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2018

Wang, X., Ang, W. T., & Fan, H. (2018). Effective properties of magnetoelectroelastic interfaces weakened by micro-cracks. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 98(5), 727-748.
doi:10.1002/zamm.201700219

<https://hdl.handle.net/10356/90273>

<https://doi.org/10.1002/zamm.201700219>

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Effective properties of magnetoelastic interfaces weakened by micro-cracks

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Abstract

The effective properties of a microscopically damaged interface between two dissimilar magnetoelastic materials are investigated using a micro-statistical model. The interface is modeled as damaged by a periodic array of micro-cracks. The micro-cracks over a period interval of the interface are randomly generated and are either all (i) magnetically and electrically impermeable, or (ii) magnetically permeable and electrically impermeable, or (iii) electrically permeable and magnetically impermeable, or (iv) magnetically and electrically permeable. The conditions on the micro-cracks are formulated in terms of hypersingular integro-differential equations with the jumps in the elastic displacements, the electric potential and the magnetic potential across opposite micro-crack faces being unknown functions to be determined. Once the unknown functions are determined, the effective properties of the microscopically damaged interface can be readily computed. The effects of the crack density and the material constants on the effective properties are investigated in detail.

Keywords: Micro-statistical model; imperfect interface; micro-cracks; magnetoelastic bimaterial; hypersingular integro-differential equations.

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1 Introduction

Materials exhibiting magneto-electro-elastic coupling effects, namely, magnetoelectroelastic materials, have extensive applications in engineering devices such as smart sensors, actuators and data storage devices (Eerenstein et al. [11] and Schmid [25]). An interface between two dissimilar magnetoelectroelastic materials may be microscopically damaged during the manufacturing and operation process of a composite structure.

At the macroscopic level, a microscopically damaged interface between two magnetoelectroelastic materials denoted by 1 and 2 may be described by a generalized spring model with interfacial conditions given by

$$\left. \begin{aligned} \underline{\underline{\sigma}}^{(1)} \cdot \underline{\mathbf{n}} &= \underline{\underline{\sigma}}^{(2)} \cdot \underline{\mathbf{n}} \\ &= \underline{\underline{\mathbf{k}}} \cdot (\underline{\mathbf{u}}^{(1)} - \underline{\mathbf{u}}^{(2)}) + \underline{\mathbf{b}}(\phi^{(1)} - \phi^{(2)}) + \underline{\mathbf{h}}(\varphi^{(1)} - \varphi^{(2)}) \\ \underline{\mathbf{D}}^{(1)} \cdot \underline{\mathbf{n}} &= \underline{\mathbf{D}}^{(2)} \cdot \underline{\mathbf{n}} \\ &= \underline{\mathbf{c}} \cdot (\underline{\mathbf{u}}^{(1)} - \underline{\mathbf{u}}^{(2)}) + w(\phi^{(1)} - \phi^{(2)}) + s(\varphi^{(1)} - \varphi^{(2)}) \\ \underline{\mathbf{B}}^{(1)} \cdot \underline{\mathbf{n}} &= \underline{\mathbf{B}}^{(2)} \cdot \underline{\mathbf{n}} \\ &= \underline{\mathbf{g}} \cdot (\underline{\mathbf{u}}^{(1)} - \underline{\mathbf{u}}^{(2)}) + d(\phi^{(1)} - \phi^{(2)}) + v(\varphi^{(1)} - \varphi^{(2)}) \end{aligned} \right\} \text{ on } \Gamma, \quad (1)$$

where Γ is the interface between the two materials, $\underline{\mathbf{n}}$ is the unit normal vector to the interface pointing into material 1, $\underline{\mathbf{u}}^{(i)}$ and $\underline{\underline{\sigma}}^{(i)}$ are respectively the elastic displacement and the elastic stress in material i , $\phi^{(i)}$ and $\underline{\mathbf{D}}^{(i)}$ are respectively the electric potential and the electric displacement in material i , $\varphi^{(i)}$ and $\underline{\mathbf{B}}^{(i)}$ are respectively the magnetic potential and the magnetic induction in material i , and the second rank tensor $\underline{\underline{\mathbf{k}}}$, the vectors $\underline{\mathbf{b}}$, $\underline{\mathbf{c}}$, $\underline{\mathbf{g}}$ and $\underline{\mathbf{h}}$ and the scalars w , s , d and v are tensorial quantities characterizing the effective properties of the microscopically damaged interface. Note that an interface defined by (1) is damaged mechanically, electrically and magnetically, that is, the elastic displacement, the electric potential and the magnetic potential are all discontinuous across the interface.

The macro-model in (1) has been widely used for analyzing imperfect interfaces between piezomagnetic/piezoelectric materials (see, for example, Fan et al. [13], Huang and Li [15], Kuo [17], Melkumyan and Mai [20], Nie et al. [22] and Sun et al. [27]). Most papers in the literature assumed that the jumps in the elastic displacement, the electric potential and the magnetic potential across the interface were decoupled from one another in (1), that is, they assumed that $\underline{\mathbf{b}}$, $\underline{\mathbf{c}}$, $\underline{\mathbf{g}}$, $\underline{\mathbf{h}}$, s and d in (1) are null quantities. Only a few papers such as Otero et al. [23] and Shi et al. [26] took into account the coupling effects of the mechanical, electric and magnetic imperfections in the macro-model of the interface.

The macro (effective) properties $\underline{\mathbf{k}}$, $\underline{\mathbf{b}}$, $\underline{\mathbf{c}}$, $\underline{\mathbf{g}}$, $\underline{\mathbf{h}}$, s and d of the interface may be estimated by using micro-models for imperfect interfaces. Currently, there are relatively few papers on micro-models for investigating the effective behaviors of imperfect interfaces. Micro-models for estimating the effective stiffness of micro-cracked interfaces between dissimilar elastic materials may be found in Wang et al. [28, 30, 31]. Recently, Ang et al. [4] and Wang et al. [29] conducted statistical simulations of micro-cracked interfaces between a piezoelectric layer and a piezoelectric half-space to study how the effective properties of the interface may be affected by the coupling of the interfacial jumps in the elastic displacement and the electric potential.

In the current paper, a micro-statistical model is employed to study the behaviors of an imperfect interface between two magnetoelectroelastic half-spaces under inplane static deformations. The interface is modeled as containing a periodical array of micro-cracks. The micro-cracks are either all (i) magnetically and electrically impermeable, or (ii) magnetically permeable and electrically impermeable, or (iii) electrically permeable and magnetically impermeable, or (iv) magnetically and electrically permeable. For statistical

simulations as in Ang et al. [4] and Wang et al. [28, 30, 31], the lengths and positions of a selected number of micro-cracks are randomly generated over a period interval of the interface. The conditions on the micro-cracks are formulated in terms of hypersingular integro-differential equations with the jumps in the elastic displacements, the electric potential and the magnetic potential across opposite micro-crack faces being unknown functions to be determined. Once the hypersingular integro-differential equations are solved numerically, the effective properties of the interface can be readily computed. Case studies are conducted to investigate the effects of the crack density of the interface and the material constants in the magnetoelectroelastic bimaterial on the effective properties of the interface. The validity of the assumption that $\underline{\mathbf{b}}$, $\underline{\mathbf{c}}$, $\underline{\mathbf{g}}$, $\underline{\mathbf{h}}$, s and d in (1) are null quantities is also examined.

2 The problem and basic equations of magnetoelectroelasticity

With reference to a Cartesian coordinate system $Ox_1x_2x_3$, consider two dissimilar magnetoelectroelastic solids occupying the half-spaces $x_2 > 0$ and $x_2 < 0$. The interface $x_2 = 0$ between the materials is microscopically damaged. The geometry of the magnetoelectroelastic bimaterial is independent of the x_3 direction.

At the microscopic level, the interface contains a periodic array of micro-cracks. More specifically, a period interval of the interface contains M arbitrarily distributed micro-cracks of arbitrary lengths. The tips of the m -th micro-cracks on the interval $0 < x_1 < L$, $x_2 = 0$ are given by $(a^{(m)}, 0)$ and $(b^{(m)}, 0)$ ($m = 1, 2, \dots, M$) such that $0 < a^{(1)} < b^{(1)} < a^{(2)} < b^{(2)} < \dots < a^{(M)} < b^{(M)} < L$, where L is the length of a period interval of the interface. On the remaining parts of the interface, the tips of the micro-cracks are given

by $a^{(k)} + nL < x_1 < b^{(k)} + nL$, for $k = 1, 2, \dots, M$ and $n = \pm 1, \pm 2, \dots$. The uncracked parts of the interface are perfectly bonded. The damage ratio or crack density of the interface is defined by

$$\rho = \frac{1}{L} \sum_{m=1}^M (b^{(m)} - a^{(m)}). \quad (2)$$

Refer to Figure 1 for a geometrical sketch of the magneto-electroelastic bi-material for $M = 3$ (three micro-cracks on the period interval $0 < x_1 < L$, $x_2 = 0$, of the interface).

Figure 1. A geometrical sketch for the magneto-electroelastic bimaterial for $M = 3$ (three micro-cracks on the period interval $0 < x_1 < L$, $x_2 = 0$, of the interface).

The bimaterial is assumed to undergo inplane magnetoelastic deformations and the electric and magnetic poling directions are both along the positive x_2 axis in such a way that the elastic displacements u_j , elastic stresses σ_{ij} , electric potential ϕ , electric displacements D_j , magnetic potential φ and magnetic inductions B_j are functions of only x_1 and x_2 and the relevant constitutive equations relating (σ_{ij}, D_j, B_j) and (u_j, ϕ, φ) are given by (see, for example, Alshits et al. [2], Benveniste [7] and Parton and Kudryavtsev [24])

$$\begin{aligned}
 \sigma_{11} &= c_{11}(x_2) \frac{\partial u_1}{\partial x_1} + c_{12}(x_2) \frac{\partial u_2}{\partial x_2} + e_{31}(x_2) \frac{\partial \phi}{\partial x_2} + q_{31}(x_2) \frac{\partial \varphi}{\partial x_2}, \\
 \sigma_{22} &= c_{12}(x_2) \frac{\partial u_1}{\partial x_1} + c_{22}(x_2) \frac{\partial u_2}{\partial x_2} + e_{33}(x_2) \frac{\partial \phi}{\partial x_2} + q_{33}(x_2) \frac{\partial \varphi}{\partial x_2}, \\
 \sigma_{12} &= c_{44}(x_2) \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) + e_{15}(x_2) \frac{\partial \phi}{\partial x_1} + q_{15}(x_2) \frac{\partial \varphi}{\partial x_1}, \\
 D_1 &= e_{15}(x_2) \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) - \kappa_{11}(x_2) \frac{\partial \phi}{\partial x_1} - \alpha_{11}(x_2) \frac{\partial \varphi}{\partial x_1}, \\
 D_2 &= e_{31}(x_2) \frac{\partial u_1}{\partial x_1} + e_{33}(x_2) \frac{\partial u_2}{\partial x_2} - \kappa_{33}(x_2) \frac{\partial \phi}{\partial x_2} - \alpha_{33}(x_2) \frac{\partial \varphi}{\partial x_2}, \\
 B_1 &= q_{15}(x_2) \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) - \alpha_{11}(x_2) \frac{\partial \phi}{\partial x_1} - \mu_{11}(x_2) \frac{\partial \varphi}{\partial x_1}, \\
 B_2 &= q_{31}(x_2) \frac{\partial u_1}{\partial x_1} + q_{33}(x_2) \frac{\partial u_2}{\partial x_2} - \alpha_{33}(x_2) \frac{\partial \phi}{\partial x_2} - \mu_{33}(x_2) \frac{\partial \varphi}{\partial x_2},
 \end{aligned} \tag{3}$$

where $c_{ij}(x_2)$, $e_{ij}(x_2)$, $q_{ij}(x_2)$, $\kappa_{ij}(x_2)$, $\alpha_{ij}(x_2)$ and $\mu_{ij}(x_2)$ are respectively the elastic moduli, piezoelectric coefficients, piezomagnetic coefficients, dielectric constants, electromagnetic coefficients and magnetic permeabilities defined by

$$\begin{aligned}
 &(c_{ij}(x_2), e_{ij}(x_2), q_{ij}(x_2), \kappa_{ij}(x_2), \alpha_{ij}(x_2), \mu_{ij}(x_2)) \\
 &= \begin{cases} (c_{ij}^{(1)}, e_{ij}^{(1)}, q_{ij}^{(1)}, \kappa_{ij}^{(1)}, \alpha_{ij}^{(1)}, \mu_{ij}^{(1)}) & \text{for } x_2 > 0, \\ (c_{ij}^{(2)}, e_{ij}^{(2)}, q_{ij}^{(2)}, \kappa_{ij}^{(2)}, \alpha_{ij}^{(2)}, \mu_{ij}^{(2)}) & \text{for } x_2 < 0, \end{cases}
 \end{aligned} \tag{4}$$

with $c_{ij}^{(p)}$, $e_{ij}^{(p)}$, $q_{ij}^{(p)}$, $\kappa_{ij}^{(p)}$, $\alpha_{ij}^{(p)}$ and $\mu_{ij}^{(p)}$ ($p = 1, 2$) being suitable constants.

According to the conservation of momentum and the Gauss laws of electricity and magnetism, the inplane deformation of the bimaterial is governed by the partial differential equations

$$\begin{aligned}
 & c_{11}(x_2) \frac{\partial^2 u_1}{\partial x_1^2} + (c_{12}(x_2) + c_{44}(x_2)) \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + c_{44}(x_2) \frac{\partial^2 u_1}{\partial x_2^2} \\
 & + (e_{31}(x_2) + e_{15}(x_2)) \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + (q_{31}(x_2) + q_{15}(x_2)) \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = 0, \\
 & c_{44}(x_2) \frac{\partial^2 u_2}{\partial x_1^2} + (c_{44}(x_2) + c_{12}(x_2)) \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + c_{22}(x_2) \frac{\partial^2 u_2}{\partial x_2^2} \\
 & + e_{15}(x_2) \frac{\partial^2 \phi}{\partial x_1^2} + e_{33}(x_2) \frac{\partial^2 \phi}{\partial x_2^2} + q_{15}(x_2) \frac{\partial^2 \varphi}{\partial x_1^2} + q_{33}(x_2) \frac{\partial^2 \varphi}{\partial x_2^2} = 0, \\
 & (e_{15}(x_2) + e_{31}(x_2)) \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + e_{15}(x_2) \frac{\partial^2 u_2}{\partial x_1^2} + e_{33}(x_2) \frac{\partial^2 u_2}{\partial x_2^2} \\
 & - \kappa_{11}(x_2) \frac{\partial^2 \phi}{\partial x_1^2} - \kappa_{33}(x_2) \frac{\partial^2 \phi}{\partial x_2^2} - \alpha_{11}(x_2) \frac{\partial^2 \varphi}{\partial x_1^2} - \alpha_{33}(x_2) \frac{\partial^2 \varphi}{\partial x_2^2} = 0, \\
 & (q_{15}(x_2) + q_{31}(x_2)) \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + q_{15}(x_2) \frac{\partial^2 u_2}{\partial x_1^2} + q_{33}(x_2) \frac{\partial^2 u_2}{\partial x_2^2} \\
 & - \alpha_{11}(x_2) \frac{\partial^2 \phi}{\partial x_1^2} - \alpha_{33}(x_2) \frac{\partial^2 \phi}{\partial x_2^2} - \mu_{11}(x_2) \frac{\partial^2 \varphi}{\partial x_1^2} - \mu_{33}(x_2) \frac{\partial^2 \varphi}{\partial x_2^2} = 0. \quad (5)
 \end{aligned}$$

At the macroscopic level, the micro-cracked interface between the two magnetoelectroelastic half-spaces is considered as a homogeneous interface with constant effective properties.

If the macro-level interface is electrically and magnetically impermeable, it has to be modeled at the micro-level as containing electrically and magnetically impermeable micro-cracks where the electric and magnetic potentials are discontinuous across opposite micro-crack faces. For such an interface, the generalized spring model in (1) can be written in Cartesian coordinates

as

$$\begin{bmatrix} k_{11}^{(\text{imp})} & k_{12}^{(\text{imp})} & b_1^{(\text{imp})} & h_1^{(\text{imp})} \\ k_{21}^{(\text{imp})} & k_{22}^{(\text{imp})} & b_2^{(\text{imp})} & h_2^{(\text{imp})} \\ c_1^{(\text{imp})} & c_2^{(\text{imp})} & w^{(\text{imp})} & s^{(\text{imp})} \\ g_1^{(\text{imp})} & g_2^{(\text{imp})} & d^{(\text{imp})} & v^{(\text{imp})} \end{bmatrix} \begin{bmatrix} \Delta \bar{u}_1(x_1) \\ \Delta \bar{u}_2(x_1) \\ \Delta \bar{\phi}(x_1) \\ \Delta \bar{\varphi}(x_1) \end{bmatrix} = \begin{bmatrix} \bar{\sigma}_{12}(x_1, 0^\pm) \\ \bar{\sigma}_{22}(x_1, 0^\pm) \\ \bar{D}_2(x_1, 0^\pm) \\ \bar{B}_2(x_1, 0^\pm) \end{bmatrix} \quad \text{for } -\infty < x_1 < \infty, \quad (6)$$

where $\Delta \bar{u}_i(x_1) = \bar{u}_i(x_1, 0^+) - \bar{u}_i(x_1, 0^-)$, $\Delta \bar{\phi}(x_1) = \bar{\phi}(x_1, 0^+) - \bar{\phi}(x_1, 0^-)$ and $\Delta \bar{\varphi}(x_1) = \bar{\varphi}(x_1, 0^+) - \bar{\varphi}(x_1, 0^-)$ are respectively the homogenized elastic displacement, electric potential and magnetic potential jumps across the interface and $k_{ij}^{(\text{imp})}$, $b_i^{(\text{imp})}$, $c_i^{(\text{imp})}$, $h_i^{(\text{imp})}$, $g_i^{(\text{imp})}$, $w^{(\text{imp})}$, $s^{(\text{imp})}$, $d^{(\text{imp})}$ and $v^{(\text{imp})}$ are the effective properties of the interface. Note that the overhead bar over the interfacial quantity $f(x_1)$ denotes that the quantity is homogenized over a macroscopic portion of the interface centered about the point x_1 , that is,

$$\bar{f}(x_1) = \frac{1}{2\ell} \int_{x_1-\ell}^{x_1+\ell} f(x) dx, \quad (7)$$

where ℓ is the length of the macroscopic portion.

If the interface is magnetically permeable and electrically impermeable interface at the macro level, the micro-cracks are magnetically permeable and electrically impermeable. The magnetic potential is continuous on the micro-cracks and the interface macro-model in (6) is modified to $\bar{B}_2(x_1, 0^+) = \bar{B}_2(x_1, 0^-)$, $\Delta \bar{\varphi}(x_1) = 0$ and

$$\begin{bmatrix} k_{11}^{(\text{mper})} & k_{12}^{(\text{mper})} & b_1^{(\text{mper})} \\ k_{21}^{(\text{mper})} & k_{22}^{(\text{mper})} & b_2^{(\text{mper})} \\ c_1^{(\text{mper})} & c_2^{(\text{mper})} & w^{(\text{mper})} \end{bmatrix} \begin{bmatrix} \Delta \bar{u}_1(x_1) \\ \Delta \bar{u}_2(x_1) \\ \Delta \bar{\phi}(x_1) \end{bmatrix} = \begin{bmatrix} \bar{\sigma}_{12}(x_1, 0^\pm) \\ \bar{\sigma}_{22}(x_1, 0^\pm) \\ \bar{D}_2(x_1, 0^\pm) \end{bmatrix} \quad \text{for } -\infty < x_1 < \infty, \quad (8)$$

where $k_{ij}^{(\text{mper})}$, $b_i^{(\text{mper})}$, $c_i^{(\text{mper})}$ and $w^{(\text{mper})}$ are the effective properties of the interface.

If the interface is electrically permeable and magnetically impermeable at the macro level, the micro-cracks are electrically permeable and magnetically impermeable. The electric potential is continuous on the interface and the interface macro-model in (6) is modified to $\overline{D}_2(x_1, 0^+) = \overline{D}_2(x_1, 0^-)$, $\Delta\overline{\phi}(x_1) = 0$ and

$$\begin{bmatrix} k_{11}^{(\text{eper})} & k_{12}^{(\text{eper})} & h_1^{(\text{eper})} \\ k_{21}^{(\text{eper})} & k_{22}^{(\text{eper})} & h_2^{(\text{eper})} \\ g_1^{(\text{eper})} & g_2^{(\text{eper})} & v^{(\text{eper})} \end{bmatrix} \begin{bmatrix} \Delta\overline{u}_1(x_1) \\ \Delta\overline{u}_2(x_1) \\ \Delta\overline{\varphi}(x_1) \end{bmatrix} = \begin{bmatrix} \overline{\sigma}_{12}(x_1, 0^\pm) \\ \overline{\sigma}_{22}(x_1, 0^\pm) \\ \overline{B}_2(x_1, 0^\pm) \end{bmatrix} \quad \text{for } -\infty < x_1 < \infty, \quad (9)$$

where $k_{ij}^{(\text{eper})}$, $h_i^{(\text{eper})}$, $g_i^{(\text{eper})}$ and $v^{(\text{eper})}$ are the effective properties.

Lastly, for the magnetically and electrically permeable interface, the magnetic and the electric potentials are both continuous across the interface and the interface macro-model is given by $\overline{D}_2(x_1, 0^+) = \overline{D}_2(x_1, 0^-)$, $\Delta\overline{\phi}(x_1) = 0$, $\overline{B}_2(x_1, 0^+) = \overline{B}_2(x_1, 0^-)$, $\Delta\overline{\varphi}(x_1) = 0$ and

$$\begin{bmatrix} k_{11}^{(\text{per})} & k_{12}^{(\text{per})} \\ k_{21}^{(\text{per})} & k_{22}^{(\text{per})} \end{bmatrix} \begin{bmatrix} \Delta\overline{u}_1(x_1) \\ \Delta\overline{u}_2(x_1) \end{bmatrix} = \begin{bmatrix} \overline{\sigma}_{12}(x_1, 0^\pm) \\ \overline{\sigma}_{22}(x_1, 0^\pm) \end{bmatrix} \quad \text{for } -\infty < x_1 < \infty, \quad (10)$$

where $k_{ij}^{(\text{per})}$ are the effective properties.

The problem of interest is to estimate the effective properties of the interface in the macro-models given in (6), (8), (9) and (10) by taking into account details of the interfacial micro-cracks. The effective properties can be computed once the jumps in the elastic displacement, the electric potential and the magnetic potentials across opposite micro-crack faces are determined by solving a boundary value problem governed by (5) subject to suitably prescribed conditions on the micro-cracks.

3 Boundary value problem

Guided by the notation in Barnett and Lothe [6] for electroelasticity, we define the generalized displacements U_J and stresses S_{Ij} as (see, for example, Athanasius and Ang [5] and Hattori and Sáez [14])

$$\begin{aligned} U_J &= \begin{cases} u_j & \text{for } J = j = 1, 2, \\ \phi & \text{for } J = 3, \\ \varphi & \text{for } J = 4. \end{cases} \\ S_{Ij} &= \begin{cases} \sigma_{Ij} & \text{for } I = i = 1, 2, \\ D_j & \text{for } I = 3, \\ B_j & \text{for } I = 4. \end{cases} \end{aligned} \quad (11)$$

Furthermore, if we define C_{IjKp} as

$$C_{IjKp} = \begin{cases} c_{11} & \text{if } (I, j, K, p) = (1, 1, 1, 1), \\ c_{12} & \text{if } (I, j, K, p) = (1, 1, 2, 2) = (2, 2, 1, 1), \\ c_{22} & \text{if } (I, j, K, p) = (2, 2, 2, 2), \\ c_{44} & \text{if } (I, j, K, p) = (1, 2, 1, 2) = (2, 1, 2, 1) \\ & \quad = (1, 2, 2, 1) = (2, 1, 1, 2), \\ e_{15} & \text{if } (I, j, K, p) = (2, 1, 3, 1) = (1, 2, 3, 1) \\ & \quad = (3, 1, 2, 1) = (3, 1, 1, 2), \\ e_{31} & \text{if } (I, j, K, p) = (1, 1, 3, 2) = (3, 2, 1, 1), \\ e_{33} & \text{if } (I, j, K, p) = (2, 2, 3, 2) = (3, 2, 2, 2), \\ -\kappa_{11} & \text{if } (I, j, K, p) = (3, 1, 3, 1), \\ -\kappa_{33} & \text{if } (I, j, K, p) = (3, 2, 3, 2), \\ q_{15} & \text{if } (I, j, K, p) = (1, 2, 4, 1) = (2, 1, 4, 1) \\ & \quad = (4, 1, 1, 2) = (4, 1, 2, 1), \\ q_{31} & \text{if } (I, j, K, l) = (1, 1, 4, 2) = (4, 2, 1, 1), \\ q_{33} & \text{if } (I, j, K, l) = (2, 2, 4, 2) = (4, 2, 2, 2), \\ -\alpha_{11} & \text{if } (I, j, K, l) = (3, 1, 4, 1) = (4, 1, 3, 1), \\ -\alpha_{33} & \text{if } (I, j, K, l) = (3, 2, 4, 2) = (4, 2, 3, 2), \\ -\mu_{11} & \text{if } (I, j, K, l) = (4, 1, 4, 1), \\ -\mu_{33} & \text{if } (I, j, K, l) = (4, 2, 4, 2), \\ 0 & \text{otherwise,} \end{cases} \quad (12)$$

then (3) and (5) may be respectively written in a more compact form as

$$S_{Ij} = C_{IjKl} \frac{\partial U_K}{\partial x_l} \quad (\text{for } I = 1, 2, 3, 4; j = 1, 2), \quad (13)$$

and

$$C_{IjKl} \frac{\partial^2 U_K}{\partial x_j \partial x_l} = 0 \text{ (for } I = 1, 2, 3, 4). \quad (14)$$

Note that C_{IjKl} are functions of x_2 as given in (4). The Einsteinian convention of summing over repeated indices is applicable here to the uppercase Latin subscripts which run from 1 to 4 and to lowercase Latin subscripts from 1 to 2.

The boundary value problem for the micro-analysis of the micro-cracked interface described in Section 2 requires the partial differential equations in (14) to be solved subject to the continuity conditions

$$\begin{aligned} S_{K2}(x_1, 0^+) &= S_{K2}(x_1, 0^-) \text{ for } -\infty < x_1 < \infty, \\ U_K(x_1, 0^+) &= U_K(x_1, 0^-) \text{ for } (x_1, 0) \text{ on uncracked parts of the interface,} \end{aligned} \quad (15)$$

and the conditions on the micro-cracks given by

$$\left. \begin{aligned} S_{12}(x_1, 0) &= -P_1(x_1) \\ S_{22}(x_1, 0) &= -P_2(x_1) \end{aligned} \right\} \text{ for } (x_1, 0) \text{ on all micro-cracks,} \quad (16)$$

$$\left. \begin{aligned} S_{32}(x_1, 0) &= -P_3(x_1) \\ S_{42}(x_1, 0) &= -P_4(x_1) \end{aligned} \right\} \text{ for } (x_1, 0) \text{ on magnetically and} \\ \text{electrically impermeable micro-cracks,} \quad (17)$$

$$\left. \begin{aligned} S_{32}(x_1, 0) &= -P_3(x_1) \\ U_4(x_1, 0^+) &= U_4(x_1, 0^-) \end{aligned} \right\} \text{ for } (x_1, 0) \text{ on magnetically permeable and} \\ \text{and electrically impermeable micro-cracks,} \quad (18)$$

$$\left. \begin{aligned} U_3(x_1, 0^+) &= U_3(x_1, 0^-) \\ S_{42}(x_1, 0) &= -P_4(x_1) \end{aligned} \right\} \text{ for } (x_1, 0) \text{ on magnetically impermeable} \\ \text{and electrically permeable micro-cracks,} \quad (19)$$

$$\left. \begin{aligned} U_3(x_1, 0^+) &= U_3(x_1, 0^-) \\ U_4(x_1, 0^+) &= U_4(x_1, 0^-) \end{aligned} \right\} \begin{array}{l} \text{for } (x_1, 0) \text{ on magnetically and} \\ \text{electrically permeable micro-cracks,} \end{array} \quad (20)$$

together with the far field conditions

$$S_{Kj}(x_1, x_2) \rightarrow 0 \text{ as } |x_2| \rightarrow \infty, \quad (21)$$

where $P_K(x_1)$ ($K = 1, 2, 3, 4$) are suitably prescribed internal loads on the micro-cracks. The internal loads $P_K(x_1)$ are taken to be periodic functions of x_1 with period L . The component $P_2(x_1)$ is assumed to be sufficiently large to ensure that all the interfacial micro-cracks open up (that is, the micro-cracks are traction free under inplane magnetoelectroelastic loads). For the conditions at far fields in (21), the generalized stresses are required to decay as $O(|x_2|^{-\alpha})$ for large $|x_2|$, where α is a real constant greater than 1.

4 Hypersingular integro-differential equations

If the micro-cracks on the interface are all magnetically and electrically impermeable, the analysis in Wang et al. [30] may be used, as outlined in Appendix A, to formulate the boundary value problem in Section 3 in terms of the hypersingular integro-differential equations

$$\begin{aligned} & \frac{1}{2\pi} \left[\int_{a^{(k)}}^{b^{(k)}} \frac{\Delta U_R(x_1) \operatorname{Re}\{W_{RI}\}}{(x_1 - \xi_1)^2} dx_1 \right. \\ & + \sum_{\substack{n=1 \\ n \neq k}}^M \int_{a^{(n)}}^{b^{(n)}} \frac{\Delta U_R(x_1) \operatorname{Re}\{W_{RI}\}}{(x_1 - \xi_1)^2} dx_1 \\ & + \sum_{n=1}^M \int_{a^{(n)}}^{b^{(n)}} \Delta U_R(x_1) \operatorname{Re}\{W_{RI}\} \Theta(x_1, \xi_1) dx_1 - \frac{d\Delta U_R(\xi_1)}{d\xi_1} \operatorname{Im}\{\pi V_{RI}\} \Big] \\ & = P_I(\xi_1) \text{ for } a^{(k)} < \xi_1 < b^{(k)} \quad (I = 1, 2, 3, 4; \quad k = 1, 2, \dots, M), \end{aligned} \quad (22)$$

where $\oint$ denotes that the integral is to be interpreted in the Hadamard finite-part sense (for details, see Ang [3]), Re and Im denote respectively the real part and the imaginary part of a complex number, $\Delta U_R(x_1) = U_R(x_1, 0^+) - U_R(x_1, 0^-)$, $\Theta(x_1, \xi_1)$ is a real function given by

$$\Theta(x_1, \xi_1) = \frac{1}{L^2} \Psi_1\left(\frac{L + x_1 - \xi_1}{L}\right) + \frac{1}{L^2} \Psi_1\left(\frac{L + \xi_1 - x_1}{L}\right), \quad (23)$$

Ψ_1 is the trigamma function, the constants W_{RI} and V_{RI} are defined by

$$\begin{aligned} W_{RI} &= \sum_{\alpha=1}^4 G_{RI\alpha} + \sum_{\alpha=1}^4 \sum_{\beta=1}^4 H_{RI\alpha\beta}, \\ V_{RI} &= \sum_{\alpha=1}^4 \sum_{\beta=1}^4 H_{RI\alpha\beta} - \sum_{\alpha=1}^4 G_{RI\alpha}, \end{aligned} \quad (24)$$

the constants $G_{RI\alpha}$ and $H_{RI\alpha\beta}$ are given by

$$\begin{aligned} G_{RI\alpha} &= [L_{R2\alpha}^{(1)} N_{\alpha P}^{(1)} (C_{I2M1}^{(1)} + \tau_{\alpha}^{(1)} C_{I2M2}^{(1)})] D_{PM}^{(1)}, \\ H_{RI\alpha\beta} &= [L_{R2\alpha}^{(1)} Q_{\alpha\beta P}^{(1)} (C_{I2M1}^{(1)} + \bar{\tau}_{\beta}^{(1)} C_{I2M2}^{(1)})] D_{PM}^{(1)}, \end{aligned} \quad (25)$$

the overhead bar denotes the conjugate of a complex number, the constants $Q_{\alpha\beta P}^{(1)}$ are implicitly defined by

$$\sum_{\alpha=1}^4 (\bar{A}_{K\alpha}^{(1)} N_{\gamma K}^{(2)} - \bar{L}_{K2\alpha}^{(1)} M_{\gamma K}^{(2)}) \bar{Q}_{\alpha\beta P}^{(1)} = (L_{K2\beta}^{(1)} M_{\gamma K}^{(2)} - A_{K\beta}^{(1)} N_{\gamma K}^{(2)}) N_{\beta P}^{(1)}, \quad (26)$$

the constants $L_{I2\alpha}^{(p)}$ ($p = 1, 2$) are given by

$$L_{I2\alpha}^{(p)} = (C_{I2K1}^{(p)} + \tau_{\alpha}^{(p)} C_{I2K2}^{(p)}) A_{K\alpha}^{(p)}, \quad (27)$$

the constants $C_{IjKp}^{(1)}$ and $C_{IjKp}^{(2)}$ denote respectively the material constants C_{IjKp} for the upper and lower half-spaces, $\tau_1^{(p)}$, $\tau_2^{(p)}$, $\tau_3^{(p)}$ and $\tau_4^{(p)}$ are complex constants with positive imaginary parts and they are solutions of the octic equations in $\tau^{(p)}$ given by

$$\det \left[C_{I1K1}^{(p)} + (C_{I1K2}^{(p)} + C_{I2K1}^{(p)}) \tau^{(p)} + C_{I2K2}^{(p)} (\tau^{(p)})^2 \right] = 0, \quad (28)$$

the constants $A_{K\alpha}^{(p)}$ are non-trivial solutions of the homogeneous linear equations

$$\left[C_{I1K1}^{(p)} + (C_{I1K2}^{(p)} + C_{I2K1}^{(p)})\tau_{\alpha}^{(p)} + C_{I2K2}^{(p)}(\tau_{\alpha}^{(p)})^2 \right] A_{K\alpha}^{(p)} = 0, \quad (29)$$

the matrices $\left[N_{\gamma K}^{(p)} \right]$ and $\left[M_{\gamma K}^{(p)} \right]$ are respectively the inverses of $\left[A_{\gamma K}^{(p)} \right]$ and $\left[L_{I2\alpha}^{(p)} \right]$, and the constants $D_{PM}^{(p)}$ are implicitly defined by

$$\text{Im} \left\{ \sum_{\alpha=1}^4 L_{I2\alpha}^{(p)} N_{\alpha R}^{(p)} \right\} D_{RP}^{(p)} = \delta_{IP}, \quad (30)$$

with δ_{IP} being the Kronecker-delta. Note that Greek subscripts run from 1 to 4 and the Einsteinian summation convention for repeated subscripts does not hold for Greek subscripts.

If the interfacial micro-cracks are magnetically permeable and electrically impermeable, the hypersingular integro-differential equations for the boundary value problem are given by (22) for $I = 1, 2, 3$ (instead of $I = 1, 2, 3, 4$) with $\Delta U_4(x_1) = 0$.

If the micro-cracks are electrically permeable and magnetically impermeable, the hypersingular integro-differential equations are given by (22) for $I = 1, 2, 4$, with $\Delta U_3(x_1) = 0$.

Lastly, the hypersingular integro-differential equations for magnetically and electrically permeable micro-cracks are given by (22) for $I = 1, 2$, with $\Delta U_3(x_1) = \Delta U_4(x_1) = 0$.

5 Computing the effective properties of an interface

The computation of the effective properties of an interface requires the hypersingular integro-differential equations in Section 4 to be solved. The equa-

tions may be solved numerically by using the technique of Kaya and Erdogan [16] outlined in Appendix B.

For a magnetically and electrically impermeable interface, the hypersingular integro-differential equations are using for four independent sets of magneto-electroelastic loads given by $P_I(x_1) = \sigma_I^{(q)}$ ($I = 1, 2, 3, 4; q = 1, 2, 3, 4$) in (22) for the unknown functions $\Delta U_R(x_1)$. We take $\sigma_I^{(q)}$ as positive constants. If the solution of the hypersingular integro-differential equations corresponding to $P_I(x_1) = \sigma_I^{(q)}$ is given by $\Delta U_R(x_1) = \Delta U_R^{(q)}(x_1)$, then the effective properties $k_{ij}^{(\text{imp})}$, $b_i^{(\text{imp})}$, $c_i^{(\text{imp})}$, $h_i^{(\text{imp})}$, $g_i^{(\text{imp})}$, $w^{(\text{imp})}$, $s^{(\text{imp})}$, $d^{(\text{imp})}$ and $v^{(\text{imp})}$ may be obtained by solving simultaneously the system of 16 equations

$$\begin{bmatrix} k_{11}^{(\text{imp})} & k_{12}^{(\text{imp})} & b_1^{(\text{imp})} & h_1^{(\text{imp})} \\ k_{21}^{(\text{imp})} & k_{22}^{(\text{imp})} & b_2^{(\text{imp})} & h_2^{(\text{imp})} \\ c_1^{(\text{imp})} & c_2^{(\text{imp})} & w^{(\text{imp})} & s^{(\text{imp})} \\ g_1^{(\text{imp})} & g_2^{(\text{imp})} & d^{(\text{imp})} & v^{(\text{imp})} \end{bmatrix} \cdot \frac{1}{L} \sum_{k=1}^M \int_{a^{(k)}}^{b^{(k)}} \begin{bmatrix} \Delta U_1^{(q)}(x_1) \\ \Delta U_2^{(q)}(x_1) \\ \Delta U_3^{(q)}(x_1) \\ \Delta U_4^{(q)}(x_1) \end{bmatrix} dx_1 = \begin{bmatrix} \sigma_1^{(q)} \\ \sigma_2^{(q)} \\ \sigma_3^{(q)} \\ \sigma_4^{(q)} \end{bmatrix} \quad \text{for } q = 1, 2, 3, 4. \quad (31)$$

For a magnetically permeable and electrically impermeable interface, the hypersingular integro-differential equations in (22) for $I = 1, 2, 3$ together with $\Delta U_4(x_1) = 0$ are solved numerically using three sets of independent loads $P_I(x_1) = \sigma_I^{(q)}$ ($I = 1, 2, 3; q = 1, 2, 3$) for the unknown functions $\Delta U_1(x_1)$, $\Delta U_2(x_1)$ and $\Delta U_3(x_1)$. Once the solution $\Delta U_R(x_1) = \Delta U_R^{(q)}(x_1)$ is obtained, the required effective properties $k_{ij}^{(\text{mper})}$, $b_i^{(\text{mper})}$, $c_i^{(\text{mper})}$ and $w^{(\text{mper})}$

may then be computed by solving the simultaneous system

$$\begin{bmatrix} k_{11}^{(\text{mper})} & k_{12}^{(\text{mper})} & b_1^{(\text{mper})} \\ k_{21}^{(\text{mper})} & k_{22}^{(\text{mper})} & b_2^{(\text{mper})} \\ c_1^{(\text{mper})} & c_2^{(\text{mper})} & w^{(\text{mper})} \end{bmatrix} \cdot \frac{1}{L} \sum_{k=1}^M \int_{a^{(k)}}^{b^{(k)}} \begin{bmatrix} \Delta U_1^{(q)}(x_1) \\ \Delta U_2^{(q)}(x_1) \\ \Delta U_3^{(q)}(x_1) \end{bmatrix} dx_1 \\ = \begin{bmatrix} \sigma_1^{(q)} \\ \sigma_2^{(q)} \\ \sigma_3^{(q)} \end{bmatrix} \quad \text{for } q = 1, 2, 3. \quad (32)$$

For an electrically permeable and magnetically impermeable interface, we solve (22) for $I = 1, 2, 4$ together with $\Delta U_3(x_1) = 0$ using $P_I(x_1) = \sigma_I^{(q)}$ ($I = 1, 2, 4$; $q = 1, 2, 3$) to determine the required effective properties $k_{ij}^{(\text{eper})}$, $h_i^{(\text{eper})}$, $g_i^{(\text{eper})}$ and $v^{(\text{eper})}$ from the system

$$\begin{bmatrix} k_{11}^{(\text{eper})} & k_{12}^{(\text{eper})} & h_1^{(\text{eper})} \\ k_{21}^{(\text{eper})} & k_{22}^{(\text{eper})} & h_2^{(\text{eper})} \\ g_1^{(\text{eper})} & g_2^{(\text{eper})} & v^{(\text{eper})} \end{bmatrix} \cdot \frac{1}{L} \sum_{k=1}^M \int_{a^{(k)}}^{b^{(k)}} \begin{bmatrix} \Delta U_1^{(q)}(x_1) \\ \Delta U_2^{(q)}(x_1) \\ \Delta U_4^{(q)}(x_1) \end{bmatrix} dx_1 \\ = \begin{bmatrix} \sigma_1^{(q)} \\ \sigma_2^{(q)} \\ \sigma_4^{(q)} \end{bmatrix} \quad \text{for } q = 1, 2, 3. \quad (33)$$

Lastly, for a magnetically and electrically permeable interface, the hypersingular integro-differential equations in (22) for $I = 1, 2$, together with $\Delta U_3(x_1) = 0$ and $\Delta U_4(x_1) = 0$, are solved for only two sets of independent loads $P_I(x_1) = \sigma_I^{(q)}$ ($I = 1, 2$; $q = 1, 2$) to obtain

$$\begin{bmatrix} k_{11}^{(\text{per})} & k_{12}^{(\text{per})} \\ k_{21}^{(\text{per})} & k_{22}^{(\text{per})} \end{bmatrix} \cdot \frac{1}{L} \sum_{k=1}^M \int_{a^{(k)}}^{b^{(k)}} \begin{bmatrix} \Delta U_1^{(q)}(x_1) \\ \Delta U_2^{(q)}(x_1) \end{bmatrix} dx_1 = \begin{bmatrix} \sigma_1^{(q)} \\ \sigma_2^{(q)} \end{bmatrix} \\ \text{for } q = 1, 2. \quad (34)$$

6 Statistical simulations

The statistical approach in Wang et al. [28], [30] and [31] is used here to estimate the effective properties of the interface.

For an interface damaged by magnetically and electrically impermeable micro-cracks of average half length $\hat{a}$, the non-dimensionalized effective properties of the interface given by $\hat{a}k_{ij}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_i^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}c_i^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_i^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}g_i^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$, $\hat{a}d^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ depend only on the damage ratio ρ defined in (2) and the properties of the magnetoelectroelastic materials in the half-spaces. To estimate these non-dimensionalized effective properties, a sample of N micro-cracked interfaces of the same damage ratio ρ are randomly generated.

To generate a micro-cracked interface randomly, the M interfacial micro-cracks on the interface are chosen to have randomly generated lengths and positions over the period interval $0 < x_1 < L$. More specifically, the lengths of the micro-cracks are chosen to follow a χ^2 distribution of a degree of freedom m , that is, $\chi^2(m)$. The $\chi^2(m)$ distribution provides a greater number of shorter micro-cracks if the degree of freedom m has a smaller value, like $m = 5$. Such a distribution (that is, a $\chi^2(m)$ distribution of a smaller value of m) can give rise to a more realistic simulation for the variation of micro-crack lengths. Thus, we use $\chi^2(5)$ to generate the lengths of the M micro-cracks. Once the lengths are generated, the micro-cracks are randomly positioned over the interval $0 < x_1 < L$. The micro-cracks on the remaining part of the interface are obtained by shifting each of the randomly generated micro-cracks to the left and right by a distance of nL ($n = 1, 2, \dots$).

The non-dimensionalized effective properties of each of the N randomly generated interfaces in the sample are calculated as explained in Section 5. The final value of a non-dimensionalized effective property is obtained by averaging the values of the non-dimensionalized effective property computed for the N randomly generated interfaces in the sample.

The statistical approach above is also used to estimate the non-dimensionalized effective properties $\hat{a}k_{ij}^{(\text{mper})}/c_{44}^{(2)}$, $\hat{a}b_i^{(\text{mper})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}c_i^{(\text{mper})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$ and $\hat{a}w^{(\text{mper})}/(-\kappa_{11}^{(2)})$ for a magnetically permeable and electrically impermeable interface, $\hat{a}k_{ij}^{(\text{eper})}/c_{44}^{(2)}$, $\hat{a}h_i^{(\text{eper})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}g_i^{(\text{eper})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ and $\hat{a}v^{(\text{eper})}/(-\mu_{11}^{(2)})$ for an electrically permeable and magnetically impermeable interface, and $\hat{a}k_{ij}^{(\text{per})}/c_{44}^{(2)}$ for a magnetically and electrically permeable interface.

Elvin [12] performed numerical experiments to determine the minimum number of grains needed in statistical simulations to calculate the effective elastic properties of polycrystalline ice, that is, the number of grains required to homogenize the elastic properties. Such a homogenization approach is adopted here to determine the minimum number of micro-cracks (over one period interval of the interface) that should be used in estimating the effective property of the micro-cracked interface.

For fixed values of ρ and material constants, if the number of micro-cracks (M) over one period interval of the interface is greater than a certain positive integer M_0 , which denotes the number of micro-cracks required for homogenizing the interface, the effective properties of the interface calculated from a sample containing a sufficiently large number of interfaces should not change very much as M increases. As in [28], [30] and [31], we perform numerical simulations for wide ranges of ρ and material constants, with 50 interfaces in each simulation sample (that is, $N = 50$), and find that the estimated effective property of the interface does not vary significantly as M exceeds 40. Thus, 40 micro-cracks over one period interval of the interface may be used to homogenize the effective properties of the interface, that is, 40 randomly generated micro-cracks over a period interval of the interface are sufficient to estimate the effective property of the interface.

7 Specific case studies

In this section, statistical simulations are conducted to investigate how the effective properties of a micro-cracked interface are influenced by the damage ratio of the interface and the properties of the magneto-electroelastic materials in the half-spaces. In each simulation, we use a sample of 50 randomly generated interfaces of the same damage ratio ρ , each of which contains 40 randomly generated micro-cracks over a period length of the interface, with the length of a micro-crack following the $\chi^2(5)$ distribution. Note that the choice of 40 micro-cracks over a period interval of the interface is determined by the homogenization procedure described in the last paragraph of Section 6 and that the central limit theorem in statistics (Mendenhall et al. [21]) requires a sample size of at least 30 randomly generated interfaces for the data for the effective property of the interfaces in the sample to be normally distributed.

In all the simulations, we find that the values of $\hat{a}k_{ij}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{ij}^{(\text{imper})}/c_{44}^{(2)}$, $\hat{a}k_{ij}^{(\text{eper})}/c_{44}^{(2)}$ and $\hat{a}k_{ij}^{(\text{per})}/c_{44}^{(2)}$ are very close to one another, the values of $\hat{a}b_i^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}c_i^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$ and $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$ are very close to $\hat{a}b_i^{(\text{imper})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}c_i^{(\text{imper})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$ and $\hat{a}w^{(\text{imper})}/(-\kappa_{11}^{(2)})$ respectively, and the values of $\hat{a}h_i^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}g_i^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ are very close to the values of $\hat{a}h_i^{(\text{eper})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}g_i^{(\text{eper})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ and $\hat{a}v^{(\text{eper})}/(-\mu_{11}^{(2)})$ respectively. Thus, in all the case studies below, we present only the effective properties for magnetically and electrically impermeable micro-cracked interfaces. The effective properties of the other types of micro-cracked interfaces can be directly deduced from the results presented.

Furthermore, the values of $\hat{a}k_{12}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{21}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_1^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}c_1^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_1^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ and $\hat{a}g_1^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ are found to be

extremely small in the order of 10^{-12} or even smaller, having the negligible magnitudes compared to the other effective properties. We also observe that the values of $c_2^{(\text{imp})}$, $g_2^{(\text{imp})}$ and $d^{(\text{imp})}$ are extremely close to the values of $b_2^{(\text{imp})}$, $h_2^{(\text{imp})}$ and $s^{(\text{imp})}$ respectively. Thus, only the effective properties $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ are presented in the case studies below.

In all the case studies reported below, the lower half-space is occupied by the magnetoelectroelastic composite $\text{BaTiO}_3\text{--CoFe}_2\text{O}_4$ with material constants given by (Zhong et al. [32])

$$\begin{aligned} c_{11}^{(2)} &= 226 \times 10^9, c_{12}^{(2)} = 124 \times 10^9, c_{22}^{(2)} = 216 \times 10^9, c_{44}^{(2)} = 44.0 \times 10^9, \\ e_{15}^{(2)} &= 5.80, e_{31}^{(2)} = -2.20, e_{33}^{(2)} = 9.30, q_{15}^{(2)} = 275, q_{31}^{(2)} = 290, \\ q_{33}^{(2)} &= 350, \kappa_{11}^{(2)} = 5.64 \times 10^{-9}, \kappa_{33}^{(2)} = 6.35 \times 10^{-9}, \alpha_{11}^{(2)} = 0.00537 \times 10^{-9}, \\ \alpha_{33}^{(2)} &= 2.74 \times 10^{-9}, \mu_{11}^{(2)} = 297000 \times 10^{-9}, \mu_{33}^{(2)} = 83500 \times 10^{-9}, \end{aligned} \quad (35)$$

where the above values of $c_{11}^{(2)}$, $c_{12}^{(2)}$, $c_{22}^{(2)}$ and $c_{44}^{(2)}$ are in N/m^2 , $e_{15}^{(2)}$, $e_{31}^{(2)}$ and $e_{33}^{(2)}$ in C/m^2 , $q_{15}^{(2)}$, $q_{31}^{(2)}$ and $q_{33}^{(2)}$ are in $\text{N}/(\text{Am})$, $\kappa_{11}^{(2)}$ and $\kappa_{33}^{(2)}$ in $\text{C}^2/(\text{Nm}^2)$, $\alpha_{11}^{(2)}$ and $\alpha_{33}^{(2)}$ in $\text{Ns}/(\text{VC})$ and $\mu_{11}^{(2)}$ and $\mu_{33}^{(2)}$ in Ns^2/C^2 . For parametric studies, selected property coefficients of the material in the upper half-space are taken to vary relative to the corresponding coefficients of $\text{BaTiO}_3\text{--CoFe}_2\text{O}_4$ in the lower half-space.

7.1 Influence of $c_{11}^{(1)}/c_{11}^{(2)}$, $c_{12}^{(1)}/c_{12}^{(2)}$, $c_{22}^{(1)}/c_{22}^{(2)}$ and $c_{44}^{(1)}/c_{44}^{(2)}$

Here we study how the mechanical strength of the magnetoelectroelastic material in the upper half-space relative to that in the lower half-space affects the effective properties of the interface.

Figure 2. Plots of the effective properties $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ against $\log_{10} c_r$ for $\rho = 0.4, 0.5$ and 0.6 .

The piezoelectric coefficients, piezomagnetic coefficients, dielectric constants, electromagnetic coefficients and magnetic permeabilities of the magneto-electroelastic material in the upper half-space are taken to have the same values as those in the lower half-space, but its elastic moduli are given by $c_{11}^{(1)}/c_{11}^{(2)} = c_{12}^{(1)}/c_{12}^{(2)} = c_{22}^{(1)}/c_{22}^{(2)} = c_{44}^{(1)}/c_{44}^{(2)} = c_r$, where c_r is a positive constant giving the mechanical strength of the material in the upper half-space relative to that in the lower half-space.

In Figure 2, the non-dimensionalized effective properties $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ are plotted against $\log_{10} c_r$ for selected values of the damage ratio ρ . From Figures 2(c), 2(d) and 2(f), it is obvious that the effective properties $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ and $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ are, in general, not zero or even extremely small for a wide range of c_r . This shows that the assumption that $b_2^{(\text{imp})} = 0$, $h_2^{(\text{imp})} = 0$ and $s^{(\text{imp})} = 0$, which is widely used many papers for macro-analyses of imperfect magneto-electroelastic interfaces, is not valid in general.

For a given ρ , Figure 2 shows that $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$ and $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ increase as $\log_{10} c_r$ increases. However, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ decrease with increasing $\log_{10} c_r$. It appears that increasing the mechanical strength of the material in the upper half-space has a greater influence on $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$ and $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$ than on other effective properties. Varying c_r appears to have relatively insignificant influence on $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ and $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$. All the effective properties appear to approach constant values as the relative mechanical strength c_r of the material in the upper half-space increases.

In Figure 2, for a fixed value of c_r , the effective properties of the interface

are observed to have larger values for smaller values of the damage ratio ρ . This observation may be explained as follows. The micro-cracks are expected to be more stable and have generalized displacement jumps that are smaller in magnitude, giving rise to effective properties that are larger in magnitude.

7.2 Influence of $e_{15}^{(1)}/e_{15}^{(2)}$, $e_{31}^{(1)}/e_{31}^{(2)}$ and $e_{33}^{(1)}/e_{33}^{(2)}$

We examine here how the effective properties of the micro-cracked are affected by the strength of the piezoelectric coupling parameters $e_{15}^{(1)}$, $e_{31}^{(1)}$ and $e_{33}^{(1)}$ in the upper half-space relative to the corresponding parameters $e_{15}^{(2)}$, $e_{31}^{(2)}$ and $e_{33}^{(2)}$ in the lower half-space.

The material in the upper half-space has piezoelectric coefficients given by $e_{15}^{(1)}/e_{15}^{(2)} = e_{31}^{(1)}/e_{31}^{(2)} = e_{33}^{(1)}/e_{33}^{(2)} = e_r$, where e_r is a positive constant giving the relative strength of the piezoelectric coupling. The other remaining material constants in the upper half-space have the same values as those in the lower half-space.

Figure 3 shows plots of the effective properties $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ against $\log_{10} e_r$ for selected values of ρ . From the plots, we observe that $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$ and $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$ increase as the relative strength of the piezoelectric coupling parameters in the upper half-space increases, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ and $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ have local maximums, and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ does not change significantly as we vary e_r . Furthermore, it appears that the effect of varying e_r is more obvious on $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$ and $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$ than on $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$.

Figure 3. Plots of the effective properties $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ against $\log_{10} e_r$ for $\rho = 0.4, 0.5$ and 0.6 .

7.3 Influence of $q_{15}^{(1)}/q_{15}^{(2)}$, $q_{31}^{(1)}/q_{31}^{(2)}$ and $q_{33}^{(1)}/q_{33}^{(2)}$

We examine here how the strength of the magnetic coupling parameters $q_{15}^{(1)}$, $q_{31}^{(1)}$ and $q_{33}^{(1)}$ in the upper half-space relative to the corresponding parameters $q_{15}^{(2)}$, $q_{31}^{(2)}$ and $q_{33}^{(2)}$ in the lower half-space influence the effective properties of the interface.

The piezomagnetic coefficients of the material in the upper half-space are given $q_{15}^{(1)}/q_{15}^{(2)} = q_{31}^{(1)}/q_{31}^{(2)} = q_{33}^{(1)}/q_{33}^{(2)} = q_r$, where q_r is a positive constant giving the relative strength of the piezomagnetic coefficients in the upper half-space. The other remaining material constants in the upper half-space have the same values as those in the lower half-space.

Figure 4 shows plots of $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ against $\log_{10} q_r$ for selected values of ρ . As $\log_{10} q_r$ increases, we find that $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ increase and $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$ decreases, and $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$ does not change very much. The graphs of $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ and $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ peak at certain values of q_r . Also, increasing q_r appears to have a greater influence on $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ than on $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$ and all the effective properties seem to tend to constants as q_r increases.

Figure 4. Plots of the effective properties $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ against $\log_{10} q_r$ for $\rho = 0.4, 0.5$ and 0.6 .

7.4 Influence of $\kappa_{11}^{(1)}/\kappa_{11}^{(2)}$ and $\kappa_{33}^{(1)}/\kappa_{33}^{(2)}$

Here we examine the influence of the relative strength of the dielectric constants in the upper half-space on the effective properties of the interface.

The dielectric constants of the material in the upper half-space are given by $\kappa_{11}^{(1)}/\kappa_{11}^{(2)} = \kappa_{33}^{(1)}/\kappa_{33}^{(2)} = \kappa_r$, where κ_r is a positive constant giving the strength of the dielectric constants in the upper half-space relative to those in the lower half-space. The remaining values of the material constants in the upper half-space are taken to be the same as those in the lower half-space.

Figure 5 shows plots of the effective properties $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ against $\log_{10} \kappa_r$ for selected values of ρ . As the relative strength of the dielectric constants in the upper half-space increases, the effective properties $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$ and $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ decrease, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$ and $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ increase and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ does not change very much in magnitude. We also observe that $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$ changes more significantly in magnitude than $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ and $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$, as κ_r increases from 0.001 to 1000. Thus, it seems that increasing the dielectric constants in the upper half-space has a much more significant effect on $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$ than on the other effective properties of the interface.

Figure 5. Plots of the effective properties $\hat{k}_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{k}_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{b}_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{h}_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{w}^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{s}^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{v}^{(\text{imp})}/(-\mu_{11}^{(2)})$ against $\log_{10} \kappa_r$ for $\rho = 0.4, 0.5$ and 0.6 .

7.5 Influence of $\alpha_{11}^{(1)}/\alpha_{11}^{(2)}$ and $\alpha_{33}^{(1)}/\alpha_{33}^{(2)}$

We investigate here the influence of the relative strength of the electromagnetic coefficients in the upper half-space on the effective properties of the interface.

The electromagnetic coefficients of the material in the upper half-space are given by $\alpha_{11}^{(1)}/\alpha_{11}^{(2)} = \alpha_{33}^{(1)}/\alpha_{33}^{(2)} = \alpha_r$, where α_r is a positive constant giving the strength of the electromagnetic coefficients in the upper half-space relative to those in the lower half-space. The other remaining material constants in the upper half-space have the same values as those in the lower half-space.

In Figure 6, $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ are plotted against $\log_{10} \alpha_r$ for selected values of ρ . It appears that increasing the relative strength of the electromagnetic coefficients in the upper half-space has obvious effects on only $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ and $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$. From Figures 6(d) and 6(f), $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ and $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ do not change very much over the interval $0.001 \leq \alpha_r \leq 1$, but increase significantly over $1 < \alpha_r < 100$. The rate of increase of $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ (with respect to α_r) is significantly greater than that of $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$. Thus, increasing the relative strength of the electromagnetic coefficients in the upper half-space has a more drastic influence on the effective properties $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ than on the other effective properties of the interface.

Figure 6. Plots of the effective properties $\hat{k}_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{k}_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ against $\log_{10} \alpha_r$ for $\rho = 0.4, 0.5$ and 0.6 .

7.6 Influence of $\mu_{11}^{(1)}/\mu_{11}^{(2)}$ and $\mu_{33}^{(1)}/\mu_{33}^{(2)}$

Here we investigate how the strength of the magnetic permeability of the material in the upper half-space affects the effective properties of the interface.

The magnetic permeability coefficients in the upper half-space are given by $\mu_{11}^{(1)}/\mu_{11}^{(2)} = \mu_{33}^{(1)}/\mu_{33}^{(2)} = \mu_r$, where μ_r is a positive constant giving the relative strength of the magnetic permeability of the material in the upper half-space relative to that in the lower half-space. The other material properties in the upper half-space have the same values as those in the lower half-space.

In Figure 7, we plot the effective properties $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ against $\log_{10} \mu_r$ for selected values of ρ . As μ_r increases from 0.001 to 1000, $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ increases significantly, $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ and $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ decrease and $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$ and $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$ do not change very much. The effective property $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ changes more drastically with $\log_{10} \mu_r$ than $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$ and $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$, that is, increasing the magnetic permeabilities in the upper half-space has much more influence on $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ than on the other effective properties of the interface.

Figure 7. Plots of the effective properties $\hat{a}k_{11}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}k_{22}^{(\text{imp})}/c_{44}^{(2)}$, $\hat{a}b_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\kappa_{11}^{(2)}}$, $\hat{a}h_2^{(\text{imp})}/\sqrt{c_{44}^{(2)}\mu_{11}^{(2)}}$, $\hat{a}w^{(\text{imp})}/(-\kappa_{11}^{(2)})$, $\hat{a}s^{(\text{imp})}/(-\sqrt{\kappa_{11}^{(2)}\mu_{11}^{(2)}})$ and $\hat{a}v^{(\text{imp})}/(-\mu_{11}^{(2)})$ against $\log_{10} \mu_r$ for $\rho = 0.4, 0.5$ and 0.6 .

8 Final remarks

The effective properties of a microscopically damaged interface between two magnetoelectroelastic half-spaces are investigated by modeling the interface as containing a periodic array of micro-cracks. The magnetoelectroelastic conditions on the micro-cracks are formulated in terms of hypersingular integro-differential equations with unknown functions given by the jumps of the elastic displacements, the electric potential and the magnetic potential across opposite faces of the micro-cracks. Formulae for the effective properties of the micro-cracked interface, expressed in terms of the average values of the displacement, the electric and the magnetic jumps over the micro-cracks, are given. Statistical simulations involving micro-cracks with randomly generated lengths and positions are conducted to estimate the effective properties.

The effects of the density of the micro-cracks and the relative strength of selected properties of the magnetoelectroelastic materials on the effective properties are investigated in detail. The results obtained are intuitively and qualitatively acceptable and provide useful physical insights into the effective behaviors of microscopically damage interface. We find that the widely used assumption that the properties $\underline{\mathbf{b}}$, $\underline{\mathbf{c}}$, $\underline{\mathbf{g}}$, $\underline{\mathbf{h}}$, s and d in (1) are null quantities is not valid in general.

Experimental studies on imperfect interfaces damaged by micro-defects such as micro-cracks may be found in the literature (see, for example, Lavrentyev and Rokhlin [18] and Margetan et al. [19]). Results from experiments are, however, limited to relatively simple cases involving particular materials. Thus, computer based simulation models, like the one used in the current

paper, offer a useful alternative for analyzing a wide range of micro-damaged interfaces between advanced materials with multiple coupling fields.

Acknowledgment

The research work reported here is supported by Singapore Ministry of Education Tier 1 Research Grant No. 81/15 awarded by Nanyang Technological University.

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Appendix A

The derivation of the hypersingular integro-differential equations in (22) is outlined below.

If the partial differential equations in (14) hold in a two-dimensional region R on the Ox_1x_2 plane then the corresponding boundary integral equations as derived in Clements [10] are

$$\begin{aligned} U_K(\xi_1, \xi_2) = & \int_C C_{IjMp} [U_I(x_1, x_2) \frac{\partial}{\partial x_p} (\Phi_{MK}(x_1, x_2; \xi_1, \xi_2)) \\ & - \Phi_{IK}(x_1, x_2; \xi_1, \xi_2) \frac{\partial}{\partial x_p} (U_M(x_1, x_2))] n_j(x_1, x_2) ds(x_1, x_2) \\ & \text{for } (\xi_1, \xi_2) \text{ in the interior of } R, \end{aligned} \quad (\text{A1})$$

where C is the curve bounding the region R and $\Phi_{IK}(x_1, x_2; \xi_1, \xi_2)$ is any given solution of

$$C_{IjKp} \frac{\partial^2 \Phi_{KR}}{\partial x_j \partial x_p} = \delta_{IR} \delta(x_1 - \xi_1, x_2 - \xi_2), \quad (\text{A2})$$

where δ_{IR} is the Kronecker-delta, $\delta(x_1, x_2)$ is the Dirac-delta function.

For analyzing the bimaterial in Section 2, we choose the Green's function $\Phi_{IK}(x_1, x_2; \xi_1, \xi_2)$ that satisfies the perfect interface conditions

$$\left. \begin{aligned} \Phi_{IK}(x_1, 0^+; \xi_1, \xi_2) &= \Phi_{IK}(x_1, 0^-; \xi_1, \xi_2) \\ C_{I2Mp}^{(1)} \frac{\partial \Phi_{MK}}{\partial x_p} \Big|_{x_2=0^+} &= C_{I2Mp}^{(2)} \frac{\partial \Phi_{MK}}{\partial x_p} \Big|_{x_2=0^-} \end{aligned} \right\} \text{for } -\infty < x_1 < \infty, \quad (\text{A3})$$

where $C_{IjKp}^{(1)}$ and $C_{IjKp}^{(2)}$ denote respectively the material constants C_{IjKp} for the upper and lower half-spaces.

The required Green's function $\Phi_{RK}(x_1, x_2; \xi_1, \xi_2)$ may be deduced from the corresponding one given in Berger and Tewary [8] and Chen and Ang [9] for plane elastostatic deformations. It is given by

$$\begin{aligned}
 & \Phi_{RK}(x_1, x_2; \xi_1, \xi_2) \\
 = & \frac{1}{2\pi} \operatorname{Re}\{H(x_2)H(\xi_2) \sum_{\alpha=1}^4 A_{R\alpha}^{(1)} [N_{\alpha P}^{(1)} \ln(x_1 - \xi_1 + \tau_{\alpha}^{(1)}(x_2 - \xi_2)) \\
 & + \sum_{\beta=1}^4 Q_{\alpha\beta P}^{(1)} \ln(x_1 - \xi_1 + \tau_{\alpha}^{(1)}x_2 - \bar{\tau}_{\beta}^{(1)}\xi_2)] D_{PK}^{(1)} \\
 & + H(-x_2)H(\xi_2) \sum_{\alpha=1}^4 A_{R\alpha}^{(2)} \sum_{\beta=1}^4 Q_{\alpha\beta P}^{(2)} \ln(x_1 - \xi_1 + \tau_{\alpha}^{(2)}x_2 - \tau_{\beta}^{(1)}\xi_2) D_{PK}^{(1)} \\
 & + H(x_2)H(-\xi_2) \sum_{\alpha=1}^4 A_{r\alpha}^{(1)} \sum_{\beta=1}^4 T_{\alpha\beta P}^{(1)} \ln(x_1 - \xi_1 + \tau_{\alpha}^{(1)}x_2 - \tau_{\beta}^{(2)}\xi_2) D_{PK}^{(2)} \\
 & + H(-x_2)H(-\xi_2) \sum_{\alpha=1}^4 A_{R\alpha}^{(2)} [N_{\alpha P}^{(2)} \ln(x_1 - \xi_1 + \tau_{\alpha}^{(2)}(x_2 - \xi_2)) \\
 & + \sum_{\beta=1}^4 T_{\alpha\beta P}^{(2)} \ln(x_1 - \xi_1 + \tau_{\alpha}^{(2)}x_2 - \bar{\tau}_{\beta}^{(2)}\xi_2)] D_{PK}^{(2)}\}, \tag{A4}
 \end{aligned}$$

where $H(x)$ is the unit-step Heaviside function, the constants $Q_{\alpha\beta P}^{(p)}$ and $T_{\alpha\beta P}^{(p)}$

are defined by

$$\begin{aligned}
 N_{\gamma K}^{(2)}(A_{K\beta}^{(1)}N_{\beta P}^{(1)} + \sum_{\alpha=1}^4 \bar{A}_{K\alpha}^{(1)}\bar{Q}_{\alpha\beta P}^{(1)}) &= Q_{\gamma\beta P}^{(2)}, \\
 M_{\gamma K}^{(2)}(L_{K2\beta}^{(1)}N_{\beta P}^{(1)} + \sum_{\alpha=1}^4 \bar{L}_{K2\alpha}^{(1)}\bar{Q}_{\alpha\beta P}^{(1)}) &= Q_{\gamma\beta P}^{(2)}, \\
 N_{\gamma K}^{(1)}(A_{K\beta}^{(2)}N_{\beta P}^{(2)} + \sum_{\alpha=1}^4 \bar{A}_{K\alpha}^{(2)}\bar{T}_{\alpha\beta P}^{(2)}) &= T_{\gamma\beta P}^{(1)}, \\
 M_{\gamma K}^{(1)}(L_{K2\beta}^{(2)}N_{\beta P}^{(2)} + \sum_{\alpha=1}^4 \bar{L}_{K2\alpha}^{(2)}\bar{T}_{\alpha\beta P}^{(2)}) &= T_{\gamma\beta P}^{(1)},
 \end{aligned} \tag{A5}$$

and $\tau_{\alpha}^{(p)}$, $A_{R\alpha}^{(p)}$, $L_{Rj\alpha}^{(p)}$, $N_{\alpha P}^{(p)}$ and $D_{RK}^{(p)}$ are constants defined in Section 4.

Following closely the analysis in Wang et al. [30], we may use (A1) together with (A4) to derive

$$\begin{aligned}
 U_J(\xi_1, \xi_2) &= -\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{k=1}^M \int_{a^{(k)}+nL}^{b^{(k)}+nL} \Delta U_R(x_1) \\
 &\quad \times \operatorname{Re}\left\{ \sum_{\alpha=1}^4 L_{R2\alpha}^{(1)} \left[\frac{N_{\alpha P}^{(1)}}{x_1 - \xi_1 - \tau_{\alpha}^{(1)}\xi_2} \right. \right. \\
 &\quad \left. \left. + \sum_{\beta=1}^4 Q_{\alpha\beta P}^{(1)} \frac{1}{x_1 - \xi_1 - \bar{\tau}_{\beta}^{(1)}\xi_2} \right] \right\} D_{PJ}^{(1)} dx_1 \\
 &\quad \text{for } \xi_2 > 0,
 \end{aligned} \tag{A6}$$

and

$$\begin{aligned}
 S_{I2}(\xi_1, \xi_2) &= -\frac{1}{2\pi} \operatorname{Re}\left\{ \sum_{n=-\infty}^{\infty} \sum_{k=1}^M \int_{a^{(k)}+nL}^{b^{(k)}+nL} \Delta U_R(x_1) \right. \\
 &\quad \times \left[\sum_{\alpha=1}^4 \frac{G_{RI\alpha}}{(x_1 - \xi_1 - \tau_{\alpha}^{(1)}\xi_2)^2} \right. \\
 &\quad \left. \left. + \sum_{\alpha=1}^4 \sum_{\beta=1}^4 \frac{H_{RI\alpha\beta}}{(x_1 - \xi_1 - \bar{\tau}_{\beta}^{(1)}\xi_2)^2} \right] dx_1 \right\} \\
 &\quad \text{for } \xi_2 > 0.
 \end{aligned} \tag{A7}$$

If the micro-cracks on the interface are all magnetically and electrically impermeable then, from (A7), the conditions on the micro-cracks are given by

$$\begin{aligned} & \frac{1}{2\pi} \lim_{\xi_2 \rightarrow 0^+} \operatorname{Re}\left\{ \sum_{n=-\infty}^{\infty} \sum_{k=1}^M \int_{a^{(k)}+nL}^{b^{(k)}+nL} \Delta U_R(x_1) \right. \\ & \times \left[\sum_{\alpha=1}^4 \frac{G_{RI\alpha}}{(x_1 - \xi_1 - \tau_{\alpha}^{(1)} \xi_2)^2} + \sum_{\alpha=1}^4 \sum_{\beta=1}^4 \frac{H_{RI\alpha\beta}}{(x_1 - \xi_1 - \bar{\tau}_{\beta}^{(1)} \xi_2)^2} \right] dx_1 \Big\} \\ & = P_I(\xi_1) \text{ for } a^{(k)} < \xi_1 < b^{(k)} \quad (I = 1, 2, 3, 4). \end{aligned} \quad (\text{A8})$$

If we evaluate the limit on the left hand side of (A8) following the procedure described in Ang [3] and use the formula

$$\sum_{n=1}^{\infty} \frac{1}{(a \pm bn)^2} = \frac{1}{b^2} \Psi_1\left(1 \pm \frac{a}{b}\right) \text{ for } \operatorname{Re}\left\{1 \pm \frac{a}{b}\right\} > 0, \quad (\text{A9})$$

where Ψ_1 is the trigamma function, we find that (A8) gives rise to the hypersingular integro-differential equations in (22).

Appendix B

The technique of Kaya and Erdogan [16] may be used to solve numerically the hypersingular integro-differential equations in (22) as outlined below.

The generalized displacement jumps $\Delta U_R(x_1)$ over the micro-cracks are approximated by

$$\begin{aligned} \Delta U_R(x_1) & \simeq \sqrt{(x_1 - a^{(k)})(b^{(k)} - x_1)} \\ & \times \sum_{m=1}^{N^{(k)}} \alpha_R^{(km)} U^{(m-1)}\left(\frac{2x_1 - b^{(k)} - a^{(k)}}{b^{(k)} - a^{(k)}}\right) \\ & \text{for } a^{(k)} < x_1 < b^{(k)} \quad (k = 1, 2, \dots, M), \end{aligned} \quad (\text{B1})$$

where $U^{(m)}(x)$ is the m -th order Chebyshev polynomial of the second kind and $\alpha_R^{(km)}$ are unknown coefficients to be determined.

Substituting (B1) into (22) and collocating the resulting equations by letting

$$\xi_1 = x^{(kp)} = \frac{b^{(k)} + a^{(k)}}{2} + \frac{b^{(k)} - a^{(k)}}{2} \cos\left(\frac{[2p-1]\pi}{2N^{(k)}}\right)$$

for $p = 1, 2, \dots, N^{(k)}$ ($k = 1, 2, \dots, M$),

we obtain

$$\begin{aligned} & -\pi \sum_{m=1}^{N^{(k)}} \alpha_R^{(km)} \operatorname{Re}\{W_{RI}\} m U^{(m-1)}\left(\frac{2x^{(kp)} - b^{(k)} - a^{(k)}}{b^{(k)} - a^{(k)}}\right) \\ & + \sum_{\substack{n=1 \\ n \neq k}}^M \sum_{m=1}^{N^{(n)}} \alpha_R^{(nm)} \operatorname{Re}\{W_{RI}\} \int_{a^{(n)}}^{b^{(n)}} \frac{\sqrt{(x_1 - a^{(n)})(b^{(n)} - x_1)}}{(x_1 - x^{(kp)})^2} \\ & \times U^{(m-1)}\left(\frac{2x_1 - b^{(n)} - a^{(n)}}{b^{(n)} - a^{(n)}}\right) dx_1 \\ & + \sum_{n=1}^M \sum_{m=1}^{N^{(n)}} \alpha_R^{(nm)} \operatorname{Re}\{W_{RI}\} \int_{a^{(n)}}^{b^{(n)}} \sqrt{(x_1 - a^{(n)})(b^{(n)} - x_1)} \\ & \times U^{(m-1)}\left(\frac{2x_1 - b^{(n)} - a^{(n)}}{b^{(n)} - a^{(n)}}\right) \Theta(x_1, x^{(kp)}) dx_1 \\ & - \sum_{m=1}^{N^{(k)}} \alpha_R^{(km)} \operatorname{Im}\{\pi V_{RI}\} \frac{(a^{(k)} - b^{(k)}) m \cos\left[m \arccos\left(\frac{2x^{(kp)} - b^{(k)} - a^{(k)}}{b^{(k)} - a^{(k)}}\right)\right]}{2\sqrt{(x^{(kp)} - a^{(k)})(b^{(k)} - x^{(kp)})}} \\ & = 2\pi P_I(x^{(kp)}) \text{ for } p = 1, 2, \dots, N^{(k)} \text{ } (k = 1, 2, \dots, M), \end{aligned} \quad (\text{B2})$$

that is, a system of $4(N^{(1)} + N^{(2)} + \dots + N^{(M)})$ linear algebraic equations which we can solve for the unknown coefficients $\alpha_R^{(km)}$ for $R = 1, 2, 3, 4$ and $m = 1, 2, \dots, N^{(k)}$ ($k = 1, 2, \dots, M$). Note that the integrals in (B2) can be accurately evaluated by using the integration quadrature (25.4.40) in Abramowitz and Stegun [1]. More details may be found in Ang [3].