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Dynamic analysis with flat-top partition of unity-based discontinuous deformation analysis

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Abstract: The flat-top partition of unity method can obtain a stiffness matrix with a small condition number and avoid the linear dependence problem. We aimed to extend the flat-top partition of unity method to rock dynamic analyses to simulate the influence of discontinuity and dynamic loading. Therefore, the flat-top partition of unity method was coupled with the discontinuous deformation analysis. In this paper, after the presentation of the mesh construction, the inertia force and contact algorithm for flat-top partition of unity-based discontinuous deformation analysis, several numerical examples are presented to verify the accuracy and efficiency of the method.

Keywords: flat-top partition of unity; discontinuous deformation analysis; dynamic analysis; contact algorithm

1 Introduction

Compared with partition of unity (PU)-based numerical methods, flat-top PU-based methods have the following advantages if the local approximation functions are linearly independent: 1) the condition number for the stiffness matrix is smaller [1-3]; 2) the property of linear independence is retained for global approximation space [1, 2, 4]. Therefore, the flat-top PU method has been coupled with other methods for numerical analyses in the continuous problem domain. For example, Oh et al. [5, 6] solved Poisson's equation on the continuous polygonal and polyhedral domains using the so-called generalized product partition of unity, i.e., partition of unity with flat-top. Oh et al. [7] applied flat-top PU-based mesh-free particle methods for approximating the displacements of continuous rectangular, triangular and circular plates. Hong and Lee [8] constructed a flat-top PU mesh within the framework of the Generalized Finite Element Method (GFEM) for analyzing an L-shape domain with a smooth solution. Hong and Lee [9] coupled the flat-top PU method with the Finite Element Method (FEM) and simulated the heat flow in a continuous heat conducting bar. He et al. [10] employed the flat-top PU method to simulate the mechanical behavior of several problems with continuous domain, such as beams with different shapes and plate and table-like structures.

However, in rock engineering, the existence of discontinuity influences the mechanical behavior of materials significantly [11-16]. Therefore, some numerical methods have been improved to simulate discontinuous systems, including the FEM [17, 18], the GFEM [19], the Extended Finite Element Method (X-FEM) [20-22], element free Galerkin methods [23] and the Distinct Element Method (DEM) [24-26]. However, compared with these numerical methods, Discontinuous Deformation Analysis (DDA) [27-31] has the following three advantages: 1) it is fundamentally discontinuous between blocks; 2) it is an implicit displacement method, which is fast, accurate and has correct energy consumption; 3) it complies with the strict postulate of equilibrium and complete kinematic theory.

The response of materials under dynamic loading is quite different from that under static loading, and dynamic loading tends to be more destructive [32, 33]. Different numerical methods have been developed to simulate the dynamic behavior of materials. Thomas [34] utilized FEM to simulate the response of unsaturated seabed under dynamic loading. Fan et al. [35] studied the influence of the incident boundary for wave propagation in jointed rock masses using DEM. Zhao et al. [36] developed the explicit dynamic DDA method by considering the dynamic contact force to improve the simulation efficiency. In this paper, the flat-top PU method is coupled with DDA to implement a dynamic analysis; the proposed method demonstrates its intrinsic advantages in modeling large displacement and

large deformation under dynamic loading, i.e., free from the linear dependence problem, fundamental discontinuous, implicit, strict postulate of equilibrium and complete kinematic. Moreover, the flat-top PU method is used to discretize the continuous parts of the problem domain, instead of dividing the whole area into many small blocks as in DDA. As a result, the time for contact detection and iteration can be reduced. In the flat-top PU-based DDA method, the contact algorithm is improved by determining the exact contact position, thereby increasing the numerical accuracy for contact simulation. Hence, through coupling the flat-top PU method and DDA, the mechanical behavior within each block can be approximated by the flat-top PU method, and the interaction between blocks can be simulated by DDA.

The remainder of this paper is organized as follows. In Section 2, the generation of the numerical model for the discontinuous problem domain is introduced systematically, followed by the derivation of the submatrix generated by inertia force in Section 3. The improved contact algorithm for the flat-top PU mesh is presented in Section 4, and several numerical examples are analyzed in Section 5. Finally, the conclusions are presented in Section 6.

2 Generation of the numerical model

The numerical model for the discontinuous problem domain can be generated in three steps: 1) construction of the flat-top PU mesh-based on the traditional finite element PU mesh; 2) intersection of the flat-top PU mesh with physical boundary and discontinuity; 3) optimization of the numerical model and the calculation of the percentage of the effective area.

2.1 Step 1: Construction of the flat-top PU mesh

Based on the traditional finite element PU mesh, the construction of the flat-top PU mesh is straightforward by extension of the original nodal supports into elemental supports. The traditional finite element PU mesh in a one-dimensional case is illustrated in **Fig. 1** (a), where the PU elements are numbered as e_{pui} ($i = 1, 2, \dots$) in sequence. The 2 end nodes for element e_{pui} are denoted by x_j ($j = i-1, i$). The shape function $w_{puj}(x)$ generated by nodal support x_j can be expressed as:

$$w_{puj}(x) = \begin{cases} (x - x_{j-1}) / (x_j - x_{j-1}) & (x \in (x_{j-1}, x_j)) \\ (x_{j+1} - x) / (x_{j+1} - x_j) & (x \in (x_j, x_{j+1})) \\ 0 & (x \notin (x_{j-1}, x_{j+1})) \end{cases} \quad (1)$$

In Eq. (1), each shape function $w_{puj}(x)$ is equal to one at the corresponding nodal support x_j and linearly decreases to zero at the adjacent nodal supports, as depicted in **Fig. 1** (a).

Next, the nodal support x_j in the traditional finite element PU mesh is extended into elemental support $[x_j, x'_j]$ to construct the flat-top element e_{fij} , as denoted with red color in **Fig. 1** (b). The value of the shape function $w_{fij}(x)$ for each flat-top element e_{fij} is inherited from the corresponding nodal support, which is equal to 1 and is expressed as:

$$w_{fij}(x) = \begin{cases} 1 & (x \in [x_j, x'_j]) \\ 0 & (x \notin [x_j, x'_j]) \end{cases} \quad (2)$$

In the flat-top PU mesh, each flat-top element e_{fij} together with its adjacent PU elements forms a flat-top PU unit u_{fpj} . The adjacent flat-top PU units overlap each other on the PU elements they share. Thus, the continuity between adjacent units can be automatically ensured through the shape functions for PU elements. The shape function w_{fpj} for the unit u_{fpj} includes all the shape functions generated by the elemental support e_{fij} . In the one-dimensional case, the shape function $w_{fpj}(x)$ is expressed as follows:

$$w_{fpj}(x) = \begin{cases} (x - x'_{j-1})/(x_j - x'_{j-1}) & (x \in (x'_{j-1}, x_j)) \\ 1 & (x \in [x_j, x'_j]) \\ (x_{j+1} - x)/(x_{j+1} - x'_j) & (x \in (x'_j, x_{j+1})) \\ 0 & (x \notin (x'_{j-1}, x_{j+1})) \end{cases} \quad (3)$$

As an example, the shape function $w_{fp2}(x)$ for unit u_{fp2} is highlighted as orange lines in **Fig. 1** (b). Note that incomplete flat-top PU units are generated at the boundary.

Fig. 1. Generation of the one-dimensional flat-top PU mesh

Based on the one-dimensional flat-top PU mesh, the two-dimensional flat-top PU mesh can be constructed straightforwardly with the tensor product:

$$(\mathbf{X}, \mathbf{Y}) = \mathbf{X} \otimes \mathbf{Y} \quad (4)$$

where $(\mathbf{X}, \mathbf{Y})$ represents the elements in the two-dimensional flat-top PU mesh, and $\mathbf{X}$ and $\mathbf{Y}$ represent the elements in the one-dimensional flat-top PU mesh with x and y as variables, respectively. As a result, the two-dimensional flat-top PU mesh is generated as in **Fig. 2** (a). The mesh includes grey-colored flat-top elements, blue-colored PU elements and white-colored PU elements. As elemental supports, the flat-top elements are numbered as e_{fik} ($k = 1, 2, \dots$) in sequence. According to Eq. (4), the relationship between the elements in the one- and two-dimensional flat-top PU meshes is presented in **Table 1**.

Table 1 Relationship between the elements in one- and two-dimensional flat-top PU meshes

Two-dimensional element ($\mathbf{X}, \mathbf{Y}$)		Flat-top	Blue- colored PU	Blue- colored PU	White- colored PU
One-dimensional element	$\mathbf{X}$	Flat-top	Flat-top	PU	PU
	$\mathbf{Y}$	Flat-top	PU	Flat-top	PU

In the two-dimensional case, a complete flat-top PU unit u_{fpk} is composed of 1 flat-top element e_{fik} , 4 blue-colored PU elements and 4 white-colored PU elements. The unit u_{fp6} in **Fig. 2** (a) is illustrated as an example, which is surrounded by the red-colored lines. The numbers of elemental supports for a flat-top element, a blue-colored PU element and a white-colored PU element are 1, 2 and 4, respectively. Note that the influence of relative size between flat-top elements and PU elements is negligible [4, 10], as will be further verified in Section 5.

According to Eqs. (3) and (4), in the two-dimensional case, the shape function $w_{fpk}(x, y)$ for the flat-top PU unit u_{fpk} is expressed as:

$$w_{fpk}(x, y) = w_{fpj1}(x) \otimes w_{fpj2}(y) \quad (x \in (x'_{j1-1}, x_{j1+1}), y \in (y'_{j2-1}, y_{j2+1})) \quad (5)$$

Where $w_{fpj2}(y)$ represents the shape function for the one-dimensional flat-top PU unit u_{fpj2} with y as the variable that

takes the same form as Eq. (3). The shape function for a complete two-dimensional flat-top PU unit is depicted in **Fig. 2 (b)**. The shape functions for a flat-top element, a blue-colored PU element and a white-colored PU element are constant, linear function and bilinear function, respectively.

Fig. 2. Two-dimensional flat-top PU mesh and the shape function for a complete flat-top PU unit

2.2 Step 2: Generation of the numerical model

The flat-top PU mesh is fixed over the discontinuous problem domain and intersected by the physical boundary and discontinuity. The problem domain is discretized into two blocks, i.e., Block 1 and Block 2, by the discontinuity, as shown in **Fig. 3 (a)**. The flat-top PU mesh in **Fig. 2 (a)** is utilized to cover the whole problem domain, as shown in **Fig. 3 (b)**. After intersection by the physical boundary and discontinuity, a new flat-top PU mesh is generated, as shown in **Fig. 3 (c)**. The curved line of the physical boundary can be approximated by straight segments for the convenience of numerical analysis. Moreover, if necessary, more vertices can be added in the straight boundary segments to improve the numerical accuracy by approximating the curved boundary line. Each flat-top PU unit cut by the discontinuity is divided into two independent units, which are located within Block 1 and Block 2. For example, the flat-top PU unit u_{fp6} in **Fig. 2 (a)** is divided into two new independent units, i.e., u_{fp6a} and u_{fp6b} , as shown in **Fig. 3 (d)**.

Fig. 3. Flat-top PU mesh intersected by the physical boundary and discontinuity

2.3 Step 3: Optimization of the numerical model

To avoid an ill-conditioned stiffness matrix in the global equilibrium equations, the numerical model is generated by optimizing the flat-top PU mesh shown in **Fig. 3** (c) with the calculation of percentage of effective area for each flat-top PU unit. According to the shape function w_{fpk} for the flat-top PU unit u_{fpk} , the percentage of effective area A_{pek} can be calculated as:

$$A_{pek} = \frac{A_{ek}}{A_e} \times 100\% \quad (6)$$

where A_{ek} is the effective area for unit u_{fpk} and is expressed as:

$$A_{ek} = \iint_{u_{fpk}} w_{fpk} dx dy \quad (7)$$

A_e is the effective area for a complete flat-top PU unit and is expressed as Eq. (7) with flat-top PU unit u_{fpk} as a complete unit.

The percentage of effective area for each flat-top PU unit shown in **Fig. 3** (c) can be calculated according to Eqs. (6) and (7). To avoid an ill-conditioned stiffness matrix in the global equilibrium equations, the flat-top PU mesh is optimized according to the following requirements. 1) The flat-top PU unit u_{fpk} , whose percentage of effective area is less than 3%, will be taken over by its adjacent unit $u_{fpk'}$, whose percentage of effective area is equal to or greater than 3%. 2) The units u_{fpk} and $u_{fpk'}$ should be located within the same continuous block. 3) For the cases in which the unit u_{fpk} has more than one adjacent unit with percentages of effective areas not less than 3%, unit $u_{fpk'}$ should be the

unit that shares the largest common area with unit u_{fpk} . Note that the parameter 3% was determined by trial-and-error in the numerical example in Section 5. If the minimum percentage of effective area in the numerical model is less than 3%, then the numerical results will be unreasonable because of an ill-conditioned stiffness matrix.

Accordingly, in **Fig. 3** (c), three flat-top PU units located within areas surrounded by the red line are to be optimized; these units are enlarged, as illustrated in **Fig. 4** (a) and **Fig. 4** (c). First, in **Fig. 4** (a), the PU element $e_{pu(27)a}$ is shared by two flat-top PU units. One of the units takes the flat-top element e_{fi9a} as elemental support and is denoted by unit u_{fi9a} . Another unit contains only one element $e_{pu(27)a}$ after cutting by the physical boundary and discontinuity and is denoted by $u_{fi(13)a}$. As the percentage of the effective area for the unit $u_{fi(13)a}$ is only 0.7%, according to the requirement of optimization for flat-top PU meshes, the unit $u_{fi(13)a}$ is taken over by its adjacent unit u_{fi9a} , as illustrated in **Fig. 4** (b). Second, for the elements in **Fig. 4** (c), the percentage of effective area for the flat-top PU unit that contains elements $e_{pu(24)b}$, $e_{pu(27)b}$ and e_{pu7} is only 1.4%. Therefore, this unit is taken over by its adjacent unit with the flat-top element $e_{fi(10)b}$ as elemental support. Moreover, the percentage of the effective area for the flat-top PU unit that is composed of elements $e_{pu(27)b}$, e_{pu7} , $e_{fi(13)b}$ and $e_{pu(31)b}$ is only 2.7%, and this unit is taken over by its adjacent unit with flat-top element $e_{fi(14)b}$ as elemental support. Therefore, after optimization, the elements in **Fig. 4** (d) are generated.

Finally, the numerical model is generated, as demonstrated in **Fig. 5**, where the percentage of effective area for every flat-top PU unit is not less than 3%.

Fig. 4. Optimization of the flat-top PU mesh

Fig. 5. Numerical model for the discontinuous problem domain

3 High-order local approximation

In the flat-top PU-based DDA method, a high-order polynomial is assigned to each flat-top PU unit as local displacement approximation, which can be expressed as:

$$\begin{cases} u_{xk} = \sum_{j=1}^m a_{kj} f_j(x, y) \\ u_{yk} = \sum_{j=1}^m b_{kj} f_j(x, y) \end{cases} \quad (8)$$

where (u_{xk}, u_{yk}) are the local displacements for the k th flat-top PU unit u_{fpk} ; (a_{kj}, b_{kj}) represent the coefficients that are unknowns in the global equilibrium equations; m is the number of monomials and can be determined by the degree p of polynomial as:

$$m = \frac{(p+1)(p+2)}{2!} \quad (9)$$

$f_j(x, y) = x^{\alpha_j} y^{\beta_j}$ represents the monomial, and α_j and β_j are the exponents of the bases x and y , respectively. These exponents include all the non-negative integers satisfying the following requirements:

$$\begin{cases} 0 \leq \alpha_j, \beta_j \leq p \\ 0 \leq \alpha_j + \beta_j \leq p \end{cases} \quad (10)$$

For convenience of reference, Eq. (8) is written in matrix form as:

$$[u_k] = [T][D_k] \quad (11)$$

where $[u_k] = (u_{xk}, u_{yk})^T$. $[T]$ is the $2 \times 2m$ matrix for monomials that can be expressed as:

$$[T] = \begin{pmatrix} f_1 & 0 & f_2 & 0 & \cdots & f_m & 0 \\ 0 & f_1 & 0 & f_2 & \cdots & 0 & f_m \end{pmatrix} \quad (12)$$

where $f_j = f_j(x, y)$ for simplification. $[D_k]$ is the $2m \times 1$ matrix for unknowns in the flat-top PU unit u_{fpk} and can be expressed as:

$$[D_k] = (a_{k1} \quad b_{k1} \quad a_{k2} \quad b_{k2} \quad \cdots \quad a_{km} \quad b_{km})^T = (d_{k1} \quad d_{k2} \quad d_{k3} \quad d_{k4} \quad \cdots \quad d_{k(2m-1)} \quad d_{k(2m)})^T \quad (13)$$

Based on the local approximation for flat-top PU units and the corresponding shape functions, the global displacement

approximation is derived from the element level. For an element e_i , which is either a flat-top element or a PU element, its global displacement approximation $[U_i] = (U_{xi}, U_{yi})^T$ can be expressed as:

$$[U_i] = \sum_k w_{fpk}(x, y)[u_k] = \sum_k w_{fpk}(x, y)[T(x, y)][D_k] = \sum_k [T'_k(x, y)][D_k] \quad ((x, y) \in e_i) \quad (14)$$

where $w_{fpk}(x, y) ((x, y) \in e_i)$ represents the shape function for element e_i , which is derived according to the elemental support in the flat-top PU unit u_{fpk} ; $[T'_k(x, y)] = w_{fpk}(x, y)[T(x, y)]$. Because the numbers of elemental supports for a flat-top element, a blue-colored PU element and a white-colored PU element are 1, 2 and 4, respectively, the parameter k in Eq. (14) has 1, 2, and 4 different values for a flat-top element, blue-colored PU element and white-colored PU element, respectively.

4 Inertia force for flat-top PU-based DDA

Compared to a static analysis, inertia force is added in the dynamic analysis. In this section, the submatrix generated by inertia force is derived for the flat-top PU-based DDA method.

According to Eq. (14), at the beginning of current time step, the displacement $[U_i(0)]$ for element e_i can be expressed as:

$$[U_i(0)] = \sum_k [T'_k][D_k(0)] = 0 \quad (15)$$

where $[D_k(0)] = 0$ is the value for unknowns at the beginning of the current time step. Correspondingly, the displacement $[U_i(\Delta t)]$ for element e_i at the end of the current time step can be expressed as:

$$[U_i(\Delta t)] = \sum_k [T'_k][D_k(\Delta t)] \quad (16)$$

where Δt is the time interval, and $[D_k(\Delta t)]$ is the value for unknowns at the end of the current time step.

From Eqs. (15) and (16), $[D_k]$ is found to be a function of Δt . Therefore, Eq. (16) can be approximated by its second-order Taylor series as:

$$\sum_k [T'_k][D_k(\Delta t)] = \sum_k [T'_k][D_k(0)] + (\Delta t) \sum_k [T'_k] \frac{\partial [D_k(0)]}{\partial (\Delta t)} + \frac{(\Delta t)^2}{2} \sum_k [T'_k] \frac{\partial^2 [D_k(0)]}{\partial (\Delta t)^2} \quad (17)$$

According to Eq. (17) and the given condition $[D_k(0)] = 0$, the second-order partial derivative of displacement $[U_i(0)]$, i.e., the acceleration of element e_i at the beginning of the current time step, can be derived as:

$$\sum_k [T'_k] \frac{\partial^2 [D_k(0)]}{\partial (\Delta t)^2} = \frac{2}{(\Delta t)^2} \sum_k [T'_k][D_k(\Delta t)] - \frac{2}{\Delta t} \sum_k [T'_k] \frac{\partial [D_k(0)]}{\partial (\Delta t)} \quad (18)$$

Moreover, the inertia force $[F_i] = (f_{xi}, f_{yi})^T$ for element e_i at the beginning of the current time step is expressed as:

$$\begin{pmatrix} f_{xi} \\ f_{yi} \end{pmatrix} = \begin{pmatrix} -\rho_i \frac{\partial^2 U_{xi}(0)}{\partial (\Delta t)^2} \\ -\rho_i \frac{\partial^2 U_{yi}(0)}{\partial (\Delta t)^2} \end{pmatrix} = -\rho_i \frac{\partial^2 U_i(0)}{\partial (\Delta t)^2} = -\rho_i \sum_k [T'_k] \frac{\partial^2 [D_k(0)]}{\partial (\Delta t)^2} \quad (19)$$

where ρ_i represents the density for element e_i . Substituting Eq. (18) into Eq. (19), the inertia force $[F_i]$ can be

expressed as:

$$[F_i] = -\rho_i \left(\frac{2}{(\Delta t)^2} \sum_k [T'_k] [D_k(\Delta t)] - \frac{2}{\Delta t} \sum_k [T'_k] \frac{\partial [D_k(0)]}{\partial (\Delta t)} \right) = -\rho_i \left(\frac{2}{(\Delta t)^2} \sum_k [T'_k] [D_k(\Delta t)] - \frac{2}{\Delta t} \sum_k [T'_k] [D'_k(0)] \right) \quad (20)$$

where $[D'_k(0)] = \partial [D_k(0)] / \partial (\Delta t)$ represents the coefficients of velocity for flat-top PU unit u_{fpk} at the beginning of the current time step.

In the flat-top PU-based DDA method, for element e_i , the relationships among the velocity at the beginning of the last time step $[V_i(0)]_{-1}$, the velocity at the end of the last time step $[V_i(\Delta t)]_{-1}$ and the displacement at the end of the last time step $[U_i(\Delta t)]_{-1}$ can be expressed as:

$$\begin{aligned} [U_i(\Delta t)]_{-1} &= \frac{\Delta t}{2} ([V_i(0)]_{-1} + [V_i(\Delta t)]_{-1}) \\ &= \frac{\Delta t}{2} ([V_i(0)]_{-1} + [V_i(0)]) \\ &= \frac{\Delta t}{2} \left(\frac{\partial [U_i(0)]_{-1}}{\partial (\Delta t)} + \frac{\partial [U_i(0)]}{\partial (\Delta t)} \right) \\ &= \frac{\Delta t}{2} \left(\sum_k [T'_k] \frac{\partial [D_k(0)]_{-1}}{\partial (\Delta t)} + \sum_k [T'_k] \frac{\partial [D_k(0)]}{\partial (\Delta t)} \right) \\ &= \frac{\Delta t}{2} \left(\sum_k [T'_k] [D'_k(0)]_{-1} + \sum_k [T'_k] [D'_k(0)] \right) \end{aligned} \quad (21)$$

where the velocity at the beginning of the current time step $[V_i(0)]$ is equal to the velocity at the end of the last time step $[V_i(\Delta t)]_{-1}$. The subscript -1 represents the corresponding matrix in the last time step. For element e_i , according to Eq. (16), the displacement at the end of the last time step $[U_i(\Delta t)]_{-1}$ can be expressed as

$$[U_i(\Delta t)]_{-1} = \sum_k [T'_k] [D_k(\Delta t)]_{-1} \quad (22)$$

Combining Eqs. (21) and (22), the following can be derived:

$$\sum_k [T'_k] [D'_k(0)] = \frac{2}{\Delta t} \sum_k [T'_k] [D_k(\Delta t)]_{-1} - \sum_k [T'_k] [D'_k(0)]_{-1} \quad (23)$$

which can be further simplified as:

$$[D'_k(0)] = \frac{2}{\Delta t} [D_k(\Delta t)]_{-1} - [D'_k(0)]_{-1} \quad (24)$$

Therefore, the coefficients of velocity $[D'_k(0)]$ in Eq. (20) can be obtained by the numerical results in the last time step, and the coefficients of velocity at the beginning of the first time step are provided as the initial boundary conditions. According to Eqs. (14) and (24), the potential energy Π_{Fi} for element e_i that is generated by inertia force in Eq. (20) can be expressed as:

$$\begin{aligned}
\Pi_{F_i} &= -\iint_{S_i} (f_{xi} \cdot U_{xi} + f_{yi} \cdot U_{yi}) dx dy \\
&= -\iint_{S_i} \begin{pmatrix} U_{xi} & U_{yi} \end{pmatrix} \begin{pmatrix} f_{xi} \\ f_{yi} \end{pmatrix} dx dy \\
&= \iint_{S_i} \left(\sum_k [T'_k] [D_k] \right)^T \left(\rho_i \begin{pmatrix} \frac{2}{(\Delta t)^2} \sum_k [T'_k] [D_k(\Delta t)] - \\ \frac{2}{\Delta t} \sum_k [T'_k] [D'_k(0)] \end{pmatrix} \right) dx dy \\
&= \frac{2\rho_i}{(\Delta t)^2} \iint_{S_i} \left(\sum_k [T'_k] [D_k] \right)^T \left(\sum_k [T'_k] [D_k] \right) dx dy - \frac{2\rho_i}{\Delta t} \iint_{S_i} \left(\sum_k [T'_k] [D_k] \right)^T \left(\sum_k [T'_k] [D'_k(0)] \right) dx dy \\
&= \frac{2\rho_i}{(\Delta t)^2} \iint_{S_i} \sum_{k1} \sum_{k2} [D_{k1}]^T [T'_{k1}]^T [T'_{k2}] [D_{k2}] dx dy - \frac{2\rho_i}{\Delta t} \iint_{S_i} \sum_{k1} \sum_{k2} [D_{k1}]^T [T'_{k1}]^T [T'_{k2}] [D'_{k2}(0)] dx dy \\
&= \frac{2\rho_i}{(\Delta t)^2} \sum_{k1} \sum_{k2} [D_{k1}]^T \iint_{S_i} [T'_{k1}]^T [T'_{k2}] dx dy [D_{k2}] - \frac{2\rho_i}{\Delta t} \sum_{k1} \sum_{k2} [D_{k1}]^T \iint_{S_i} [T'_{k1}]^T [T'_{k2}] dx dy [D'_{k2}(0)]
\end{aligned} \tag{25}$$

where S_i is the area for element e_i . $[D_k] = [D_k(\Delta t)]$ are unknowns in the current time step when assembling the global equilibrium equations. According to the principle of virtual work, the derivative of Π_{F_i} with respect to the unknown $d_{(k1)r}$ ($r = 1, 2, \dots, 2m$) can be obtained as:

$$\frac{\partial \Pi_{F_i}}{\partial d_{(k1)r}} = k_{(k1)r} - f_{(k1)r} = \frac{\partial}{\partial d_{(k1)r}} \left\{ \frac{2\rho_i}{(\Delta t)^2} \sum_{k1} \sum_{k2} [D_{k1}]^T \iint_{S_i} [T'_{k1}]^T [T'_{k2}] dx dy [D_{k2}] - \frac{2\rho_i}{\Delta t} \sum_{k1} \sum_{k2} [D_{k1}]^T \iint_{S_i} [T'_{k1}]^T [T'_{k2}] dx dy [D'_{k2}(0)] \right\} \tag{26}$$

where

$$k_{(k1)r} = \frac{\partial}{\partial d_{(k1)r}} \left\{ \frac{2\rho_i}{(\Delta t)^2} \sum_{k1} \sum_{k2} [D_{k1}]^T \iint_{S_i} [T'_{k1}]^T [T'_{k2}] dx dy [D_{k2}] \right\} \tag{27}$$

$$f_{(k1)r} = \frac{\partial}{\partial d_{(k1)r}} \left\{ \frac{2\rho_i}{\Delta t} \sum_{k1} \sum_{k2} [D_{k1}]^T \iint_{S_i} [T'_{k1}]^T [T'_{k2}] dx dy [D'_{k2}(0)] \right\} \tag{28}$$

Eq. (27) indicates that the unknown $[D_{k2}]$ is still included after the first-order derivative. Therefore, another derivative of $k_{(k1)r}$ with respect to the unknown $d_{(k2)s}$ ($s = 1, 2, \dots, 2m$) is obtained for the following result:

$$k_{(k1)(k2)rs} = \frac{\partial k_{(k1)r}}{\partial d_{(k2)s}} = \frac{2\rho_i}{(\Delta t)^2} \frac{\partial^2}{\partial d_{(k1)r} \partial d_{(k2)s}} \left\{ \sum_{k1} \sum_{k2} [D_{k1}]^T \iint_{S_i} [T'_{k1}]^T [T'_{k2}] dx dy [D_{k2}] \right\} \tag{29}$$

where $k_{(k1)(k2)rs}$ is added into the element in the r th row and s th column of the stiffness matrix $[K_{(k1)(k2)}]$ in the global equilibrium equations. Note that the elements generated by Eq. (29) are added into the 1, 4 and 16 matrices for the flat-top element, blue-colored PU element and white-colored PU element, respectively. In summary, the elements generated by Eq. (29) are added into the corresponding matrices as follows:

$$\begin{aligned}
\frac{2\rho_i}{(\Delta t)^2} \iint_{S_i} [T'_{k1}]^T [T'_{k1}] dx dy &\rightarrow [K_{(k1)(k1)}] & \frac{2\rho_i}{(\Delta t)^2} \iint_{S_i} [T'_{k1}]^T [T'_{k2}] dx dy &\rightarrow [K_{(k1)(k2)}] \\
\frac{2\rho_i}{(\Delta t)^2} \iint_{S_i} [T'_{k2}]^T [T'_{k1}] dx dy &\rightarrow [K_{(k2)(k1)}] & \frac{2\rho_i}{(\Delta t)^2} \iint_{S_i} [T'_{k2}]^T [T'_{k2}] dx dy &\rightarrow [K_{(k2)(k2)}]
\end{aligned} \tag{30}$$

Moreover, the $f_{(k1)r}$ in Eq. (28) is added into the element in the r th row of the force vector $[F_{k1}]$ in the global equilibrium equations, and the elements generated by Eq. (28) are added into the 1, 2 and 4 vectors for the flat-top element, blue-colored PU element and white-colored PU element, respectively. In summary, the elements generated by Eq. (28) are added into the corresponding matrices as:

$$\frac{2\rho_i}{\Delta t} \sum_{k2} \iint_{S_i} [T'_{k1}]^T [T'_{k2}] dx dy [D'_{k2}(0)] \rightarrow [F_{k1}] \tag{31}$$

5 Contact algorithm for the flat-top PU mesh

In this section, the numerical model in **Fig. 5** is taken as an example to introduce the contact algorithm in the flat-top PU mesh. As shown in **Fig. 6** (a), all the red-colored vertices distributed along the boundary are potential penetration nodes, and all the green-colored sides connecting two adjacent potential penetration nodes are potential penetration lines. In every time step, after solving the global equilibrium equations, all the potential penetration nodes are checked against each potential penetration line. As illustrated in **Fig. 6** (b), the red dot P_1 and green solid line P_2P_3 represent the positions for potential penetration node and potential penetration line, respectively, before contact occurs. The distance d_0 between node P_1 and line P_2P_3 is calculated as:

$$d_0 = \frac{1}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{S}{l} \leq 0 \tag{32}$$

where (x_i, y_i) are the coordinates for vertex P_i ; S represents the 3×3 determinant; and l is the length for the potential penetration line P_2P_3 . After solving the global equilibrium equations, the new coordinates (x'_i, y'_i) for vertex P_i can be obtained as:

$$\begin{cases} x'_i = x_i + U_{xP_i} \\ y'_i = y_i + U_{yP_i} \end{cases} \tag{33}$$

where (U_{xP_i}, U_{yP_i}) represent the displacement for vertex P_i obtained by solving the global equilibrium equations. When substituting the coordinates (x_i, y_i) in Eq. (32) with the new coordinates (x'_i, y'_i) in Eq. (33), if the inequality is still true, then no penetration occurs between the node P_1 and line P_2P_3 . Otherwise, the node P_1 and line P_2P_3 will be determined as the penetration node and penetration line, which are represented by the red circle and green dashed line in **Fig. 6** (b). In this case, a stiff spring is added between node P_1 and line P_2P_3 in the direction normal to line P_2P_3 .

(a) Potential penetration nodes and lines in the numerical model (b) Determination of the contact position

Fig. 6. Contact algorithm in the flat-top PU mesh

In the original DDA method, the stiff spring is added at the beginning of the current time step, i.e., from the moment when the coordinates for vertex P_i are (x_i, y_i) . In this situation, because the stiffness for the contact spring is quite large, energy loss occurs to overcome the excessive constraint, thereby reducing the numerical accuracy, especially during the simulation of collision. Therefore, with the flat-top PU-based DDA method, to improve the numerical accuracy of energy consumption, the exact contact position for node P_1 and line P_2P_3 is calculated, and the stiff spring is added from this exact contact position.

After solving the global equilibrium equations, the penetration distance d_1 , i.e., the distance between the red circle and green dashed line in **Fig. 6** (b) can be expressed as:

$$d_1 = \frac{1}{l} \begin{vmatrix} 1 & x_1 + U_{xP_1} & y_1 + U_{yP_1} \\ 1 & x_2 + U_{xP_2} & y_2 + U_{yP_2} \\ 1 & x_3 + U_{xP_3} & y_3 + U_{yP_3} \end{vmatrix} = \frac{1}{l} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} + \frac{1}{l} \begin{vmatrix} 1 & x_1 & U_{yP_1} \\ 1 & x_2 & U_{yP_2} \\ 1 & x_3 & U_{yP_3} \end{vmatrix} + \frac{1}{l} \begin{vmatrix} 1 & U_{xP_1} & y_1 \\ 1 & U_{xP_2} & y_2 \\ 1 & U_{xP_3} & y_3 \end{vmatrix} + \frac{1}{l} \begin{vmatrix} 1 & U_{xP_1} & U_{yP_1} \\ 1 & U_{xP_2} & U_{yP_2} \\ 1 & U_{xP_3} & U_{yP_3} \end{vmatrix} \quad (34)$$

where $d_1 > 0$. Because the time step Δt is short, the displacement (U_{xP_i}, U_{yP_i}) is so small that the last term in Eq. (34) is a second-order infinitesimal and is negligible. Therefore, Eq. (34) can be simplified as:

$$d_1 = \frac{1}{l} \left(S + \begin{vmatrix} 1 & x_1 & U_{yP_1} \\ 1 & x_2 & U_{yP_2} \\ 1 & x_3 & U_{yP_3} \end{vmatrix} + \begin{vmatrix} 1 & U_{xP_1} & y_1 \\ 1 & U_{xP_2} & y_2 \\ 1 & U_{xP_3} & y_3 \end{vmatrix} \right) \quad (35)$$

According to Eq. (35), the contact between node P_1 and line P_2P_3 occurs when:

$$d_{t_0} = \frac{1}{l} \left(S + \begin{vmatrix} 1 & x_1 & \lambda U_{yP_1} \\ 1 & x_2 & \lambda U_{yP_2} \\ 1 & x_3 & \lambda U_{yP_3} \end{vmatrix} + \begin{vmatrix} 1 & \lambda U_{xP_1} & y_1 \\ 1 & \lambda U_{xP_2} & y_2 \\ 1 & \lambda U_{xP_3} & y_3 \end{vmatrix} \right) = 0 \quad (36)$$

where $\lambda = t_0/\Delta t$, and t_0 is the time needed for the node P_1 and line P_2P_3 to move from the red dot to the purple dot and from the green solid line to the blue solid line, respectively, as shown in **Fig. 6** (b). By solving Eq. (36), the following can be obtained:

$$\lambda = -S / \left(\begin{vmatrix} 1 & x_1 & U_{yP_1} \\ 1 & x_2 & U_{yP_2} \\ 1 & x_3 & U_{yP_3} \end{vmatrix} + \begin{vmatrix} 1 & U_{xP_1} & y_1 \\ 1 & U_{xP_2} & y_2 \\ 1 & U_{xP_3} & y_3 \end{vmatrix} \right) \quad (37)$$

Therefore, at the exact contact position, the coordinates (x''_i, y''_i) for vertex P_i can be calculated as:

$$\begin{cases} x''_i = x_i + \lambda U_{xP_i} \\ y''_i = y_i + \lambda U_{yP_i} \end{cases} \quad (38)$$

Thus, a normal spring with stiffness k_n is added from the moment when the vertex P_i moves to the contact position (x''_i, y''_i) . Therefore, the potential energy Π_{kn} contributed by this stiff spring can be derived as:

$$\begin{aligned} \Pi_{k_n} &= \frac{k_n}{2l^2} \begin{vmatrix} 1 & x''_1 + \lambda' U_x(x_1, y_1) & y''_1 + \lambda' U_y(x_1, y_1) \\ 1 & x''_2 + \lambda' U_x(x_2, y_2) & y''_2 + \lambda' U_y(x_2, y_2) \\ 1 & x''_3 + \lambda' U_x(x_3, y_3) & y''_3 + \lambda' U_y(x_3, y_3) \end{vmatrix}^2 \\ &\approx \frac{k_n}{2l^2} \left(\begin{vmatrix} 1 & x''_1 & \lambda' U_y(x_1, y_1) \\ 1 & x''_2 & \lambda' U_y(x_2, y_2) \\ 1 & x''_3 & \lambda' U_y(x_3, y_3) \end{vmatrix} + \begin{vmatrix} 1 & \lambda' U_x(x_1, y_1) & y''_1 \\ 1 & \lambda' U_x(x_2, y_2) & y''_2 \\ 1 & \lambda' U_x(x_3, y_3) & y''_3 \end{vmatrix} \right)^2 \\ &= \frac{k_n (\lambda')^2}{2l^2} \left\{ (a_1 \quad a_2) \begin{pmatrix} U_x(x_1, y_1) \\ U_y(x_1, y_1) \end{pmatrix} + (a_3 \quad a_4) \begin{pmatrix} U_x(x_2, y_2) \\ U_y(x_2, y_2) \end{pmatrix} + (a_5 \quad a_6) \begin{pmatrix} U_x(x_3, y_3) \\ U_y(x_3, y_3) \end{pmatrix} \right\}^2 \end{aligned} \quad (39)$$

$\lambda' = (1 - \lambda)$; $(\lambda' U_x(x_i, y_i), \lambda' U_y(x_i, y_i))$ represents the displacement of vertex P_i after the node P_1 and line P_2P_3 contact each other. As in Eq. (34), the second-order infinitesimal of displacement is omitted when expanding the determinant in the first row. Because node P_1 is located on line P_2P_3 when they contact each other, the following equality is adopted when expanding the determinant in the first row:

$$\begin{vmatrix} 1 & x''_1 & y''_1 \\ 1 & x''_2 & y''_2 \\ 1 & x''_3 & y''_3 \end{vmatrix} = 0 \quad (40)$$

The coefficients a_i ($i = 1 \sim 6$) can be calculated by the coordinates of the purple dot and the two vertices of the blue solid line in **Fig. 6** (b) as follows:

$$a_1 = y''_2 - y''_3 \quad a_2 = x''_3 - x''_2 \quad a_3 = y''_3 - y''_1 \quad a_4 = x''_1 - x''_3 \quad a_5 = y''_1 - y''_2 \quad a_6 = x''_2 - x''_1 \quad (41)$$

According to Eq. (14), $U(x_i, y_i) = (U_x(x_i, y_i), U_y(x_i, y_i))^T$ can be expressed as:

$$U(x_i, y_i) = \sum_k [T'_{ik}(x_i, y_i)] [D_{ik}] \quad (42)$$

where the subscript ik refers to the k th flat-top PU unit u_{fpk} that contains the element with point P_i as the vertex. Substituting Eq. (42) into Eq. (39), the following can be obtained:

$$\Pi_{k_n} = \frac{k_n (\lambda')^2}{2l^2} \left\{ (a_1 \quad a_2) \sum_k [T'_{1k}(x_1, y_1)] [D_{1k}] + (a_3 \quad a_4) \sum_k [T'_{2k}(x_2, y_2)] [D_{2k}] + (a_5 \quad a_6) \sum_k [T'_{3k}(x_3, y_3)] [D_{3k}] \right\}^2 \quad (43)$$

where $[T'_{2k}(x, y)] = [T'_{3k}(x, y)]$ and $[D_{2k}] = [D_{3k}]$ because points P_2 and P_3 are the two vertices of line P_2P_3 and are located within the same element. Therefore, Eq. (43) can be written as:

$$\Pi_{k_n} = \frac{k_n(\lambda')^2}{2l^2} \left\{ (a_1 \ a_2) \sum_k [T'_{1k}(x_1, y_1)] [D_{1k}] + \sum_k \{ (a_3 \ a_4) [T'_{2k}(x_2, y_2)] + (a_5 \ a_6) [T'_{2k}(x_3, y_3)] \} [D_{2k}] \right\}^2 \quad (44)$$

According to the principle of virtual work, the derivative of Π_{kn} (with respect to the unknowns $[D_{1k}]$ and $[D_{2k}]$) can be obtained as follows:

$$\begin{aligned} & \frac{k_n(\lambda')^2}{l^2} [T'_{1(k1)}(x_1, y_1)]^T (a_1 \ a_2)^T (a_1 \ a_2) [T'_{1(k2)}(x_1, y_1)] \rightarrow [K_{[1(k1)][1(k2)]}] \\ & \frac{k_n(\lambda')^2}{l^2} [T'_{1(k1)}(x_1, y_1)]^T (a_1 \ a_2)^T \{ (a_3 \ a_4) [T'_{2(k2)}(x_2, y_2)] + (a_5 \ a_6) [T'_{2(k2)}(x_3, y_3)] \} \rightarrow [K_{[1(k1)][2(k2)]}] \\ & \frac{k_n(\lambda')^2}{l^2} \{ (a_3 \ a_4) [T'_{2(k1)}(x_2, y_2)] + (a_5 \ a_6) [T'_{2(k1)}(x_3, y_3)] \}^T (a_1 \ a_2) [T'_{1(k2)}(x_1, y_1)] \rightarrow [K_{[2(k1)][1(k2)]}] \\ & \frac{k_n(\lambda')^2}{l^2} \left\{ (a_3 \ a_4) [T'_{2(k1)}(x_2, y_2)] + (a_5 \ a_6) [T'_{2(k1)}(x_3, y_3)] \right\}^T \left\{ (a_3 \ a_4) [T'_{2(k2)}(x_2, y_2)] + (a_5 \ a_6) [T'_{2(k2)}(x_3, y_3)] \right\} \rightarrow [K_{[2(k1)][2(k2)]}] \end{aligned} \quad (45)$$

Based on the different element types, the penetration can be classified into nine groups, i.e., Types I - IX, as listed in **Table 2**.

Within each iteration, the contact force for each previously added stiff spring is calculated, and the spring will be removed if the contact force is determined as tension. The iteration of adding and removing stiff springs continues within the current time step until the requirement of no penetration and no tension is reached.

6 Numerical examples

In this section, three examples are presented to verify the numerical accuracy of the proposed flat-top PU-based DDA method in the dynamic analysis.

6.1 Wave propagation within a half-infinite plane

Because of the property of symmetry for stress wave propagation in a half-infinite plate, only the right half of the problem domain is simulated. The numerical model is illustrated in **Fig. 7**, where the $10\text{ m} \times 10\text{ m}$ plate is discretized into 289 flat-top PU units, including 289 flat-top elements and 800 PU elements. The coordinate system is defined as follows: the origin is located at the lower left corner of the plate, with the x and y axes pointing rightward and upward. The left boundary is fixed in the x direction, and the right boundary and lower boundary are applied with dashpots to prevent stress reflection. The plate is subjected to point loading $P = 3\text{ kN}$ at the point $(0\text{ m}, 10\text{ m})$ from the moment $t = 0\text{ s}$ onwards. The density ρ , Young's modulus E and Poisson's ratio ν for this example are $2.65 \times 10^3\text{ kg/m}^3$, 66.25 GPa and 0.3 , respectively. In the numerical simulation, the time step Δt is set as 0.00001 s , and the local approximation degree p is equal to 2. Point B with coordinates $(0\text{ m}, 5\text{ m})$ is selected as the observation point, and line AC with end nodes $(0\text{ m}, 8\text{ m})$ and $(0\text{ m}, 2\text{ m})$ is recognized as the observation line.

Table 2 Nine types of penetrations

Penetration type	I	II	III	IV	V	VI	VII	VIII	IX
Element type for node P_1	Flat-top	Flat-top	Flat-top	Blue-colored PU	Blue-colored PU	Blue-colored PU	White-colored PU	White-colored PU	White-colored PU
No. of values for k_1	1	1	1	2	2	2	4	4	4
Element type for line P_2P_3	Flat-top	Blue-colored PU	White-colored PU	Flat-top	Blue-colored PU	White-colored PU	Flat-top	Blue-colored PU	White-colored PU
No. of values for k_2	1	2	4	1	2	4	1	2	4
No. of matrices generated by Eq. (45)	4	9	25	9	16	36	25	36	64

Fig. 7. Numerical model for stress wave propagation within a half-infinite plate

According to Fung [37], when $x = 0$ m, the analytical solution for stress σ_y can be expressed as:

$$\sigma_y = -\frac{Q}{2\pi} \mathbf{1}(t - v_1 r) \left[M_1(k_+^{(1)}) \frac{\partial k_+^{(1)}}{\partial t} - M_1(k_-^{(1)}) \frac{\partial k_-^{(1)}}{\partial t} \right] + \frac{2Q}{\pi} \mathbf{1}(t - v_2 r) \left[M_2(k_+^{(2)}) \frac{\partial k_+^{(2)}}{\partial t} - M_2(k_-^{(2)}) \frac{\partial k_-^{(2)}}{\partial t} \right] \quad (46)$$

where $Q = 2P = 6$ kN is the point loading applied in the whole problem domain. The parameters v_1 and v_2 can be calculated as:

$$v_1 = \frac{1}{V_p} = \sqrt{\frac{\rho(1+\nu)(1-2\nu)}{E(1-\nu)}} \quad v_2 = \frac{1}{V_s} = \sqrt{\frac{2\rho(1+\nu)}{E}} \quad (47)$$

where V_p and V_s are the P -wave and S -wave velocities. r represents the distance from wave source (0 m, 10 m). $\mathbf{1}(t)$ is the unit step function defined as:

$$\mathbf{1}(t) = \begin{cases} 0 & (t < 0) \\ 1 & (t \geq 0) \end{cases} \quad (48)$$

when $x = 0$ m, $k_{\pm}^{(1)}$ and $k_{\pm}^{(2)}$ are calculated as:

$$k_{\pm}^{(1)} = \pm \sqrt{\frac{t^2}{r^2} - v_1^2} \quad k_{\pm}^{(2)} = \pm \sqrt{\frac{t^2}{r^2} - v_2^2} \quad (49)$$

and $M_1(k)$ and $M_2(k)$ are expressed as:

$$\begin{cases} M_1(k) = \frac{(2k^2 + v_2^2)^2}{(2k^2 + v_2^2)^2 - 4k^2 \sqrt{(k^2 + v_1^2)(k^2 + v_2^2)}} \\ M_2(k) = \frac{k^2 \sqrt{(k^2 + v_1^2)(k^2 + v_2^2)}}{(2k^2 + v_2^2)^2 - 4k^2 \sqrt{(k^2 + v_1^2)(k^2 + v_2^2)}} \end{cases} \quad (50)$$

From Eq. (47), it is easy to obtain that $v_1 < v_2$. Therefore, Eq. (46) indicates that for an arbitrary point D located over line $x = 0$ m and with distance r away from the wave source: 1) when $0 \leq t < v_1 r$, the stress $\sigma_y = 0$ kPa; 2) when $t = v_1 r$, the leading front of the P -wave arrives at point D and the stress $\sigma_y \rightarrow \infty$; 3) when $v_1 r < t < v_2 r$, the value of stress σ_y is a finite number and is caused by the arrival of P -wave; 4) when $t = v_2 r$, the leading front of the S -wave arrives at point D, and the value of stress σ_y increases to a finite number; 5) when $t > v_2 r$, the value of stress σ_y is a finite number and is caused by the P -wave and S -wave together.

We compared the numerical results for waveform at point B within the first 0.003 s with the analytical solution shown in **Fig. 8**. The trend for stress σ_y in the numerical results is similar with the analytical solution. Based on Eq. (46), when the P -wave reaches point B at $t = 0.00086$ s, the analytical solution increases from 0 kPa to negative infinity; it becomes -0.97 kPa within 0.00001 s. However, it is impossible for the numerical method to demonstrate this stress singularity exactly. The numerical results are found to gradually increase to -1.14 kPa from $t = 0.00082$ s to $t = 0.000105$ s. Afterwards, both the numerical results and the analytical solution tend to decrease and then gradually increase before the arrival of the S -wave. In the analytical solution, the S -wave reaches point B at $t = 0.00161$ s, and the second peak pressure is $\sigma_y = -1.24$ kPa. In the numerical simulation, the stress σ_y reaches its second peak pressure of -1.13 kPa when $t = 0.00149$ s. Afterwards, both the numerical results and the analytical solution decrease slowly. In summary, for the numerical results shown in **Fig. 8**, the overall trend of stress σ_y , the arrival times of the P -wave and S -wave and the value of the second peak pressure agree well with the analytical solution. Although the negative infinity for peak pressure of the P -wave cannot be produced by numerical simulation, the corresponding peak pressure in the numerical results can be up to 17.5% higher than the stress σ_y obtained 0.00001 s after the arrival of the P -wave in the analytical solution.

Fig. 8. Waveform at Point B within first 0.003 s

When the S -wave arrives, the stress σ_y reaches its second peak pressure. The attenuation of the second peak pressure along the observation line AC in the numerical results and in the analytical solution are compared in **Fig. 9**. The numerical results are found to agree well with the analytical solution. Moreover, in general, the difference between these results becomes smaller when the distance from the wave source is longer because, according to Eq. (46), at

the location of the wave source, the stress σ_y is negative infinity, which is a singular point. Therefore, when the observation point is located further away from the wave source, the numerical approximation will be more accurate.

Fig. 9. Attenuation of the second peak pressure along the observation line AC

Furthermore, the numerical results and the analytical solution for the distribution of stress σ_y along the observation line AC when $t = 0.003$ s are compared in **Fig. 10**. The numerical results agree well with the analytical solution. At this stage, the second peak pressure has passed through all the points located over line AC. Therefore, through further comparison with **Fig. 9**, it can be concluded that the difference between the numerical results and analytical solution becomes smaller as time increases, especially for the points located closer to the wave source.

Fig. 10. Distribution of stress σ_y along the observation line AC when $t = 0.003$ s

6.2 Collision

Two blocks with the distance of 1 m apart were simulated as a collision example. The dimensions of these two blocks are illustrated in **Fig. 11**. For both blocks, the density ρ , Young's modulus E and Poisson's ratio ν are 2.65×10^3 kg/m², 66.25 GPa and 0.3, respectively. Both blocks have the same acceleration $a = 10$ m/s² towards each other, and the initial velocity is equal to 0 m/s.

Fig. 11. Problem domain for two collision blocks

For easy derivation of the analytical solutions for the displacement U_x and velocity V_x , it is assumed that these two blocks are non-deformable rigid bodies. Therefore, according to the conservation of total momentum and total kinetic energy before and after collision, the analytical solutions for displacement U_x and velocity V_x before collision can be expressed as follows:

$$\begin{cases} U_x = at^2/2 \\ V_x = at \end{cases} \quad (\text{for Block 1}) \\ \begin{cases} U_x = -at^2/2 \\ V_x = -at \end{cases} \quad (\text{for Block 2}) \quad t \leq \sqrt{d/a} \quad (51)$$

where the direction from Block 1 towards Block 2 is defined as the positive direction. $d = 1 \text{ m}$ is the distance between two blocks. The analytical solutions for displacement U_x and velocity V_x after collision can be expressed as:

$$\begin{cases} U_x = C_1 \cdot \sqrt{d \cdot a} \left(t - \sqrt{d/a} \right) + a \left(t - \sqrt{d/a} \right)^2 / 2 \\ V_x = C_1 \cdot \sqrt{d \cdot a} + a \left(t - \sqrt{d/a} \right) \end{cases} \quad (\text{for Block 1}) \\ \begin{cases} U_x = C_2 \cdot \sqrt{d \cdot a} \left(t - \sqrt{d/a} \right) - a \left(t - \sqrt{d/a} \right)^2 / 2 \\ V_x = C_2 \cdot \sqrt{d \cdot a} - a \left(t - \sqrt{d/a} \right) \end{cases} \quad (\text{for Block 2}) \quad t > \sqrt{d/a} \quad (52)$$

where C_1 and C_2 are coefficients determined by the mass of Block 1 (m_1) and Block 2 (m_2) as follows:

$$C_1 = (m_1 - 3m_2) / (m_1 + m_2) \quad C_2 = (3m_1 - m_2) / (m_1 + m_2) \quad (53)$$

Regarding the numerical analysis, the results obtained from both the flat-top PU-based DDA and original DDA are presented and analyzed. In the flat-top PU-based DDA method, a flat-top PU mesh is covered over the problem domain, as shown in **Fig. 12** (a). After intersection with the physical boundary, the numerical model is generated, as shown in **Fig. 12** (b). There are 2 flat-top PU units within Block 1, including 2 flat-top elements and 1 PU element. There are 4 flat-top PU units within Block 2 containing 4 flat-top elements and 5 PU elements. The 8 red-colored vertices and 6 green-colored sides in **Fig. 12** (b) are recognized as potential penetration nodes and potential penetration lines, respectively. In the original DDA method, the numerical model is composed of two individual blocks, as shown in **Fig. 11**. The time step Δt is set as 0.0005 s in both numerical methods.

In **Fig. 13**, the displacement U_x and velocity V_x at the center of each block within first 0.5 s are shown and compared with the analytical solutions in Eqs. (51) and (52). Before the collision of the two blocks, the numerical results from both methods are found to be almost identical with the corresponding analytical solutions. The maximum error for displacement in the original DDA method is 2.22%, except in the first time step, which is 20% for both blocks. The error of the velocities in the original DDA method is negligible. In the flat-top PU-based DDA method, before

collision, the maximum error for both displacements and velocities is less than 0.00002% at all time steps. Therefore, these two numerical methods are accurate, and the difference between them is negligible before collision.

(a) Two collision blocks covered by a flat-top PU mesh (b) Numerical model in the flat-top PU-based DDA method

Fig. 12. Generation of the numerical model for two colliding blocks

However, after collision, the difference between the two numerical methods is obvious, which can be evidently observed in **Fig. 13**. In the flat-top PU-based DDA method, for both blocks, the displacement difference between the numerical results and analytical solutions increases with time. The velocity difference between them remains relatively the same, and the average differences for velocity were 0.368 m/s and -0.092 m/s for Block 1 and Block 2. Therefore, it is easy to calculate that after the moment of collision, the error for momentum is less than 0.01% and the loss of kinetic energy is approximately 8.97% (unlike the analytical analysis, the blocks in the numerical model are deformable). Therefore, a portion of the kinetic energy is transferred into elastic potential energy. Hence, further study was conducted to enhance the rigidity of the two blocks in the numerical model. If the Young's modulus E increases to 100.00 GPa , then the loss of momentum and kinetic energy can be reduced to 0.006% and 6.12%, respectively. However, in the original DDA method, two blocks move towards the left side with almost the same velocity after collision. The average velocity difference between Block 1 and Block 2 is -0.0008 m/s . After the moment of collision, the error of momentum is negligible. In contrast, the loss of kinetic energy is up to 63.72%, which is much greater than the result obtained using the flat-top PU-based DDA method because, without determining the exact position for contacting in the original DDA method, the stiff spring between the two blocks is added before contact happens. As a result, besides the kinetic energy transferred into elastic potential energy, a large part of the kinetic energy is used to overcome the high resistance force between the two blocks before collision.

(a) Displacement U_x for Block 1

(b) Velocity V_x for Block 1

Fig. 13. Displacement U_x and velocity V_x for Block 1 and Block 2

Therefore, it can be concluded that when simulating collision, the improved contact algorithm in the flat-top PU-based DDA method is more accurate than that in the original DDA method.

6.3 Block sliding on a slope

A slope with the dimensions shown in Figure 5.17 (a) is analyzed in this section. The density ρ , Young's modulus E , Poisson's ratio ν and gravitational acceleration g are $2.65 \times 10^3 \text{ kg/m}^2$, 66.25 GPa , 0.3 and 9.8 m/s^2 , respectively. Block 2 is fixed, and Block 1 slides along the boundary of Block 2 under the effect of gravity. The friction between two blocks is assumed to be equal to zero. Therefore, the analytical solutions of displacement (U_x , U_y) and velocity (V_x , V_y) for Block 1 can be expressed as follows:

$$\begin{cases} U_x = -gt^2/5 \\ U_y = -gt^2/10 \end{cases} \quad \begin{cases} V_x = -2gt/5 \\ V_y = -gt/5 \end{cases} \quad (54)$$

Regarding the numerical analysis, a flat-top PU mesh is utilized to discretize the problem domain, as demonstrated in **Fig. 14** (b). However, after intersected by boundary and discontinuity, the maximum and minimum percentages of the effective area are 99.9% and 0.2%, respectively, resulting in an ill-conditioned stiffness matrix. Therefore, in this example, optimization was conducted regarding the flat-top PU units with percentages of effective area being 0.2% and 0.4%. Consider the flat-top PU unit surrounded by the red-colored lines in **Fig. 14** (b) as an example. Its percentage of effective area is 0.4%, which is less than 3.0%. According to the requirement of optimization in Section 2.3, the flat-top PU unit is taken over by the unit with element $e_{\mu 1}$ as an elemental support. Thus, the new flat-top PU unit is generated, as shown in **Fig. 14** (c); this unit is surrounded by the orange-colored lines.

The optimized numerical model is generated as illustrated in **Fig. 14** (c). At this time, the maximum and minimum percentages of effective area are 99.9% and 3.0%, respectively. There are 4 flat-top PU units within Block 1, including 4 flat-top elements and 5 PU elements. There are 12 flat-top PU units within Block 2, including 12 flat-top elements and 21 PU elements. The 15 red-colored vertices and 13 green-colored sides are recognized as potential penetration nodes and potential penetration lines, respectively. **Fig. 14** (c) shows that some potential penetration nodes and potential penetration lines for Block 1 overlap with some in Block 2. In this numerical simulation, during the sliding of Block 1, all nine types of penetrations, as listed in **Table 2**, are involved.

(a) Slope with a square-shaped block and a right-triangle-shaped block

(b) Flat-top PU mesh for the slope

(c) Numerical model

Fig. 14. Generation of the numerical model for slope analysis

In **Fig. 15**, the numerical results of displacement (U_x , U_y) and velocity (V_x , V_y) at the center of Block 1 within the first 0.9 s are shown and compared with the analytical solution. The numerical results are almost identical to the

corresponding analytical solutions. The errors for horizontal displacement U_x , horizontal velocity V_x , vertical displacement U_y and vertical velocity V_y are 0.00%, 0.00%, 0.03% and 0.16%, respectively, at $t = 0.9$ s. Therefore, the contact algorithm for all nine types of penetrations is efficient and accurate. In the static analysis for this numerical example, the displacement (U_x, U_y) at the center of Block 1 is $(-4.95 \times 10^{-4} \text{ mm}, -6.17 \times 10^{-4} \text{ mm})$, which is much smaller than the dynamic numerical results that belong to the large deformation and large displacement problem.

Fig. 15. Displacement (U_x, U_y) and velocity (V_x, V_y) for Block 1

7 Conclusions

In this paper, the flat-top PU mesh is coupled with the DDA method to achieve the advantages of both methods: 1) obtaining the stiffness matrix with small condition number; 2) avoiding the linear dependence problem; 3) ensuring that it is fundamentally discontinuous between blocks; 4) complying with strict postulate of equilibrium and complete kinematic theory. Furthermore, with the construction of the flat-top PU mesh, it is not necessary to discretize the problem domain into many small discontinuous blocks as in the original DDA method, thereby reducing the computing time for contact detection and adding/subtracting springs. Therefore, only the dominant discontinuities are considered in the proposed method, while the displacement and stress fields within blocks are approximated with the flat-top PU mesh.

After introducing the construction of the flat-top PU mesh within a discontinuous system, the derivation of the inertia force and the contact algorithm for the flat-top PU mesh were presented in detail. Three numerical examples were analyzed to verify the following: 1) the accuracy of numerical approximation with the flat-top PU mesh; 2) the improvement of numerical accuracy for the contact algorithm by determining the exact contact position; 3) the numerical accuracy for nine different types of penetration; 4) the efficiency of optimizing the flat-top PU mesh with the calculation of the percentage of effective area. In conclusion, the flat-top PU-based DDA method is accurate and

efficient when simulating the dynamic response of both continuous and discontinuous systems.

Note that we selected 3 simple verification examples with available analytical solutions. Further studies are required to refine these analysis procedures and parameter settings when applying the method to complex rock engineering problems.

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