cos(n\theta_{12} + Q_{12}), \quad (14)$$

Here q_i is the magnitude of $\vec{q}_i$; $\Delta_m^n = h(n) - h(m)$ is the energy difference between LLs; $\theta_{12} = \theta_2 - \theta_1$ is the angle between $\vec{q}_1$ and $\vec{q}_2$; $L_k(x)$ is the k th Laguerre polynomial. We also defined $P_{12} = \frac{1}{2}q_1 q_2$ and $Q_{12} = \frac{1}{2}|\vec{q}_1 \times \vec{q}_2|$. The form factor from virtual excitations in Eq. (14) is a well-behaved function that can be very well approximated by keeping a few terms in the summation. It also has an equivalent compact form expressed in terms of incomplete gamma functions when we have Galilean invariance [42], although the physics is more transparent with the explicit sum. One can also easily introduce nontrivial metrics into Eq. (12) for systems with anisotropy, e.g., with phosphorene systems [43].

FIG. 1. Evolution of three-body PPs as a function of the softening parameter k . (a) $V_{\vec{q}} = 1/\sqrt{q^2 + k^2}$ in LLL. (b) $V_{\vec{q}} = 1/\sqrt{q^2 + k^2}$ in 1LL. (c) $V_{\vec{q}} = e^{-kq}/q$ in LLL. (d) $V_{\vec{q}} = e^{-kq}/q$ in 1LL.

The three-body interaction in the first LL (1LL) can also be readily derived as follows:

$$V_{3\text{bdy}}^{(1)}(\vec{q}_1, \vec{q}_2) = V_{\vec{q}_1} V_{\vec{q}_2} \mathcal{F}_1\left(\frac{q_1^2}{2}, \frac{q_2^2}{2}\right) \mathcal{F}_1(\vec{q}_1, \vec{q}_2), \quad (15)$$

$$\mathcal{F}_1(\vec{q}_1, \vec{q}_2) = -\frac{P_{12}}{\Delta_0^1} \cos(\theta_{12} - Q_{12}) - \sum_{n=1}^{\infty} \frac{(-P_{12})^n}{\Delta_1^{n+1} n!} \cos(n\theta_{12} + Q_{12}) L_{12}^{(n)}, \quad (16)$$

$$L_{12}^{(n)} = n + \frac{P_{12}^2}{n+1} + \frac{1}{2} [L_1(q_1^2) + L_1(q_2^2)]. \quad (17)$$

We can readily verify the correctness of Eqs. (12) and (15) by evaluating the three-body PPs via calculating the energy expectations of antisymmetric three-particle wave functions in LLL and 1LL [44]. We have thoroughly and rigorously performed the checking both for three-body and two-body interaction corrections from LL mixing at $\mathcal{O}(\Delta^{-1})$; the results agree perfectly with those reported in the literature if we take $h(x) = \hbar\omega_c(x + 1/2)$, where ω_c is the cyclotron frequency [16,17,42]. In addition, Eqs. (12)–(15) are valid for any types of bare interactions $V_{\vec{q}}$, both for isotropic and anisotropic systems. They are also valid for any LL spectrum, which is useful for graphenelike systems [45].

Realistic interactions. We first look at Galilean invariant systems with $h(x) = \hbar\omega_c(x + 1/2)$ so that LLs are equally spaced. We can now replace $V_{\vec{q}}$ in Eqs. (12) and (15) with various different experimentally realizable bare interactions. With the explicit three-body interactions, it is very convenient to explore the behaviors of three-body PPs directly. In Fig. 1 we take $V_{\vec{q}} = \frac{1}{q} e^{-kq}$ and $V_{\vec{q}} = \frac{1}{\sqrt{q^2 + k^2}}$, respectively, modeling the two-dimensional electron gas in a sample with a finite

thickness [21], or with screening [46]. One can see that in both cases, increasing the sample thickness or screening reduce the effects of LL mixing. It is thus a general trend that LL mixing is suppressed when the Coulomb interaction is softened one way or another.

For Galilean invariant, isotropic systems, we analyzed a large class of interactions, and found that as long as the bare interaction $V_{\vec{q}}$ decays for large q monotonically, V_3 is generally the dominant three-body PP. The sign of V_3 is of particular importance for the MR state at half-filling. This is because in addition to breaking the PH symmetry, a positive V_3 gives the model Hamiltonian for the Pf state (of which it is the exact zero energy state), and a negative V_3 plus a specific two-body interaction [47] is the model Hamiltonian for the APf state. All the interactions we analyzed have negative V_3 , and all PPs become weaker when the bare interaction is softened (e.g., by the finite thickness). We thus expect that in realistic systems, the non-Abelian topological phase is mostly stabilized by the effective two-body interactions. However, small three-body interactions from LL mixing could be crucial in determining if we have Pf, APf, or PH-Pf phases, or even a mixture of two phases with domain walls [31,32,48–52]. Given that all varieties of bare interactions we have analyzed give negative V_3 , this is strong evidence that in Galilean invariant, isotropic experimental systems, the Pf is *not* the preferred MR state, even with realistic interactions having details not exactly captured in our analysis (e.g., detailed confining potentials in the vertical direction).

Non-Galilean invariant systems. Apart from degrees of freedom such as spins, different experimental systems are fully characterized by the bare interaction profile $V_{\vec{q}}$ (which could be anisotropic relative to the effective mass tensor), and Δ_m^n intrinsically determined by $h(x)$, the single-particle LL spectrum. In general, Galilean invariance is only an approximation in real materials, when the Fermi level is near the bottom (top) of the conduction (valence) band [42]. Higher order terms in momenta generally exist, however, leading to LL spectrums not equally spaced.

It is worth noting from Eqs. (12) and (15) that virtual excitations into different LLs are suppressed in two ways: the factorial suppression from the Taylor expansion of the density operators, and the suppression due to the energy gaps between LLs. The latter can in principle be tuned in experiments, so it is instructive to first look at the “bare contribution” from the virtual excitations to each LL, without the suppression of the LL energy gaps. As one can see from Fig. 2, even without energy suppressions, bare contributions from higher LLs decrease rapidly with the LL index. In the LLL, most of the effects come from virtual excitations into the 1LL, which gives a large negative V_3 contribution. As a result, we do not expect nonlinearity in $h(x)$ could overcome this dominant contribution. We thus expect Pf to be always disfavored in the LLL (if an incompressible phase exists) when other factors are ignored.

It is interesting to see that in the 1LL, the virtual excitation to and from the LLL gives a positive contribution to V_3 , while those to all higher LLs give negative contributions. We can thus in principle tune the LL spacing to enhance the suppression into higher LLs, to bring the overall V_3 from negative to positive. In Fig. 3(a) we plot various three-body

FIG. 2. Contributions from each LL to different three-body PPs, computed from Eqs. (12) and (15), without the energy gap suppressions in Eqs. (14) and (16). Only one term in the summation in Eq. (14) or Eq. (16) is used, indexed by n which is the x axis. (a) Contributions to effective interactions in LLL. (b) Contributions to effective interactions in 1LL; $n = -1$ is given by virtual excitations to and from LLL.

PPs with the simple model of $h(x) = x + 1/2 + k(x + 1/2)^4$, with Coulomb interaction $V_{\vec{q}} = 1/q$. While clearly we have a dominant negative V_3 at $k = 0$, for very small nonlinearity there is a transition to a dominant positive V_3 . If the two-body interactions support a sizable incompressibility gap at half-filling, we take this as strong evidence of a possible APf to Pf transition, driven by the breaking of Galilean invariance, or the higher orders of the band structure. In Fig. 3(b) we look at $h(x) = x + 1/2 + k_1(x + 1/2)^2 + k_2(x + 1/2)^3$, and plot the relative strength of V_3 as a function of Δ_0^1, Δ_1^2 (uniquely determined by k_1, k_2). Contributions from higher LLs can be safely ignored based on Fig. 2, and one can see possible regions where Pf or APf phases can be stabilized by LL mixing. More extensive numerical analyses are needed to further investigate such phase transitions, especially for systems with disorder, particularly with the recent proposal of particle-hole Pfaffian (PH-Pf) [31,32].

Summary and discussions. In this work, we first show that effective three-body interactions from LL mixing have illuminating analytic expressions that can be readily derived from Shrieffer-Wolff transformation in the first quantized form. This allows us to get around sophisticated perturbative

FIG. 3. In 1LL with $V_{\vec{q}} = 1/q$. (a) Three-body PPs with $h(x) = x + 1/2 + k(x + 1/2)^4$. (b) V_3 with 1LL $\rightarrow$ LLL and 1LL $\rightarrow$ 2LL contributions, with different energy gaps. All other LL contributions are ignored because they are small. The two shaded regions are $V_3 > 0.016$ and $|V_3/V_6| > 1.3$, where V_6 is the next largest PP. The numbers are chosen arbitrarily to tentatively indicate the possibility of stabilizing Pf and APf phases.

Feynman diagram computations that are more suitable for computing individual PPs. Moreover, the analytic expressions can be used for a wide variety of physical systems, both with or without rotational or Galilean invariance. Different theoretical models or experimental systems are completely characterized by the single-particle LL spectrum and the bare interactions between electrons, which can be easily plugged into Eq. (12) or Eq. (15). While we only focus on three-body interactions in the lowest two LLs, the formalism works for all LLs and for effective two-body interaction corrections to the leading order of Δ^{-1} as well.

Both Eqs. (12) and (15) could be useful for numerical computations especially on torus geometry. We also found in general three-body interactions from LL mixing are strongest for infinitesimally thin, unscreened Coulomb interaction. Softening the Coulomb interaction tends to significantly suppress three-body interaction. Thus experimentally for MR states at $\nu = 1/2$ or $5/2$, most probably the incompressibility gaps of Pf or APf are dominated by two-body interactions, at least for cases where spin or valley degrees of freedom are unimportant. Nevertheless, for experimental realizations of robust MR states, the breaking of particle-hole symmetry by LL mixing plays an important role. If the three-body interaction only weakly prefers one phase (Pf or APf) over the other, the experimental systems may have complicated arrangements of Pf and APf domains, or PH-Pf phases due to various factors such as disorder [31,32,48–52]. We find that for clean, Galilean invariant systems, physical interactions generally favor the APf phase, although the suppression of the three-body interactions by softening of the Coulomb interaction (very common in experimental systems) can easily make the symmetry breaking ambiguous, which may explain some of the experimental difficulties in realizing very robust APf or Pf phases [53].

We also particularly explored systems without Galilean invariance, which is a common feature in most materials, although not very well investigated before. When the Fermi level moves away from the bottom (top) of the conduction (valence) band, higher order terms (beyond the quadratic term) of the band dispersion are generally present, leading to LLs not equally spaced. Such physics can be explicitly captured in the analytical expressions of three-body interactions we derived. Most interestingly, in 1LL (and higher LLs in general, but not in LLL), we found that higher order terms in the band dispersion can drive the three-body interactions from being dominated by negative V_3 to being dominated by positive V_3 . As long as the Coulomb interaction is not overly softened, this will very possibly lead to a phase transition from APf to Pf via band engineering. We thus propose that detailed band dispersions in the experimental systems deserve more attention (even for cases where quadratic dispersion dominates) for the MR state, because they could be essential ingredients in determining the nature of the topological phases found in experiments at $\nu = 5/2$.

The formalism we have derived is in the limit of zero disorder (clean, translationally invariant systems), while it is recently pointed out [32,50,51] that disorder could play an important role for stabilizing the PH-Pf state. Weak disorder (as compared to electron-electron interaction) that does not mix LL can be added in a straightforward way as one-body orbital

potentials to the effective Hamiltonian within a single LL. For relatively strong disorder such that the induced LL mixing becomes important, more perturbative computations are needed. The study of interplay between LL mixing and correlated disorders in realistic samples could be crucial for the complete understanding of the topological phases at $\mu = 5/2$.

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