ANALYTICITY OF WEIGHTED CENTRAL PATHS AND ERROR BOUNDS FOR SEMIDEFINITE PROGRAMMING

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Abstract. The purpose of this paper is two-fold. Firstly, we show that every Cholesky-based weighted central path for semidefinite programming is analytic under strict complementarity. This result is applied to homogeneous cone programming to show that the central paths defined by the known class of optimal self-concordant barriers are analytic in the presence of strictly complementary solutions. Secondly, we consider a sequence of primal-dual solutions that lies within a prescribed neighborhood of the central path of a pair of primal-dual semidefinite programming problems, and converges to the respective optimal faces. Under the additional assumption of strict complementarity, we derive two necessary and sufficient conditions for the sequence of primal-dual solutions to converge linearly with their duality gaps.

1. Introduction

This paper is motivated by the works of Kojima et. al. [10, 11], Luo et. al. [15], Potra and Sheng [18, 19], and Potra et. al. [9] on primal-dual superlinearly convergent interior-point algorithms for semidefinite programming (SDP) and semidefinite linear complementarity problems. A common feature among these papers is the assumption of the existence of a pair of strictly complementary solutions. This assumption, while not without loss of generality, is somewhat necessary as it is noted by Anstreicher and Ye [2], and further generalized by Monteiro and Wright [16], that strict complementarity is necessary for the local superlinear convergence of a large class of algorithms for monotone linear complementarity problems.

Besides strict complementarity, each of the papers [9, 10, 15, 18, 19] assumes at least one of the following additional conditions:

(1) There exists a pair of primal-dual nondegenerate solutions.
(2) The iterates converge tangentially to the central path in the sense that the size of the central path neighborhood decreases towards zero.
(3) The products of the iterates converge to the zero matrix faster than the square-root of their duality gaps.

In the exception [11], global convergence and local superlinear convergence were established using the Alizadeh-Haeberly-Overton (AHO) search direction (see, e.g., [1]) without any further assumptions. However, global linear convergence and polynomial-time convergence of the algorithm was not proven. Thus the existence of a polynomial-time primal-dual interior-point algorithm for SDP, that converges

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locally at a superlinear rate under the sole assumption of strict complementarity, remains an open question.

The success of the AHO search direction was explained by Monteiro and Lu in [13], where they showed that under a certain proximity condition, the distance of the primal-dual iterates to the primal-dual optimal solution set is bounded above by a linear function of their duality gap. Such Lipschitzian-type error bound was well-studied for linear programming (see [8]), and for convex quadratic programming and SDP under Slater's condition (see [22] and references therein). Their proof uses the analyticity of weighted AHO central paths at the optimal solutions, which was established in the same paper. It is, however, difficult to deduce polynomial complexity for algorithms based solely on the AHO direction. This difficulty can be partly explained by the ill behavior of the natural neighborhood associated with the AHO direction — for each fixed AHO neighborhood around the central path, there exists a pair of primal-dual feasible solutions arbitrarily close to the central path, and yet falls outside the AHO neighborhood.

We thus consider a different notion of weighted centers based on Cholesky factors, which was first introduced by the author in [4]. We show in Section 2 that Cholesky weighted centers depend analytically on the weights and the duality gap, and that the analyticity extends to pairs of strictly complementary solutions. This result is then generalized to extended Cholesky weighted centers, whose collection covers the entire relative interior of the primal-dual feasible region. We then consider conic optimization problems over homogeneous cones, and extend the analyticity result to the analytic centers associated with the only known class of optimal barriers for such cones. In Section 3, we show an error bound result that parallels the above-mentioned Lipschitzian-type error bound, and provide two different characterizations of a Lipschitzian error bound for semidefinite programming under strict complementarity.

Notations and conventions. Throughout this paper, we use the following notations and conventions.

We use uppercase bold letters (e.g., $\mathbf{X}, \mathbf{L}$, etc.) to represent matrices, and use lowercase bold letters (e.g., $\mathbf{y}, \mathbf{b}$, etc.) to represent vectors.

The space of real n-vectors is denoted by $\mathbb{R}^n$, and the nonnegative (resp. positive) orthant in $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ (resp. $\mathbb{R}^n_{++}$).

The space of real n-by-n matrices is denoted by $\mathbb{M}^n$, and the group of invertible matrices in $\mathbb{M}^n$ is denoted by $\mathbb{M}^n_\star$. We equip $\mathbb{M}^n$ with the inner product $\bullet : (\mathbf{A}, \mathbf{B}) \in \mathbb{M}^n \oplus \mathbb{M}^n \mapsto trace(\mathbf{A}^T\mathbf{B})$. It induces the Frobenius norm $\| \cdot \|_F : \mathbf{M} \in \mathbb{M}^n \mapsto (trace(\mathbf{M}^T\mathbf{M}))^{1/2}$.

The subspace of lower triangular (resp. upper triangular) matrices in $\mathbb{M}^n$ is denoted by $\mathbb{L}^n$ (resp. $\mathbb{U}^n$), and the subgroup of lower triangular matrices in $\mathbb{M}^n_\star$ is denoted by $\mathbb{L}^n_\star$.

The subgroup of orthogonal matrices in $\mathbb{M}^n_\star$ is denoted by $\mathbb{O}^n$.

The subspace of symmetric matrices in $\mathbb{M}^n$ is denoted by $\mathbb{S}^n$, and the cone of positive semidefinite (resp. positive definite) matrices in $\mathbb{S}^n$ is denoted by $\mathbb{S}^n_+$ (resp. $\mathbb{S}^n_{++}$). The dimension of $\mathbb{S}^n$ is denoted by $n^{\bar{2}}$.

The subspace of diagonal matrices in $\mathbb{S}^n$ is denoted by $\mathbb{D}^n$, and its intersection with $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$ are respectively denoted by $\mathbb{D}^n_+$ and $\mathbb{D}^n_{++}$.

For any matrix $\mathbf{M} \in \mathbb{M}^n$ and any two subsets of indices $I, J \subseteq \{1, \ldots, n\}$, the submatrix of $\mathbf{M}$ with row indices in I and column indices in J is denoted by $\mathbf{M}_{IJ}$.

If $I = \{i\}$ (or $J = \{j\}$) is a singleton, we may also write i (or j) in place of $\{i\}$ (or $\{j\}$). If $I = \{1, \ldots, n\}$ (or $J = \{1, \ldots, n\}$) is the full index set, we may also write $*$ in place of $\{1, \ldots, n\}$. For instance, $\mathbf{M}_{1*}$ denotes the first row of $\mathbf{M}$.

For any matrix $\mathbf{M} \in \mathbb{M}^n$, the unique lower triangular matrix $\mathbf{L}$ satisfying $\mathbf{M} - \mathbf{L} \in \mathbb{U}^n$ and $\mathbf{L}_{ii} = \mathbf{M}_{ii}/2$ for $i = 1, \ldots, n$, is denoted by $\langle\!\langle \mathbf{M} \rangle\!\rangle$. For any matrix $\mathbf{M} \in \mathbb{M}^n$, we denote by $\mathbf{M}_H$ the symmetric matrix $\mathbf{M} + \mathbf{M}^T$. Consequently $\langle\!\langle \mathbf{M} \rangle\!\rangle_H$ denotes the unique symmetric matrix whose lower triangular entries coincide with those of $\mathbf{M}$.

The zero matrix and the identity matrix of appropriate sizes (in the context used) are denoted by $\mathbf{0}$ and $\mathbf{I}$ respectively.

For any diagonal matrix $\mathbf{D} \in \mathbb{D}^n$ and any subset B of positive integer indices, $\mathbf{D}_B$ denotes the diagonal matrix in $\mathbb{D}^n$ with $(\mathbf{D}_B)_{ii} = \mathbf{D}_{ii}$ if $i \in B$ and $(\mathbf{D}_B)_{ii} = 0$ otherwise.

For any symmetric, positive definite matrix $\mathbf{X} \in \mathbb{S}^n_{++}$, its unique Cholesky factor (i.e., the unique lower triangular matrix $\mathbf{L} \in \mathbb{L}^n$ with positive diagonal entries satisfying $\mathbf{L}\mathbf{L}^T = \mathbf{X}$) is denoted by $\mathbf{L}_{\mathbf{X}}$, and its unique inverse Cholesky factor (i.e., the unique upper triangular matrix $\mathbf{U} \in \mathbb{U}^n$ with positive diagonal entries satisfying $\mathbf{U}\mathbf{U}^T = \mathbf{X}$) is denoted by $\mathbf{U}_{\mathbf{X}}$.

For each linear map $\boldsymbol{\mathcal{A}} : \mathbb{E} \to \mathbb{F}$ between two finite dimensional vector spaces, $\boldsymbol{\mathcal{A}}^H : \mathbb{F} \to \mathbb{E}$ denotes its adjoint map, $Im(\boldsymbol{\mathcal{A}})$ denotes its range $\boldsymbol{\mathcal{A}}(\mathbb{E})$, and $Ker(\boldsymbol{\mathcal{A}})$ denotes its kernel $\boldsymbol{\mathcal{A}}^{-1}(\{\mathbf{0}\})$.

For each topological subspace S, $relint(S)$ denotes the relative interior of S and $cl(S)$ denotes the closure of S.

For each sequence $x_1, \ldots, x_n$ of real numbers, $Diag(x_1, \ldots, x_n)$ denotes the diagonal matrix in $\mathbb{D}^n$ with $x_1, \ldots, x_n$ on its diagonal.

For two functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, the notation $f(t) = O(g(t))$ means there exists a $C > 0$ such that $f(t) < Cg(t)$ for all $t \in \mathbb{R}_+$ sufficiently small. The notation $f(t) = \Theta(g(t))$ means $f(t) = O(g(t))$ and $g(t) = O(f(t))$.

2. Analyticity of Central Paths

We consider the primal-dual pair of SDP problems

$$\inf_{\mathbf{X}} \{\widehat{\mathbf{S}} \bullet \mathbf{X} : \mathbf{X} \in \mathcal{L} + \widehat{\mathbf{X}},\ \mathbf{X} \in \mathbb{S}^n_+\} \quad \text{and} \quad \inf_{\mathbf{S}} \{\widehat{\mathbf{X}} \bullet \mathbf{S} : \mathbf{S} \in \mathcal{L}^\perp + \widehat{\mathbf{S}},\ \mathbf{S} \in \mathbb{S}^n_+\}, \tag{2.1}$$

where $\widehat{\mathbf{X}}, \widehat{\mathbf{S}} \in \mathbb{S}^n_{++}$, $\mathcal{L} \subseteq \mathbb{S}^n$ is a linear subspace and $\mathcal{L}^\perp$ denotes its orthogonal complement in $\mathbb{S}^n$.

Let m denote the dimension of $\mathcal{L}^\perp$, and let $\boldsymbol{\mathcal{A}} : \mathbb{S}^n \to \mathbb{R}^m$ and $\boldsymbol{\mathcal{C}} : \mathbb{S}^n \to \mathbb{R}^{n^{\bar{2}} - m}$ denote linear maps satisfying $Ker(\boldsymbol{\mathcal{A}}) = \mathcal{L}$ and $Ker(\boldsymbol{\mathcal{C}}) = \mathcal{L}^\perp$, and let $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{d} \in \mathbb{R}^{n^{\bar{2}} - m}$ denote the vectors $\boldsymbol{\mathcal{A}}\widehat{\mathbf{X}}$ and $\boldsymbol{\mathcal{C}}\widehat{\mathbf{S}}$ respectively. Thus (2.1) can be rewritten as

$$\inf_{\mathbf{X}} \{\widehat{\mathbf{S}} \bullet \mathbf{X} : \boldsymbol{\mathcal{A}}\mathbf{X} = \mathbf{b},\ \mathbf{X} \in \mathbb{S}^n_+\} \quad \text{and} \quad \inf_{\mathbf{S}} \{\widehat{\mathbf{X}} \bullet \mathbf{S} : \boldsymbol{\mathcal{C}}\mathbf{S} = \mathbf{d},\ \mathbf{S} \in \mathbb{S}^n_+\}.$$

Let $\mathcal{F}_p$, $\mathcal{F}_p^\circ$ and $\mathcal{O}_p$ denote respectively the primal feasible region, the primal strictly feasible region and the set of primal optimal solutions. Their dual counterparts are denoted by $\mathcal{F}_d$, $\mathcal{F}_d^\circ$ and $\mathcal{O}_d$ respectively.

Note that since $\widehat{\mathbf{X}} \in \mathbb{S}^n_{++} \cap \mathcal{F}_p$ and $\widehat{\mathbf{S}} \in \mathbb{S}^n_{++} \cap \mathcal{F}_d$, the SDP problems satisfy Slater's condition.

2.1. **Cholesky Weighted Centers for SDP.** It was shown in [4] that for each $\mathbf{w} = (\nu, \mathbf{D}) \in \mathbb{R}_{++} \oplus \mathbb{D}^n_{++}$, the system of weighted center equations

$$\boldsymbol{\mathcal{A}}\mathbf{X} = \mathbf{b}, \quad \boldsymbol{\mathcal{C}}\mathbf{S} = \mathbf{d},$$
$$\mathbf{L}_{\mathbf{X}}^T \mathbf{S} \mathbf{L}_{\mathbf{X}} = \nu \mathbf{D}, \quad \mathbf{X}, \mathbf{S} \in \mathbb{S}^n_{++}$$

has a unique pair of solutions $(\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$, which we called the *Cholesky weighted centers determined by* $\mathbf{w}$. Moreover, when the primal-dual SDP problems (2.1) have strictly complementary solutions, each *primal-dual Cholesky weighted central path*

$$\{(\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w})) : \nu > 0\}$$

converges as $\nu \downarrow 0$. We denote the limit by $(\mathbf{X}(0, \mathbf{D}), \mathbf{S}(0, \mathbf{D}))$.

This system of weighted center equations is equivalent to

$$\boldsymbol{\mathcal{A}}\mathbf{X} = \mathbf{b}, \quad \boldsymbol{\mathcal{C}}\mathbf{S} = \mathbf{d}, \tag{2.2a}$$
$$\langle\!\langle \mathbf{S}\mathbf{X} \rangle\!\rangle_H = \nu \mathbf{D}, \quad \mathbf{X}, \mathbf{S} \in \mathbb{S}^n_{++}. \tag{2.2b}$$

To see this, we pre- and post-multiplying $\mathbf{L}_{\mathbf{X}}^{-T}$ and $\mathbf{L}_{\mathbf{X}}^T$ respectively to both sides of the equation $\mathbf{L}_{\mathbf{X}}^T \mathbf{S} \mathbf{L}_{\mathbf{X}} = \nu \mathbf{D}$ to get $\mathbf{S}\mathbf{X} = \nu \mathbf{L}_{\mathbf{X}}^{-T} \mathbf{D} \mathbf{L}_{\mathbf{X}}^{-1}$, which implies that the lower-triangular part of $\mathbf{S}\mathbf{X}$ is the diagonal matrix $\nu\mathbf{D}$. On the other hand, if the lower-triangular part of $\mathbf{S}\mathbf{X}$ is the diagonal matrix $\nu\mathbf{D}$, pre- and post-multiplying $\mathbf{L}_{\mathbf{X}}^T$ and $\mathbf{L}_{\mathbf{X}}^{-T}$ to it maintains this lower-triangular part, which implies that $\mathbf{L}_{\mathbf{X}}^T \mathbf{S} \mathbf{L}_{\mathbf{X}}$, which is symmetric at the same time, is the diagonal matrix $\nu\mathbf{D}$. Thus we have shown that the equation $\mathbf{L}_{\mathbf{X}}^T \mathbf{S} \mathbf{L}_{\mathbf{X}} = \nu \mathbf{D}$ is equivalent to $\mathbf{S}\mathbf{X}$ having the diagonal matrix $\nu\mathbf{D}$ as its lower-triangular part, which in turn is equivalent to $\langle\!\langle \mathbf{S}\mathbf{X} \rangle\!\rangle_H = \nu \mathbf{D}$.

Henceforth, we assume that the primal-dual SDP problems (2.1) *have strictly complementary solutions.*

The main result of this section states that under the assumption of strict complementarity, the map

$$\mathbf{w} \in \mathbb{R}_{+} \oplus \mathbb{D}^n_{++} \mapsto (\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$$

is analytic; i.e., for each $\widehat{\mathbf{w}} = (\widehat{\nu}, \widehat{\mathbf{D}}) \in \mathbb{R}_{+} \oplus \mathbb{D}^n_{++}$, there exists an open subset $\widehat{\mathbf{w}} \in \mathcal{U} \oplus \mathcal{V} \subseteq \mathbb{R} \oplus \mathbb{D}^n$ and a real analytic map $\mathbf{w} \in \mathcal{U} \oplus \mathcal{V} \mapsto (\mathbf{X}'(\mathbf{w}), \mathbf{S}'(\mathbf{w}))$ that agrees with $\mathbf{w} \mapsto (\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$ over $(\mathcal{U} \oplus \mathcal{V}) \cap (\mathbb{R}_{+} \oplus \mathbb{D}^n_{++})$.

First consider the case $\widehat{\nu} > 0$. For each $\mathbf{w} \in \mathbb{R}_{+} \oplus \mathbb{D}^n_{++}$, the pair $(\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$ solves

$$\boldsymbol{\mathcal{F}}(\mathbf{X}, \mathbf{S}, \widehat{\nu}, \widehat{\mathbf{D}}) = (\mathbf{0}, \mathbf{0}, \mathbf{0}), \tag{2.3}$$

where $\boldsymbol{\mathcal{F}} : \mathbb{S}^n \oplus \mathbb{S}^n \oplus \mathbb{R} \oplus \mathbb{D}^n \to \mathbb{R}^m \oplus \mathbb{R}^{\bar{n^2}-m} \oplus \mathbb{S}^n$ is the analytic map

$$(\mathbf{X}, \mathbf{S}, \nu, \mathbf{D}) \mapsto (\boldsymbol{\mathcal{A}}\mathbf{X} - \mathbf{b}, \boldsymbol{\mathcal{C}}\mathbf{S} - \mathbf{d}, \langle\!\langle \mathbf{S}\mathbf{X} \rangle\!\rangle_H - \nu \mathbf{D}),$$

It follows from [5, Theorem 6.1] that the Jacobian

$$\frac{\partial \boldsymbol{\mathcal{F}}}{\partial(\mathbf{X}, \mathbf{S})}(\mathbf{X}, \mathbf{S}, \mathbf{w}) : (\boldsymbol{\Delta}\mathbf{X}, \boldsymbol{\Delta}\mathbf{S}) \mapsto \begin{bmatrix} \boldsymbol{\mathcal{A}}(\boldsymbol{\Delta}\mathbf{X}) \\ \boldsymbol{\mathcal{C}}(\boldsymbol{\Delta}\mathbf{S}) \\ \langle\!\langle (\boldsymbol{\Delta}\mathbf{S})\mathbf{X} + \mathbf{S}(\boldsymbol{\Delta}\mathbf{X}) \rangle\!\rangle_H \end{bmatrix}.$$

is nonsingular at $(\mathbf{X}(\widehat{\mathbf{w}}), \mathbf{S}(\widehat{\mathbf{w}}), \widehat{\mathbf{w}})$. Alternatively we can deduce the nonsingularity of the Jacobian directly from the following lemma by taking $(\widehat{\boldsymbol{\mathcal{A}}}, \widehat{\boldsymbol{\mathcal{C}}}, \widehat{\mathbf{X}}, \widehat{\mathbf{S}})$ in the lemma to be $(\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{C}}, \mathbf{X}(\widehat{\mathbf{w}}), \mathbf{S}(\widehat{\mathbf{w}}))$.

Lemma 1. *If* $\widehat{\mathcal{A}} : \mathbb{S}^n \to \mathbb{R}^p$ *and* $\widehat{\mathcal{C}} : \mathbb{S}^n \to \mathbb{R}^q$ *are linear maps such that* $Ker(\widehat{\mathcal{A}}) \perp Ker(\widehat{\mathcal{C}})$, *and* $\widehat{\mathbf{X}}, \widehat{\mathbf{S}} \in \mathbb{S}^n_{++}$ *satisfy* $\mathbf{L}_{\widehat{\mathbf{X}}}^T \widehat{\mathbf{S}} \mathbf{L}_{\widehat{\mathbf{X}}} \in \mathbb{D}^n_{++}$, *then the linear map*

$$\mathcal{J} : (\Delta\mathbf{X}, \Delta\mathbf{S}) \in \mathbb{S}^n \oplus \mathbb{S}^n \mapsto \begin{bmatrix} \widehat{\mathcal{A}}(\Delta\mathbf{X}) \\ \widehat{\mathcal{C}}(\Delta\mathbf{S}) \\ \langle\!\langle (\Delta\mathbf{S})\widehat{\mathbf{X}} + \widehat{\mathbf{S}}(\Delta\mathbf{X}) \rangle\!\rangle_H \end{bmatrix}$$

is injective.

Proof. Suppose that $\mathcal{J}(\Delta\mathbf{X}, \Delta\mathbf{S}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$. Then the matrix

$$(\Delta\mathbf{S})\widehat{\mathbf{X}} + \widehat{\mathbf{S}}(\Delta\mathbf{X})$$

is upper triangular, from which it follows that

$$\mathbf{L}_{\widehat{\mathbf{X}}}^T ((\Delta\mathbf{S})\widehat{\mathbf{X}} + \widehat{\mathbf{S}}(\Delta\mathbf{X}))\mathbf{L}_{\widehat{\mathbf{X}}}^{-T} = \mathbf{L}_{\widehat{\mathbf{X}}}^T (\Delta\mathbf{S})\mathbf{L}_{\widehat{\mathbf{X}}} + \mathbf{D}\mathbf{L}_{\widehat{\mathbf{X}}}^{-1}(\Delta\mathbf{X})\mathbf{L}_{\widehat{\mathbf{X}}}^{-T}$$

is also upper triangular, where $\mathbf{D} = \mathbf{L}_{\widehat{\mathbf{X}}}^T \widehat{\mathbf{S}} \mathbf{L}_{\widehat{\mathbf{X}}} \in \mathbb{D}^n_{++}$ by assumption. Thus

$$\begin{bmatrix} \widetilde{\mathcal{A}} & \\ & \widetilde{\mathcal{C}} \\ \mathcal{D} & \mathcal{I} \end{bmatrix} \begin{bmatrix} \mathbf{L}_{\widehat{\mathbf{X}}}^{-1}(\Delta\mathbf{X})\mathbf{L}_{\widehat{\mathbf{X}}}^{-T} \\ \mathbf{L}_{\widehat{\mathbf{X}}}^T(\Delta\mathbf{S})\mathbf{L}_{\widehat{\mathbf{X}}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

where $\widetilde{\mathcal{A}} : \mathbb{S}^n \to \mathbb{R}^p$ and $\widetilde{\mathcal{C}} : \mathbb{S}^n \to \mathbb{R}^q$ are linear maps defined respectively by

$$\mathbf{X} \mapsto \widehat{\mathcal{A}}(\mathbf{L}_{\widehat{\mathbf{X}}}\mathbf{X}\mathbf{L}_{\widehat{\mathbf{X}}}^T) \quad \text{and} \quad \mathbf{S} \mapsto \widehat{\mathcal{C}}(\mathbf{L}_{\widehat{\mathbf{X}}}^{-T}\mathbf{S}\mathbf{L}_{\widehat{\mathbf{X}}}^{-1}),$$

and $\mathcal{D} : \mathbb{S}^n \to \mathbb{S}^n$ is the map

$$\mathbf{X} \mapsto \mathbf{D}\langle\!\langle \mathbf{X} \rangle\!\rangle + \langle\!\langle \mathbf{X} \rangle\!\rangle^T \mathbf{D}.$$

For every $\mathbf{X}, \mathbf{Y} \in \mathbb{S}^n$,

$$\begin{aligned} \mathcal{D}(\mathbf{X}) \bullet \mathbf{Y} &= (\mathbf{D}\langle\!\langle \mathbf{X} \rangle\!\rangle) \bullet \mathbf{Y} + (\langle\!\langle \mathbf{X} \rangle\!\rangle^T \mathbf{D}) \bullet \mathbf{Y} \\ &= \sum_{i=1}^n \mathbf{D}_{ii}\mathbf{X}_{ii}\mathbf{Y}_{ii} + 2 \sum_{1 \le j < i \le n} \mathbf{D}_{ii}\mathbf{X}_{ij}\mathbf{Y}_{ij}. \end{aligned}$$

Since $\mathbf{D} \in \mathbb{D}^n_{++}$, the bilinear function $(\mathbf{X}, \mathbf{Y}) \mapsto \mathcal{D}(\mathbf{X}) \bullet \mathbf{Y}$ is an inner product on $\mathbb{S}^n$.

Eliminating $\mathbf{L}_{\widehat{\mathbf{X}}}^T(\Delta\mathbf{S})\mathbf{L}_{\widehat{\mathbf{X}}}$ using the last two equations gives

$$\begin{aligned} \widetilde{\mathcal{A}}(\mathbf{L}_{\widehat{\mathbf{X}}}^{-1}(\Delta\mathbf{X})\mathbf{L}_{\widehat{\mathbf{X}}}^{-T}) &= \mathbf{0} \\ (\widetilde{\mathcal{C}} \circ \mathcal{D})(\mathbf{L}_{\widehat{\mathbf{X}}}^{-1}(\Delta\mathbf{X})\mathbf{L}_{\widehat{\mathbf{X}}}^{-T}) &= \mathbf{0} \end{aligned}$$

Since $Ker(\widehat{\mathcal{A}}) \perp Ker(\widehat{\mathcal{C}})$ implies that $Ker(\widetilde{\mathcal{A}}) \perp Ker(\widetilde{\mathcal{C}})$, it follows that $Ker(\widetilde{\mathcal{A}})$ and $Ker(\widetilde{\mathcal{C}} \circ \mathcal{D}) = \mathcal{D}^{-1}Ker(\widetilde{\mathcal{C}})$ are orthogonal to each other under the inner product $(\mathbf{X}, \mathbf{Y}) \mapsto (\mathcal{D}\mathbf{X}) \bullet \mathbf{Y}$. Consequently $\mathbf{L}_{\widehat{\mathbf{X}}}^{-1}(\Delta\mathbf{X})\mathbf{L}_{\widehat{\mathbf{X}}}^{-T}$, whence $\Delta\mathbf{X}$ and $\Delta\mathbf{S}$, are zero matrices. □

With a nonsingular Jacobian, the analytic version of the Implicit Function Theorem (see, e.g., [12]) states that there exist an open set $\widehat{\mathbf{w}} \in \mathcal{U} \oplus \mathcal{V} \subseteq \mathbb{R} \oplus \mathbb{D}^n$ and a unique continuous map

$$\mathbf{w} \in \mathcal{U} \oplus \mathcal{V} \mapsto (\mathbf{X}'(\mathbf{w}), \mathbf{S}'(\mathbf{w}))$$

such that $(\mathbf{X}'(\mathbf{w}), \mathbf{S}'(\mathbf{w}))$ solves (2.3) for each $\mathbf{w} = (\nu, \mathbf{D}) \in \mathcal{U} \oplus \mathcal{V}$, and the pair $(\mathbf{X}'(\widehat{\mathbf{w}}), \mathbf{S}'(\widehat{\mathbf{w}}))$ coincides with $(\mathbf{X}(\widehat{\mathbf{w}}), \mathbf{S}(\widehat{\mathbf{w}}))$. Moreover the analyticity of $\mathcal{F}$ implies that this continuous map is analytic. By the continuity of the map, we may assume without loss of generality, by restricting $\mathcal{U} \oplus \mathcal{V}$ further, that $(\mathbf{X}'(\mathbf{w}), \mathbf{S}'(\mathbf{w})) \in \mathbb{S}^n_{++} \oplus \mathbb{S}^n_{++}$ for each $\mathbf{w} \in (\mathcal{U} \oplus \mathcal{V}) \cap (\mathbb{R}_+ \oplus \mathbb{D}^n_{++})$. Since (2.3) has a unique solution in $\mathbb{S}^n_{++} \oplus \mathbb{S}^n_{++}$ for each $\mathbf{w} \in \mathcal{U} \oplus \mathcal{V}$, it follows that the pair $(\mathbf{X}'(\mathbf{w}), \mathbf{S}'(\mathbf{w}))$ agrees with $(\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$ for all $\mathbf{w} \in (\mathcal{U} \oplus \mathcal{V}) \cap (\mathbb{R}_+ \oplus \mathbb{D}^n_{++})$. Consequently we have shown (without any assumption of strict complementarity) that

Theorem 2. *The map* $\mathbf{w} \mapsto (\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$ *is analytic on* $\mathbb{R}_{++} \oplus \mathbb{D}^n_{++}$.

For the case $\widehat{\nu} = 0$, the above approach fails due to a possible singular Jacobian. We shall use the following lemmas, whose proofs can be found in [4], to demonstrate this possible singularity.

Lemma 3 (Lemma 6 of [4])**.** *For each* $(\mathbf{X}, \mathbf{S}) \in relint(\mathcal{O}_p) \oplus relint(\mathcal{O}_d)$, *there exists* $\mathbf{L} \in \mathbb{L}^n$ *with positive diagonal such that*

$$\mathbf{L}\mathbf{X}\mathbf{L}^T = \mathbf{I}_B \qquad \textit{and} \qquad \mathbf{L}^{-T}\mathbf{S}\mathbf{L}^{-1} = \mathbf{I}_N$$

for some disjoint subsets $B, N \subseteq \{1, \dots, n\}$.

Lemma 4 (Lemma 10 of [4])**.** *For each* $\mathbf{w} \in \{0\} \oplus \mathbb{D}^n_{++}$,

$$(\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w})) \in relint(\mathcal{O}_p) \oplus relint(\mathcal{O}_d).$$

As a consequence of Lemmas 3 and 4, there exists $\widehat{\mathbf{L}} \in \mathbb{L}^n$ with positive diagonal such that

$$\widehat{\mathbf{L}}\mathbf{X}(\widehat{\mathbf{w}})\widehat{\mathbf{L}}^T = \mathbf{I}_B \qquad \text{and} \qquad \widehat{\mathbf{L}}^{-T}\mathbf{S}(\widehat{\mathbf{w}})\widehat{\mathbf{L}}^{-1} = \mathbf{I}_N$$

for some disjoint subsets $B, N \subseteq \{1, \dots, n\}$. It follows from strict complementarity that $B \cup N = \{1, \dots, n\}$. Let k denote $|B|$. For each $\mathbf{w} = (\nu, \mathbf{D}) \in \mathbb{R}_{++} \oplus \mathbb{D}^n_{++}$, let $\mathbf{X}_{\widehat{\mathbf{L}}}(\mathbf{w})$ and $\mathbf{S}_{\widehat{\mathbf{L}}}(\mathbf{w})$ denote the matrices $\widehat{\mathbf{L}}\mathbf{X}(\mathbf{w})\widehat{\mathbf{L}}^T$ and $\widehat{\mathbf{L}}^{-T}\mathbf{S}(\mathbf{w})\widehat{\mathbf{L}}^{-1}$ respectively. It is easily checked that the pair $(\mathbf{X}_{\widehat{\mathbf{L}}}(\mathbf{w}), \mathbf{S}_{\widehat{\mathbf{L}}}(\mathbf{w}))$ solves the Cholesky weighted center equations for

$$\inf_{\mathbf{X}}\{\widehat{\mathbf{S}}_{\widehat{\mathbf{L}}} \bullet \mathbf{X} : \boldsymbol{\mathcal{A}}_{\widehat{\mathbf{L}}}\mathbf{X} = \mathbf{b},\ \mathbf{X} \in \mathbb{S}^n_+\} \quad \text{and} \quad \inf_{\mathbf{S}}\{\widehat{\mathbf{X}}_{\widehat{\mathbf{L}}} \bullet \mathbf{S} : \boldsymbol{\mathcal{C}}_{\widehat{\mathbf{L}}}\mathbf{S} = \mathbf{d},\ \mathbf{S} \in \mathbb{S}^n_+\}, \tag{2.4}$$

where $\boldsymbol{\mathcal{A}}_{\widehat{\mathbf{L}}}$ and $\boldsymbol{\mathcal{C}}_{\widehat{\mathbf{L}}}$ denote respectively the maps $\mathbf{X} \in \mathbb{S}^n \mapsto \boldsymbol{\mathcal{A}}(\widehat{\mathbf{L}}^{-1}\mathbf{X}\widehat{\mathbf{L}}^{-T})$ and $\mathbf{S} \in \mathbb{S}^n \mapsto \boldsymbol{\mathcal{C}}(\widehat{\mathbf{L}}^T\mathbf{S}\widehat{\mathbf{L}})$, and $\widehat{\mathbf{X}}_{\widehat{\mathbf{L}}}, \widehat{\mathbf{S}}_{\widehat{\mathbf{L}}} \in \mathbb{S}^n$ denote respectively the matrices $\widehat{\mathbf{L}}\widehat{\mathbf{X}}\widehat{\mathbf{L}}^T$ and $\widehat{\mathbf{L}}^{-T}\widehat{\mathbf{S}}\widehat{\mathbf{L}}^{-1}$. The primal-dual SDP problems (2.4) have strictly complementary solutions $\mathbf{I}_B$ and $\mathbf{I}_N$. Thus each primal-dual Cholesky weighted central path of (2.4) converges to a limit which we denote by $(\mathbf{X}_{\widehat{\mathbf{L}}}(0, \mathbf{D}), \mathbf{S}_{\widehat{\mathbf{L}}}(0, \mathbf{D}))$ as before. Clearly the Jacobian $\frac{\partial}{\partial(\mathbf{X},\mathbf{S})}\boldsymbol{\mathcal{F}}$ is nonsingular at $(\mathbf{X}(\widehat{\mathbf{w}}), \mathbf{S}(\widehat{\mathbf{w}}), \widehat{\mathbf{w}}))$ if and only if the Jacobian $\frac{\partial}{\partial(\mathbf{X},\mathbf{S})}\boldsymbol{\mathcal{F}}_{\widehat{\mathbf{L}}}$ is nonsingular at $(\mathbf{X}_{\widehat{\mathbf{L}}}(\widehat{\mathbf{w}}), \mathbf{S}_{\widehat{\mathbf{L}}}(\widehat{\mathbf{w}}), \widehat{\mathbf{w}})$, where $\boldsymbol{\mathcal{F}}_{\widehat{\mathbf{L}}} : \mathbb{S}^n \oplus \mathbb{S}^n \oplus \mathbb{R} \oplus \mathbb{D}^n \to \mathbb{R}^m \oplus \mathbb{R}^{n^2-m} \oplus \mathbb{S}^n$ is the analytic map

$$(\mathbf{X}, \mathbf{S}, \nu, \mathbf{D}) \mapsto (\boldsymbol{\mathcal{A}}_{\widehat{\mathbf{L}}}\mathbf{X} - \mathbf{b}, \boldsymbol{\mathcal{C}}_{\widehat{\mathbf{L}}}\mathbf{S} - \mathbf{d}, \langle\!\langle \mathbf{S}\mathbf{X} \rangle\!\rangle_H - \nu\mathbf{D}).$$

Thus by considering (2.4) instead, we may assume, without loss of generality, that $\mathbf{X}(\widehat{\mathbf{w}}) = \mathbf{I}_B$ and $\mathbf{S}(\widehat{\mathbf{w}}) = \mathbf{I}_N$. Since $\mathbf{I}_B \in relint(\mathcal{O}_p)$ and $\mathbf{I}_N \in relint(\mathcal{O}_d)$ by assumption, it follows from a characterization of faces of $\mathbb{S}^n_+$ that

$$\begin{aligned} &relint(\mathcal{O}_p) \subseteq \{\mathbf{X} \in \mathbb{S}^n_+ : \mathbf{X}_{BB} \in \mathbb{S}^k_{++}, \mathbf{X}_{NB} = \mathbf{0}, \mathbf{X}_{NN} = \mathbf{0}\} \quad \text{and} \\ &relint(\mathcal{O}_d) \subseteq \{\mathbf{S} \in \mathbb{S}^n_+ : \mathbf{S}_{NN} \in \mathbb{S}^{n-k}_{++}, \mathbf{S}_{NB} = \mathbf{0}, \mathbf{S}_{BB} = \mathbf{0}\}. \end{aligned} \tag{2.5}$$

We then conclude from Lemma 4 that for every $\mathbf{w} \in \{0\} \oplus \mathbb{D}^n_{++}$,

$$\begin{aligned} \mathbf{X}(\mathbf{w})_{BB} \in \mathbb{S}^k_{++}, \quad \mathbf{S}(\mathbf{w})_{NN} \in \mathbb{S}^{n-k}_{++}, \\ \mathbf{X}(\mathbf{w})_{NN} = \mathbf{0}, \quad \mathbf{S}(\mathbf{w})_{BB} = \mathbf{0}, \\ \mathbf{X}(\mathbf{w})_{NB} = \mathbf{S}(\mathbf{w})_{NB} = \mathbf{0}. \end{aligned} \tag{2.6}$$

Thus the Jacobian $\frac{\partial}{\partial(\mathbf{X},\mathbf{S})}\boldsymbol{\mathcal{F}}$ at $(\mathbf{X}(\widehat{\mathbf{w}}), \mathbf{S}(\widehat{\mathbf{w}}), \widehat{\mathbf{w}})$ is

$$(\boldsymbol{\Delta}\mathbf{X}, \boldsymbol{\Delta}\mathbf{S}, \boldsymbol{\Delta}\mathbf{y}) \mapsto \begin{bmatrix} \boldsymbol{\mathcal{A}}(\boldsymbol{\Delta}\mathbf{X}) \\ \boldsymbol{\mathcal{C}}(\boldsymbol{\Delta}\mathbf{S}) \\ \langle\!\langle (\boldsymbol{\Delta}\mathbf{S})\mathbf{I}_B + \mathbf{I}_N(\boldsymbol{\Delta}\mathbf{X}) \rangle\!\rangle_H \end{bmatrix}.$$

If $\frac{\partial}{\partial(\mathbf{X},\mathbf{S})}\boldsymbol{\mathcal{F}}(\mathbf{X}(\widehat{\mathbf{w}}), \mathbf{S}(\widehat{\mathbf{w}}), \widehat{\mathbf{w}})(\boldsymbol{\Delta}\mathbf{X}, \boldsymbol{\Delta}\mathbf{S}, \boldsymbol{\Delta}\mathbf{y}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$, then

$$\begin{bmatrix} \boldsymbol{\mathcal{A}}_{BB} & \boldsymbol{\mathcal{A}}_{NB} & \boldsymbol{\mathcal{A}}_{NN} & & & \\ & & & \boldsymbol{\mathcal{C}}_{BB} & \boldsymbol{\mathcal{C}}_{NB} & \boldsymbol{\mathcal{C}}_{NN} \\ & & & \boldsymbol{\mathcal{I}} & & \\ & \boldsymbol{\mathcal{S}}_{NB} & & & \boldsymbol{\mathcal{X}}_{NB} & \\ & & \boldsymbol{\mathcal{I}} & & & \end{bmatrix} \begin{bmatrix} (\boldsymbol{\Delta}\mathbf{X})_{BB} \\ (\boldsymbol{\Delta}\mathbf{X})_{NB} \\ (\boldsymbol{\Delta}\mathbf{X})_{NN} \\ (\boldsymbol{\Delta}\mathbf{S})_{BB} \\ (\boldsymbol{\Delta}\mathbf{S})_{NB} \\ (\boldsymbol{\Delta}\mathbf{S})_{NN} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

where $\boldsymbol{\mathcal{A}}_{BB} : \mathbb{S}^k \to \mathbb{R}^m$, $\boldsymbol{\mathcal{A}}_{NB} : \mathbb{R}^{(n-k)\times k} \to \mathbb{R}^m$ and $\boldsymbol{\mathcal{A}}_{NN} : \mathbb{S}^{n-k} \to \mathbb{R}^m$ are maps satisfying

$$\boldsymbol{\mathcal{A}}\mathbf{X} = \boldsymbol{\mathcal{A}}_{BB}\mathbf{X}_{BB} + \boldsymbol{\mathcal{A}}_{NB}\mathbf{X}_{NB} + \boldsymbol{\mathcal{A}}_{NN}\mathbf{X}_{NN}$$

for all $\mathbf{X} \in \mathbb{S}^n$, $\boldsymbol{\mathcal{C}}_{BB} : \mathbb{S}^k \to \mathbb{R}^{\bar{n}^2-m}$, $\boldsymbol{\mathcal{C}}_{NB} : \mathbb{R}^{(n-k)\times k} \to \mathbb{R}^{\bar{n}^2-m}$ and $\boldsymbol{\mathcal{C}}_{NN} : \mathbb{S}^{n-k} \to \mathbb{R}^{\bar{n}^2-m}$ are similarly defined based on $\boldsymbol{\mathcal{C}}$, and $\boldsymbol{\mathcal{S}}_{NB} : \mathbb{R}^{(n-k)\times k} \to \mathbb{R}^{(n-k)\times k}$ and $\boldsymbol{\mathcal{X}}_{NB} : \mathbb{R}^{(n-k)\times k} \to \mathbb{R}^{(n-k)\times k}$ are maps defined by

$$(\boldsymbol{\mathcal{S}}_{NB}(\boldsymbol{\Delta}\mathbf{X}))_{ij} = \begin{cases} (\boldsymbol{\Delta}\mathbf{X})_{ij} & N_i > B_j, \\ 0 & N_i < B_j, \end{cases}$$

and

$$(\boldsymbol{\mathcal{X}}_{NB}(\boldsymbol{\Delta}\mathbf{S}))_{ij} = \begin{cases} (\boldsymbol{\Delta}\mathbf{S})_{ij} & N_i > B_j, \\ 0 & N_i < B_j, \end{cases}$$

where N_i and B_j denote respectively the i-th least element of N and the j-th least element of B.

Clearly if $\boldsymbol{\mathcal{A}}_{BB}$ or $\boldsymbol{\mathcal{C}}_{NN}$ is not injective, then the Jacobian is singular. If $\boldsymbol{\mathcal{A}}_{BB}$ is injective, then we say that $\mathbf{S}(\widehat{\mathbf{w}})$ is *dual nondegenerate*, and if $\boldsymbol{\mathcal{C}}_{NN}$ is injective, then we say that $\mathbf{X}(\widehat{\mathbf{w}})$ is *primal nondegenerate* (see, e.g., [1]). Thus in the case of either primal or dual degeneracy, our approach does not work.

One way to overcome a singular Jacobian is to

1. define new variables $\widetilde{\mathbf{X}}(\mathbf{w})$ and $\widetilde{\mathbf{S}}(\mathbf{w})$ so that the analyticity of the map $\mathbf{w} \mapsto (\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ at $\widehat{\mathbf{w}}$ implies the analyticity of $\mathbf{w} \mapsto (\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$ at $\widehat{\mathbf{w}}$, and
2. rewrite the Cholesky weighted center equations in terms of the new variables so that the resulting Jacobian at $(\widetilde{\mathbf{X}}(\widehat{\mathbf{w}}), \widetilde{\mathbf{S}}(\widehat{\mathbf{w}}), \widehat{\mathbf{w}})$ is nonsingular.

This technique was used by Stoer and Wechs [21] on weighted analytic centers of sufficient linear complementarity problems, by Halická [6, 7] on weighted analytic centers of linear programming problems and central paths of semidefinite programming problems, by Neto, Ferreira and Monteiro [17] on the central paths of degenerate semidefinite programming problems, and by Monteiro and Lu [13, 14] and Preiß and Stoer [20] on weighted centers of SDP problems and semidefinite linear complementarity problems respectively. In both [13] and [20], a pre-requisite for their application of this technique is the boundedness of $\nu^{-1}\mathbf{X}(\mathbf{w})_{NB}$ and $\nu^{-1}\mathbf{S}(\mathbf{w})_{NB}$ as functions of ν — this was proved directly in [13], and shown indirectly in [20] by applying the above technique to deduce the analyticity of a weighted AHO central path as a function of $\sqrt{\nu}$. We, however, do not require the boundedness of $\nu^{-1}\mathbf{X}(\mathbf{w})_{NB}$ and $\nu^{-1}\mathbf{S}(\mathbf{w})_{NB}$. Instead we use the following characterizations of $\mathbf{X}(\widehat{\mathbf{w}})$ and $\mathbf{S}(\widehat{\mathbf{w}})$ to deduce a solution pair of the weighted center equations for the new variables at each $\mathbf{w} = (0, \mathbf{D}) \in \{0\} \oplus \mathbb{D}^n_{++}$, and show that this solution pair is actually the limit $\lim_{\nu\downarrow 0}(\widetilde{\mathbf{X}}(\nu, \mathbf{D}), \widetilde{\mathbf{S}}(\nu, \mathbf{D}))$.

Theorem 5. *Suppose, without loss of generality, that the pair of primal-dual SDP problems* (2.1) *satisfies* (2.5). *If the SDP problems* (2.1) *have strictly complementary solutions, then for each* $\mathbf{D} \in \mathbb{D}^n_{++}$, *the limit of the primal-dual Cholesky weighted central path* $\{(\mathbf{X}(\nu, \mathbf{D}), \mathbf{S}(\nu, \mathbf{D})) : \nu > 0\}$ *is the pair of unique minimizers of* $\mathbf{X} \mapsto -\sum_{i\in B} \mathbf{D}_{ii} \log(\mathbf{L}_\mathbf{X})^2_{ii}$ *and* $\mathbf{S} \mapsto -\sum_{i\in N} \mathbf{D}_{ii} \log(\mathbf{U}_\mathbf{S})^2_{ii}$ *respectively over the primal and dual optimal faces.*

Proof. The statement on $\mathbf{X}(0, \mathbf{D})$ is [4, Theorem 14]. The statement on $\mathbf{S}(0, \mathbf{D})$ follows from applying the proof of [4, Theorem 14] to the dual problem. □

Observe that a contributing factor to the singularity of the Jacobian is the lack of positive definiteness of both $\mathbf{X}(\widehat{\mathbf{w}})$ and $\mathbf{S}(\widehat{\mathbf{w}})$, or more specifically, $\mathbf{X}(\widehat{\mathbf{w}})_{NN} = \mathbf{0}$ and $\mathbf{S}(\widehat{\mathbf{w}})_{BB} = \mathbf{0}$. As an attempt to remove this lack of positive definiteness, we define $(\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ for $\mathbf{w} \in \mathbb{R}_{++} \oplus \mathbb{D}^n_{++}$ by

$$\begin{aligned} \widetilde{\mathbf{X}}(\mathbf{w})_{BB} &= \mathbf{X}(\mathbf{w})_{BB}, & \widetilde{\mathbf{S}}(\mathbf{w})_{BB} &= \nu^{-1}\mathbf{S}(\mathbf{w})_{BB}, \\ \widetilde{\mathbf{X}}(\mathbf{w})_{NB} &= \nu^{-1}\mathbf{X}(\mathbf{w})_{NB}, & \widetilde{\mathbf{S}}(\mathbf{w})_{NB} &= \nu^{-1}\mathbf{S}(\mathbf{w})_{NB}, \\ \widetilde{\mathbf{X}}(\mathbf{w})_{NN} &= \nu^{-1}\mathbf{X}(\mathbf{w})_{NN}, & \widetilde{\mathbf{S}}(\mathbf{w})_{NN} &= \mathbf{S}(\mathbf{w})_{NN}, \end{aligned} \tag{2.7}$$

and for $\mathbf{w} \in \{0\} \oplus \mathbb{D}^n_{++}$ by

$$\widetilde{\mathbf{X}}(\mathbf{w})_{BB} = \mathbf{X}(\mathbf{w})_{BB}, \quad \widetilde{\mathbf{S}}(\mathbf{w})_{NN} = \mathbf{S}(\mathbf{w})_{NN}, \tag{2.8}$$

where $\widetilde{\mathbf{X}}(\mathbf{w})_{NN}$, $\widetilde{\mathbf{X}}(\mathbf{w})_{NB}$, $\widetilde{\mathbf{S}}(\mathbf{w})_{BB}$ and $\widetilde{\mathbf{S}}(\mathbf{w})_{NB}$ will be defined later. It follows from (2.6) that regardless of how $\widetilde{\mathbf{X}}(\mathbf{w})_{NN}$, $\widetilde{\mathbf{X}}(\mathbf{w})_{NB}$, $\widetilde{\mathbf{S}}(\mathbf{w})_{BB}$ and $\widetilde{\mathbf{S}}(\mathbf{w})_{NB}$ are defined for $\mathbf{w} \in \{0\} \oplus \mathbb{D}^n_{++}$, we have

$$\begin{aligned} \widetilde{\mathbf{X}}(\mathbf{w})_{BB} &= \mathbf{X}(\mathbf{w})_{BB}, & \nu\widetilde{\mathbf{S}}(\mathbf{w})_{BB} &= \mathbf{S}(\mathbf{w})_{BB}, \\ \nu\widetilde{\mathbf{X}}(\mathbf{w})_{NB} &= \mathbf{X}(\mathbf{w})_{NB}, & \nu\widetilde{\mathbf{S}}(\mathbf{w})_{NB} &= \mathbf{S}(\mathbf{w})_{NB}, \\ \nu\widetilde{\mathbf{X}}(\mathbf{w})_{NN} &= \mathbf{X}(\mathbf{w})_{NN}, & \widetilde{\mathbf{S}}(\mathbf{w})_{NN} &= \mathbf{S}(\mathbf{w})_{NN}, \end{aligned}$$

for each $\mathbf{w} \in \mathbb{R}_+ \oplus \mathbb{D}^n_{++}$, and consequently the analyticity of $\mathbf{w} \mapsto (\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ at $\widehat{\mathbf{w}}$ will imply the same for $\mathbf{w} \mapsto (\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$.

From (2.2a), we see that $(\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ satisfies

$$\begin{aligned} \mathbf{0} &= \boldsymbol{\mathcal{A}}_{BB}(\mathbf{X}_{BB} - \mathbf{I}) + \nu\boldsymbol{\mathcal{A}}_{NB}\mathbf{X}_{NB} + \nu\boldsymbol{\mathcal{A}}_{NN}\mathbf{X}_{NN}, \\ \mathbf{0} &= \nu\boldsymbol{\mathcal{C}}_{BB}\mathbf{S}_{BB} + \nu\boldsymbol{\mathcal{C}}_{NB}\mathbf{S}_{NB} + \boldsymbol{\mathcal{C}}_{NN}(\mathbf{S}_{NN} - \mathbf{I}), \end{aligned} \tag{2.9}$$

where the respective primal and dual feasibility of $\mathbf{X}(\widehat{\mathbf{w}}) = \mathbf{I}_B$ and $\mathbf{S}(\widehat{\mathbf{w}}) = \mathbf{I}_N$ are used to derive the equations. If we use (2.9) in our defining equations for the new variables, then we may run into another cause of singularity: One of the maps $\mathbf{X} \in \mathbb{S}^n \mapsto \boldsymbol{\mathcal{A}}_{BB}\mathbf{X}_{BB} + \nu\boldsymbol{\mathcal{A}}_{NB}\mathbf{X}_{NB} + \nu\boldsymbol{\mathcal{A}}_{NN}\mathbf{X}_{NN}$ and $\mathbf{S} \in \mathbb{S}^n \mapsto \nu\boldsymbol{\mathcal{C}}_{BB}\mathbf{S}_{BB} + \nu\boldsymbol{\mathcal{C}}_{NB}\mathbf{S}_{NB} + \boldsymbol{\mathcal{C}}_{NN}\mathbf{S}_{NN}$ may no longer be surjective when $\nu = 0$, in which case their kernels will have a non-trivial intersection.

This can be rectified using linear automorphisms $\boldsymbol{\mathcal{T}}$ on $\mathbb{R}^m$ and $\boldsymbol{\mathcal{U}}$ on $\mathbb{R}^{n^2-m}$ that respectively reduce $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{C}}$ to row-echelon forms

$$\boldsymbol{\mathcal{T}} \circ \boldsymbol{\mathcal{A}} : \begin{bmatrix} \mathbf{X}_{BB} \\ \mathbf{X}_{NB} \\ \mathbf{X}_{NN} \end{bmatrix} \mapsto \begin{bmatrix} \boldsymbol{\mathcal{A}}_{11}\mathbf{X}_{BB} + \boldsymbol{\mathcal{A}}_{12}\mathbf{X}_{NB} + \boldsymbol{\mathcal{A}}_{13}\mathbf{X}_{NN} \\ \boldsymbol{\mathcal{A}}_{22}\mathbf{X}_{NB} + \boldsymbol{\mathcal{A}}_{23}\mathbf{X}_{NN} \\ \boldsymbol{\mathcal{A}}_{33}\mathbf{X}_{NN} \end{bmatrix}$$

and

$$\boldsymbol{\mathcal{U}} \circ \boldsymbol{\mathcal{C}} : \begin{bmatrix} \mathbf{S}_{BB} \\ \mathbf{S}_{NB} \\ \mathbf{S}_{NN} \end{bmatrix} \mapsto \begin{bmatrix} \boldsymbol{\mathcal{C}}_{11}\mathbf{S}_{BB} \\ \boldsymbol{\mathcal{C}}_{21}\mathbf{S}_{BB} + \boldsymbol{\mathcal{C}}_{22}\mathbf{S}_{NB} \\ \boldsymbol{\mathcal{C}}_{31}\mathbf{S}_{BB} + \boldsymbol{\mathcal{C}}_{32}\mathbf{S}_{NB} + \boldsymbol{\mathcal{C}}_{33}\mathbf{S}_{NN} \end{bmatrix},$$

so that $\boldsymbol{\mathcal{A}}_{ii}$, $\boldsymbol{\mathcal{C}}_{ii}$ are surjective for each $i \in \{1, 2, 3\}$ (see [14, Lemma 3.6] and [20, Page 513]). Since $Ker(\boldsymbol{\mathcal{T}} \circ \boldsymbol{\mathcal{A}}) = Ker(\boldsymbol{\mathcal{A}})$ and $Ker(\boldsymbol{\mathcal{U}} \circ \boldsymbol{\mathcal{C}}) = Ker(\boldsymbol{\mathcal{C}})$, we may assume, by using $\{\mathbf{X} : (\boldsymbol{\mathcal{T}} \circ \boldsymbol{\mathcal{A}})\mathbf{X} = \boldsymbol{\mathcal{T}}\mathbf{b}\}$ and $\{\mathbf{S} : (\boldsymbol{\mathcal{U}} \circ \boldsymbol{\mathcal{C}})\mathbf{S} = \boldsymbol{\mathcal{U}}\mathbf{d}\}$ to describe the affine spaces $\mathcal{L} + \widehat{\mathbf{X}}$ and $\mathcal{L}^\perp + \widehat{\mathbf{S}}$ respectively, that both $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{C}}$ are in the above respective row-echelon form. By rewriting (2.9) using the row-echelon form for $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{C}}$, and dividing the appropriate rows by ν, we get

$$\mathbf{0} = \boldsymbol{\mathcal{A}}_{11}(\mathbf{X}_{BB} - \mathbf{I}) + \nu\boldsymbol{\mathcal{A}}_{12}\mathbf{X}_{NB} + \nu\boldsymbol{\mathcal{A}}_{13}\mathbf{X}_{NN}, \tag{2.10a}$$
$$\mathbf{0} = \boldsymbol{\mathcal{A}}_{22}\mathbf{X}_{NB} + \boldsymbol{\mathcal{A}}_{23}\mathbf{X}_{NN}, \tag{2.10b}$$
$$\mathbf{0} = \boldsymbol{\mathcal{A}}_{33}\mathbf{X}_{NN}, \tag{2.10c}$$
$$\mathbf{0} = \boldsymbol{\mathcal{C}}_{11}\mathbf{S}_{BB}, \tag{2.10d}$$
$$\mathbf{0} = \boldsymbol{\mathcal{C}}_{21}\mathbf{S}_{BB} + \boldsymbol{\mathcal{C}}_{22}\mathbf{S}_{NB}, \tag{2.10e}$$
$$\mathbf{0} = \nu\boldsymbol{\mathcal{C}}_{31}\mathbf{S}_{BB} + \nu\boldsymbol{\mathcal{C}}_{32}\mathbf{S}_{NB} + \boldsymbol{\mathcal{C}}_{33}(\mathbf{S}_{NN} - \mathbf{I}), \tag{2.10f}$$

which are still satisfied by $(\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ for each $\mathbf{w} \in \mathbb{R}_{++} \oplus \mathbb{D}^n_{++}$. For $\mathbf{w} \in \{0\} \oplus \mathbb{D}^n_{++}$, we shall define $\widetilde{\mathbf{X}}(\mathbf{w})_{NN}$, $\widetilde{\mathbf{X}}(\mathbf{w})_{NB}$, $\widetilde{\mathbf{S}}(\mathbf{w})_{BB}$ and $\widetilde{\mathbf{S}}(\mathbf{w})_{NB}$ so that the pair $(\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ satisfies (2.10).

Before that, let us look at the remaining equation (2.2b). From (2.2b), we see that $(\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ satisfies

$$\begin{aligned} \nu\mathbf{D}_{BB} &= \langle\!\langle \nu^2\mathbf{S}^T_{NB}\mathbf{X}_{NB} + \nu\mathbf{S}_{BB}\mathbf{X}_{BB} \rangle\!\rangle_H, \\ \nu\mathbf{D}_{NN} &= \langle\!\langle \nu^2\mathbf{S}_{NB}\mathbf{X}^T_{NB} + \nu\mathbf{S}_{NN}\mathbf{X}_{NN} \rangle\!\rangle_H, \\ 0 &= \nu(\mathbf{S}_{N*}\mathbf{X}_{*B})_{ij} \quad \text{for } N_i > B_j, \\ 0 &= \nu^2(\mathbf{S}^T_{*B}\mathbf{X}^T_{N*})_{ij} \quad \text{for } B_i > N_j. \end{aligned}$$

Dividing the appropriate equations by ν and ν^2 gives

$$\mathbf{0} = \langle\!\langle \nu\mathbf{S}^T_{NB}\mathbf{X}_{NB} + \mathbf{S}_{BB}\mathbf{X}_{BB} \rangle\!\rangle_H - \mathbf{D}_{BB}, \tag{2.11a}$$
$$\mathbf{0} = \langle\!\langle \nu\mathbf{S}_{NB}\mathbf{X}^T_{NB} + \mathbf{S}_{NN}\mathbf{X}_{NN} \rangle\!\rangle_H - \mathbf{D}_{NN}, \tag{2.11b}$$
$$0 = (\mathbf{S}_{N*}\mathbf{X}_{*B})_{ij}, \quad \text{for } N_i > B_j, \tag{2.11c}$$
$$0 = (\mathbf{S}^T_{*B}\mathbf{X}^T_{N*})_{ij}, \quad \text{for } B_i > N_j, \tag{2.11d}$$

which are still satisfied by $(\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ for each $\mathbf{w} \in \mathbb{R}_{++} \oplus \mathbb{D}^n_{++}$. We now define $\widetilde{\mathbf{X}}(\mathbf{w})_{NN}$, $\widetilde{\mathbf{X}}(\mathbf{w})_{NB}$, $\widetilde{\mathbf{S}}(\mathbf{w})_{BB}$ and $\widetilde{\mathbf{S}}(\mathbf{w})_{NB}$ for $\mathbf{w} \in \{0\} \oplus \mathbb{D}^n_{++}$ so that the pair $(\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ satisfies both (2.10) and (2.11).

Equations (2.10a) and (2.10f) are clearly satisfied by definitions of $\widetilde{\mathbf{X}}(\mathbf{w})_{BB}$ and $\widetilde{\mathbf{S}}(\mathbf{w})_{NN}$ in (2.8). Since $\nu = 0$, equations (2.11a) and (2.11b) are satisfied if and only if

(2.12)
$$\widetilde{\mathbf{S}}(\mathbf{w})_{BB} = \mathbf{L}^{-T}_{\mathbf{X}(\mathbf{w})_{BB}} \mathbf{D}_{BB} \mathbf{L}^{-1}_{\mathbf{X}(\mathbf{w})_{BB}} \quad \text{and} \quad \widetilde{\mathbf{X}}(\mathbf{w})_{NN} = \mathbf{U}^{-T}_{\mathbf{S}(\mathbf{w})_{NN}} \mathbf{D}_{NN} \mathbf{U}^{-1}_{\mathbf{S}(\mathbf{w})_{NN}}.$$

With these definitions of $\widetilde{\mathbf{S}}(\mathbf{w})_{BB}$ and $\widetilde{\mathbf{X}}(\mathbf{w})_{NN}$, (2.10c) and (2.10d) are satisfied as consequences of Theorem 5. To see this, note that the definition of $\widetilde{\mathbf{S}}(\mathbf{w})_{BB}$ is precisely the gradient of $\mathbf{X} \mapsto -\sum_{i \in B} \mathbf{D}_{ii} \log(\mathbf{L}_{\mathbf{X}})^2_{ii}$ at $\mathbf{X}(\mathbf{w})_{BB}$ (see [5, Section 6]). Since Lemma 6 below states that $Ker(\boldsymbol{\mathcal{C}}_{11})$ is the orthogonal complement of $Ker(\boldsymbol{\mathcal{A}}_{11})$ in the domain of $\boldsymbol{\mathcal{A}}_{11}$, (2.10d) is exactly the statement that the gradient of $\mathbf{X} \mapsto -\sum_{i \in B} \mathbf{D}_{ii} \log(\mathbf{L}_{\mathbf{X}})^2_{ii}$ at $\widetilde{\mathbf{X}}(\mathbf{w})_{BB}$ is orthogonal to the kernel of $\boldsymbol{\mathcal{A}}_{11}$, which is a consequence of Theorem 5. Equation (2.10c) can be deduced similarly.

Lemma 6. *For $i = 1, 2, 3$, $Ker(\boldsymbol{\mathcal{A}}_{ii}) \perp Ker(\boldsymbol{\mathcal{C}}_{ii})$.*

Consequently, for $i = 1, 2, 3$, $Ker(\boldsymbol{\mathcal{C}}_{ii})$ is the orthogonal complement of $Ker(\boldsymbol{\mathcal{A}}_{ii})$ in the domain of $\boldsymbol{\mathcal{A}}_{ii}$ (or equivalently, the domain of $\boldsymbol{\mathcal{C}}_{ii}$).

Proof. We prove the case $i = 2$ and remark that the proofs of other cases are similar.

Suppose $\mathbf{Y}$ and $\mathbf{W}$ are matrices satisfying $\boldsymbol{\mathcal{A}}_{22}\mathbf{Y} = \mathbf{0}$ and $\boldsymbol{\mathcal{C}}_{22}\mathbf{W} = \mathbf{0}$. Let $\mathbf{X}, \mathbf{S} \in \mathbb{S}^n$ be matrices satisfying

$$\mathbf{X}_{NB} = \mathbf{Y},\ \mathbf{S}_{NB} = \mathbf{W},\ \mathbf{X}_{NN} = \mathbf{0},\ \mathbf{S}_{BB} = \mathbf{0},$$
$$\boldsymbol{\mathcal{A}}_{11}\mathbf{X}_{BB} = -\boldsymbol{\mathcal{A}}_{12}\mathbf{X}_{NB},\ \boldsymbol{\mathcal{C}}_{33}\mathbf{S}_{NN} = -\boldsymbol{\mathcal{C}}_{32}\mathbf{S}_{NB}.$$

The existence of $\mathbf{X}$ and $\mathbf{S}$ is ensured by the surjectiveness of $\boldsymbol{\mathcal{A}}_{11}$ and $\boldsymbol{\mathcal{C}}_{33}$ respectively. Consequently $\mathbf{X} \in Ker(\boldsymbol{\mathcal{T}} \circ \boldsymbol{\mathcal{A}}) = Ker(\boldsymbol{\mathcal{A}}) \perp Ker(\boldsymbol{\mathcal{C}}) = Ker(\boldsymbol{\mathcal{U}} \circ \boldsymbol{\mathcal{C}}) \ni \mathbf{S}$ implies that $0 = \mathbf{X} \bullet \mathbf{S} = \mathbf{Y} \bullet \mathbf{W}$.

For the last statement of the lemma, we observe that $Ker(\boldsymbol{\mathcal{A}}_{ii}) \perp Ker(\boldsymbol{\mathcal{C}}_{ii})$ implies that

$$\sum_{i=1}^{3} (\text{nullity of } \boldsymbol{\mathcal{A}}_{ii} + \text{nullity of } \boldsymbol{\mathcal{C}}_{ii}) \le n^{\bar{2}}.$$

On the other hand, $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{C}}$ were chosen to satisfy

$$\text{nullity of } \boldsymbol{\mathcal{A}} + \text{nullity of } \boldsymbol{\mathcal{C}} = n^{\bar{2}}.$$

Since nullity of $\boldsymbol{\mathcal{A}} = \sum_{i=1}^{3}(\text{nullity of } \boldsymbol{\mathcal{A}}_{ii})$ and nullity of $\boldsymbol{\mathcal{C}} = \sum_{i=1}^{3}(\text{nullity of } \boldsymbol{\mathcal{C}}_{ii})$, we necessarily have

$$\text{nullity of } \boldsymbol{\mathcal{A}}_{ii} + \text{nullity of } \boldsymbol{\mathcal{C}}_{ii} = dim(domain(\boldsymbol{\mathcal{A}}_{ii})),$$

whence $Ker(\boldsymbol{\mathcal{C}}_{ii})$ is the orthogonal complement of $Ker(\boldsymbol{\mathcal{A}}_{ii})$ in the domain of $\boldsymbol{\mathcal{A}}_{ii}$, for $i = 1, 2, 3$. □

It remains to consider (2.10b), (2.10e), (2.11c) and (2.11d).

Lemma 7. *If* $\widehat{\boldsymbol{\mathcal{A}}} : \mathbb{R}^{(n-k)\times k} \to \mathbb{R}^p$ *and* $\widehat{\boldsymbol{\mathcal{C}}} : \mathbb{R}^{(n-k)\times k} \to \mathbb{R}^{(n-k)k-p}$ *are linear maps satisfying* $Ker(\widehat{\boldsymbol{\mathcal{A}}}) \perp Ker(\widehat{\boldsymbol{\mathcal{C}}})$, *and* $\widehat{\mathbf{X}}, \widehat{\mathbf{S}} \in \mathbb{S}^n_{++}$ *are matrices satisfying*

$$\widehat{\mathbf{X}}_{NB} = \widehat{\mathbf{S}}_{NB} = \mathbf{0}, \quad \mathbf{L}^T_{\widehat{\mathbf{X}}_{BB}} \widehat{\mathbf{S}}_{BB} \mathbf{L}_{\widehat{\mathbf{X}}_{BB}} \in \mathbb{D}^k_{++} \quad \text{and} \quad \mathbf{L}^T_{\widehat{\mathbf{X}}_{NN}} \widehat{\mathbf{S}}_{NN} \mathbf{L}_{\widehat{\mathbf{X}}_{NN}} \in \mathbb{D}^{n-k}_{++},$$

then the linear map

$$\boldsymbol{\mathcal{J}} : (\mathbf{Y}, \mathbf{W}) \in \mathbb{R}^{(n-k)\times k} \oplus \mathbb{R}^{(n-k)\times k} \mapsto \begin{bmatrix} \widehat{\boldsymbol{\mathcal{A}}}\mathbf{Y} \\ \widehat{\boldsymbol{\mathcal{C}}}\mathbf{W} \\ \mathbf{Z} \end{bmatrix},$$

where $\mathbf{Z} \in \mathbb{R}^{(n-k)\times k}$ *is defined by*

$$\mathbf{Z}_{ij} = \begin{cases} (\widehat{\mathbf{S}}_{NN}\mathbf{Y} + \mathbf{W}\widehat{\mathbf{X}}_{BB})_{ij} & \text{if } N_i > B_j, \\ (\widehat{\mathbf{S}}_{BB}\mathbf{Y}^T + \mathbf{W}^T\widehat{\mathbf{X}}_{NN})_{ji} & \text{if } N_i < B_j, \end{cases}$$

is bijective.

Proof. It suffices to show that $\boldsymbol{\mathcal{J}}$ is injective.

Let $\mathbf{X}, \mathbf{S} \in \mathbb{S}^n$ be the matrices defined by

$$\mathbf{X}_{NB} = \mathbf{Y}, \quad \mathbf{S}_{NB} = \mathbf{W},$$
$$\mathbf{X}_{BB} = \mathbf{S}_{BB} = \mathbf{0} \quad \text{and} \quad \mathbf{X}_{NN} = \mathbf{S}_{NN} = \mathbf{0}.$$

Suppose $\boldsymbol{\mathcal{J}}(\mathbf{Y}, \mathbf{W}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$. Then $\mathbf{Z} = \mathbf{0}$ implies

$$\langle\!\langle \mathbf{S}\widehat{\mathbf{X}} + \widehat{\mathbf{S}}\mathbf{X} \rangle\!\rangle_H = \mathbf{0}.$$

Consequently $(\mathbf{X}, \mathbf{S})$ satisfies

$$\begin{aligned} \widetilde{\boldsymbol{\mathcal{A}}}\mathbf{X} &= (\mathbf{0}, \mathbf{0}), \\ \widetilde{\boldsymbol{\mathcal{C}}}\mathbf{S} &= (\mathbf{0}, \mathbf{0}), \\ \langle\!\langle \mathbf{S}\widehat{\mathbf{X}} + \widehat{\mathbf{S}}\mathbf{X} \rangle\!\rangle_H &= \mathbf{0}. \end{aligned}$$

where $\widetilde{\boldsymbol{\mathcal{A}}} : \mathbb{S}^n \to \mathbb{R}^p \oplus \mathbb{S}^{n-k}$ and $\widetilde{\boldsymbol{\mathcal{C}}} : \mathbb{S}^n \to \mathbb{R}^{(n-k)k-p} \oplus \mathbb{S}^k$ denote respectively the maps

$$\mathbf{X} \mapsto (\widehat{\boldsymbol{\mathcal{A}}}\mathbf{X}_{NB}, \mathbf{X}_{NN}) \quad \text{and} \quad \mathbf{S} \mapsto (\widehat{\boldsymbol{\mathcal{C}}}\mathbf{S}_{NB}, \mathbf{S}_{BB}).$$

Note that $Ker(\widetilde{\boldsymbol{\mathcal{A}}}) \perp Ker(\widetilde{\boldsymbol{\mathcal{C}}})$ since $Ker(\widehat{\boldsymbol{\mathcal{A}}}) \perp Ker(\widehat{\boldsymbol{\mathcal{C}}})$. Let $\mathbf{P}$ denote the permutation matrix satisfying

$$(\mathbf{P}^T\mathbf{v})_i = \begin{cases} \mathbf{v}_{B_i} & \text{if } i \le k, \\ \mathbf{v}_{N_{i-k}} & \text{if } i > k, \end{cases}$$

for all $\mathbf{v} \in \mathbb{R}^n$. It follows that

$$\mathbf{P}^T\widehat{\mathbf{X}}\mathbf{P} = \begin{bmatrix} \widehat{\mathbf{X}}_{BB} & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{X}}_{NN} \end{bmatrix} \quad \text{and} \quad \mathbf{P}^T\widehat{\mathbf{S}}\mathbf{P} = \begin{bmatrix} \widehat{\mathbf{S}}_{BB} & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{S}}_{NN} \end{bmatrix}.$$

Moreover

$$\widehat{\mathbf{S}}\widehat{\mathbf{X}} = \mathbf{P} \begin{bmatrix} \widehat{\mathbf{S}}_{BB}\widehat{\mathbf{X}}_{BB} & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{S}}_{NN}\widehat{\mathbf{X}}_{NN} \end{bmatrix} \mathbf{P}^T$$

is upper-triangular since $\widehat{\mathbf{S}}_{BB}\widehat{\mathbf{X}}_{BB}$ and $\widehat{\mathbf{S}}_{NN}\widehat{\mathbf{X}}_{NN}$ are upper-triangular, and the order of the indices in B and the order of those in N remain unchanged by $\mathbf{P}$. Thus we may apply Lemma 1 to deduce that $\mathbf{X}$ and $\mathbf{S}$, whence $\mathbf{Y}$ and $\mathbf{W}$, are zero matrices. $\square$

As a consequence of Lemmas 6 and 7, the system of equations

$$\begin{aligned}
-\boldsymbol{\mathcal{A}}_{23}\widetilde{\mathbf{X}}(\mathbf{w})_{NN} &= \boldsymbol{\mathcal{A}}_{22}\mathbf{X}_{NB},\\
-\boldsymbol{\mathcal{C}}_{21}\widetilde{\mathbf{S}}(\mathbf{w})_{BB} &= \boldsymbol{\mathcal{C}}_{22}\mathbf{S}_{NB},\\
0 &= (\widetilde{\mathbf{S}}(\mathbf{w})_{NN}\mathbf{X}_{NB} + \mathbf{S}_{NB}\widetilde{\mathbf{X}}(\mathbf{w})_{BB})_{ij} \quad \text{for } N_i > B_j,\\
0 &= (\widetilde{\mathbf{S}}(\mathbf{w})_{BB}\mathbf{X}_{NB}^T + \mathbf{S}_{NB}^T\widetilde{\mathbf{X}}(\mathbf{w})_{NN})_{ij} \quad \text{for } B_i > N_j,
\end{aligned}$$

has a unique pair of solutions $(\widehat{\mathbf{Y}}(\mathbf{w}), \widehat{\mathbf{W}}(\mathbf{w}))$. Thus by defining $\widetilde{\mathbf{X}}(\mathbf{w})_{NB} = \widehat{\mathbf{Y}}(\mathbf{w})$ and $\widetilde{\mathbf{S}}(\mathbf{w})_{NB} = \widehat{\mathbf{W}}(\mathbf{w})$, we have that $(\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ satisfies (2.10) and (2.11).

Lemma 8. *The Jacobian of the system of equations* (2.10) *and* (2.11) *is nonsingular at* $(\widetilde{\mathbf{X}}(\widehat{\mathbf{w}}), \widetilde{\mathbf{S}}(\widehat{\mathbf{w}}), \widehat{\mathbf{w}})$ *for each* $\widehat{\mathbf{w}} \in \{0\} \oplus \mathbb{D}^n_{++}$.

Proof. Suppose that $(\boldsymbol{\Delta}\mathbf{X}, \boldsymbol{\Delta}\mathbf{S})$ is in the null space of the Jacobian of the system (2.10) and (2.11) at $(\widetilde{\mathbf{X}}(\widehat{\mathbf{w}}), \widetilde{\mathbf{S}}(\widehat{\mathbf{w}}), \widehat{\mathbf{w}})$. Then

$$\mathbf{0} = \boldsymbol{\mathcal{A}}_{11}(\boldsymbol{\Delta}\mathbf{X})_{BB}, \tag{2.13a}$$

$$\mathbf{0} = \boldsymbol{\mathcal{A}}_{22}(\boldsymbol{\Delta}\mathbf{X})_{NB} + \boldsymbol{\mathcal{A}}_{23}(\boldsymbol{\Delta}\mathbf{X})_{NN}, \tag{2.13b}$$

$$\mathbf{0} = \boldsymbol{\mathcal{A}}_{33}(\boldsymbol{\Delta}\mathbf{X})_{NN}, \tag{2.13c}$$

$$\mathbf{0} = \boldsymbol{\mathcal{C}}_{11}(\boldsymbol{\Delta}\mathbf{S})_{BB}, \tag{2.13d}$$

$$\mathbf{0} = \boldsymbol{\mathcal{C}}_{21}(\boldsymbol{\Delta}\mathbf{S})_{BB} + \boldsymbol{\mathcal{C}}_{22}(\boldsymbol{\Delta}\mathbf{S})_{NB}, \tag{2.13e}$$

$$\mathbf{0} = \boldsymbol{\mathcal{C}}_{33}(\boldsymbol{\Delta}\mathbf{S})_{NN}, \tag{2.13f}$$

$$\mathbf{0} = \langle\!\langle \widetilde{\mathbf{S}}(\widehat{\mathbf{w}})_{BB}(\boldsymbol{\Delta}\mathbf{X})_{BB} + (\boldsymbol{\Delta}\mathbf{S})_{BB}\widetilde{\mathbf{X}}(\widehat{\mathbf{w}})_{BB} \rangle\!\rangle_H, \tag{2.13g}$$

$$\mathbf{0} = \langle\!\langle \widetilde{\mathbf{S}}(\widehat{\mathbf{w}})_{NN}(\boldsymbol{\Delta}\mathbf{X})_{NN} + (\boldsymbol{\Delta}\mathbf{S})_{NN}\widetilde{\mathbf{X}}(\widehat{\mathbf{w}})_{NN} \rangle\!\rangle_H, \tag{2.13h}$$

$$\begin{aligned} 0 = {} & (\widetilde{\mathbf{S}}(\widehat{\mathbf{w}})_{NN}(\boldsymbol{\Delta}\mathbf{X})_{NB} + (\boldsymbol{\Delta}\mathbf{S})_{NB}\widetilde{\mathbf{X}}(\widehat{\mathbf{w}})_{BB})_{ij} \\ & + ((\boldsymbol{\Delta}\mathbf{S})_{NN}\widetilde{\mathbf{X}}(\widehat{\mathbf{w}})_{NB} + \widetilde{\mathbf{S}}(\widehat{\mathbf{w}})_{NB}(\boldsymbol{\Delta}\mathbf{X})_{BB})_{ij} \quad \text{for } N_i > B_j, \end{aligned} \tag{2.13i}$$

$$\begin{aligned} 0 = {} & (\widetilde{\mathbf{S}}(\widehat{\mathbf{w}})_{BB}(\boldsymbol{\Delta}\mathbf{X})_{NB}^T + (\boldsymbol{\Delta}\mathbf{S})_{NB}^T\widetilde{\mathbf{X}}(\widehat{\mathbf{w}})_{NN})_{ij} \\ & + ((\boldsymbol{\Delta}\mathbf{S})_{BB}\widetilde{\mathbf{X}}(\widehat{\mathbf{w}})_{NB}^T + \widetilde{\mathbf{S}}(\widehat{\mathbf{w}})_{NB}^T(\boldsymbol{\Delta}\mathbf{X})_{NN})_{ij} \quad \text{for } B_i > N_j. \end{aligned} \tag{2.13j}$$

By Lemma 6, we may apply Lemma 1 to (2.13a), (2.13d) and (2.13g) to conclude that $(\boldsymbol{\Delta}\mathbf{X})_{BB} = (\boldsymbol{\Delta}\mathbf{S})_{BB} = \mathbf{0}$. Similarly, applying Lemma 1 to (2.13c), (2.13f) and (2.13h) gives $(\boldsymbol{\Delta}\mathbf{X})_{NN} = (\boldsymbol{\Delta}\mathbf{S})_{NN} = \mathbf{0}$. Thus (2.13b), (2.13e), (2.13i) and (2.13j) respectively simplifies to

$$\begin{aligned}
&\mathbf{0} = \boldsymbol{\mathcal{A}}_{22}(\boldsymbol{\Delta}\mathbf{X})_{NB},\\
&\mathbf{0} = \boldsymbol{\mathcal{C}}_{22}(\boldsymbol{\Delta}\mathbf{S})_{NB},\\
&0 = (\widetilde{\mathbf{S}}(\widehat{\mathbf{w}})_{NN}(\boldsymbol{\Delta}\mathbf{X})_{NB} + (\boldsymbol{\Delta}\mathbf{S})_{NB}\widetilde{\mathbf{X}}(\widehat{\mathbf{w}})_{BB})_{ij} \quad \text{for } N_i > B_j, \quad \text{and}\\
&0 = (\widetilde{\mathbf{S}}(\widehat{\mathbf{w}})_{BB}(\boldsymbol{\Delta}\mathbf{X})_{NB}^T + (\boldsymbol{\Delta}\mathbf{S})_{NB}^T\widetilde{\mathbf{X}}(\widehat{\mathbf{w}})_{NN})_{ij} \quad \text{for } B_i > N_j.
\end{aligned}$$

By applying Lemma 7 to the above equations, we conclude that $(\boldsymbol{\Delta}\mathbf{X})_{NB} = (\boldsymbol{\Delta}\mathbf{S})_{NB} = \mathbf{0}$. □

We are now ready to prove the main theorem of this section.

Theorem 9. *If* (2.1) *has strictly complementary solutions, then the map*

$$\mathbf{w} \in \mathbb{R}_+ \oplus \mathbb{D}^n_{++} \mapsto (\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$$

is analytic.

Proof. Since it was established in Theorem 2 that the above map is analytic over $\mathbb{R}_{++} \oplus \mathbb{D}^n_{++}$, it remains to show that it is analytic at each $\widehat{\mathbf{w}} \in \{0\} \oplus \mathbb{D}^n_{++}$. To this end, as discussed in the paragraph following Theorem 5, it suffices to show that the map $\mathbf{w} \in \mathbb{R}_+ \oplus \mathbb{D}^n_{++} \mapsto (\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ defined above is analytic at each $\widehat{\mathbf{w}} \in \{0\} \oplus \mathbb{D}^n_{++}$. We have established that $(\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ satisfies the system of equations (2.10) and (2.11) for all $\mathbf{w} \in \mathbb{R}_+ \oplus \mathbb{D}^n_{++}$. We now fixed an $\widehat{\mathbf{w}} = (0, \widehat{\mathbf{D}}) \in \{0\} \oplus \mathbb{D}^n_{++}$. Lemma 8 states that the Jacobian of the system is nonsingular at $(\widetilde{\mathbf{X}}(\widehat{\mathbf{w}}), \widetilde{\mathbf{S}}(\widehat{\mathbf{w}}), \widehat{\mathbf{w}})$. Thus applying the Implicit Function Theorem on (2.10) and (2.11) gives an open set $\widehat{\mathbf{w}} \in \mathcal{U} \oplus \mathcal{V} \subseteq \mathbb{R} \oplus \mathbb{D}^n$ and a unique continuous map

$$\mathbf{w} \in \mathcal{U} \oplus \mathcal{V} \mapsto (\widetilde{\mathbf{X}}'(\mathbf{w}), \widetilde{\mathbf{S}}'(\mathbf{w})) \tag{2.14}$$

such that $(\widetilde{\mathbf{X}}'(\mathbf{w}), \widetilde{\mathbf{S}}'(\mathbf{w}))$ solves (2.10) and (2.11) for each $\mathbf{w} = (\nu, \mathbf{D}) \in \mathcal{U} \oplus \mathcal{V}$, and the pair $(\widetilde{\mathbf{X}}'(\widehat{\mathbf{w}}), \widetilde{\mathbf{S}}'(\widehat{\mathbf{w}}))$ coincides with $(\widetilde{\mathbf{X}}(\widehat{\mathbf{w}}), \widetilde{\mathbf{S}}(\widehat{\mathbf{w}}))$. Moreover the analyticity of the system (2.10) and (2.11) implies that the map (2.14) is analytic. The definitions in (2.8) and (2.12) imply that $\widetilde{\mathbf{X}}(\widehat{\mathbf{w}})_{BB}$, $\widetilde{\mathbf{S}}(\widehat{\mathbf{w}})_{NN}$, $\widetilde{\mathbf{X}}(\widehat{\mathbf{w}})_{NN}$ and $\widetilde{\mathbf{S}}(\widehat{\mathbf{w}})_{BB}$ are positive definite. Using the continuity of (2.14), we may assume without loss of generality, by restricting $\mathcal{U} \oplus \mathcal{V}$ further, that $\widetilde{\mathbf{X}}'(\mathbf{w})_{BB}$, $\widetilde{\mathbf{S}}'(\mathbf{w})_{NN}$, $\widetilde{\mathbf{X}}(\mathbf{w})_{NN}$ and $\widetilde{\mathbf{S}}(\mathbf{w})_{BB}$ are positive definite for each $\mathbf{w} \in \mathcal{U} \oplus \mathcal{V}$. Again by continuity of (2.14) (and possibly restricting $\mathcal{U} \oplus \mathcal{V}$ further), we may assume without loss of generality that both $\widetilde{\mathbf{X}}'(\mathbf{w})_{BB} - \nu \widetilde{\mathbf{X}}'(\mathbf{w})^T_{NB} \widetilde{\mathbf{X}}'(\mathbf{w})^{-1}_{NN} \widetilde{\mathbf{X}}'(\mathbf{w})_{NB}$ and $\widetilde{\mathbf{S}}'(\mathbf{w})_{NN} - \nu \widetilde{\mathbf{S}}'(\mathbf{w})_{NB} \widetilde{\mathbf{S}}'(\mathbf{w})^{-1}_{BB} \widetilde{\mathbf{S}}'(\mathbf{w})^T_{NB}$ are positive definite for all $\mathbf{w} = (\nu, \mathbf{D}) \in \mathcal{U} \oplus \mathcal{V}$. Hence the symmetric matrices $\mathbf{X}'(\mathbf{w})$ and $\mathbf{S}'(\mathbf{w})$ defined by

$$\begin{aligned} \mathbf{X}'(\mathbf{w})_{BB} &= \widetilde{\mathbf{X}}'(\mathbf{w})_{BB}, & \mathbf{S}'(\mathbf{w})_{BB} &= \nu \widetilde{\mathbf{S}}'(\mathbf{w})_{BB}, \\ \mathbf{X}'(\mathbf{w})_{NB} &= \nu \widetilde{\mathbf{X}}'(\mathbf{w})_{NB}, & \mathbf{S}'(\mathbf{w})_{NB} &= \nu \widetilde{\mathbf{S}}'(\mathbf{w})_{NB}, \\ \mathbf{X}'(\mathbf{w})_{NN} &= \nu \widetilde{\mathbf{X}}'(\mathbf{w})_{NN}, & \mathbf{S}'(\mathbf{w})_{NN} &= \widetilde{\mathbf{S}}'(\mathbf{w})_{NN}, \end{aligned}$$

are positive definite for each $\mathbf{w} = (\nu, \mathbf{D}) \in \mathcal{U} \oplus \mathcal{V}$. Since the pair $(\mathbf{X}'(\mathbf{w}), \mathbf{S}'(\mathbf{w}))$ defined above is a pair of positive definite solutions to (2.3), which is unique for each $(\nu, \mathbf{D}) \in (\mathcal{U} \oplus \mathcal{V}) \cap (\mathbb{R}_+ \oplus \mathbb{D}^n_{++})$, it follows that the pair $(\mathbf{X}'(\mathbf{w}), \mathbf{S}'(\mathbf{w}))$ agrees with $(\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$ for every $\mathbf{w} \in (\mathcal{U} \oplus \mathcal{V}) \cap (\mathbb{R}_+ \oplus \mathbb{D}^n_{++})$. We then deduce from (2.7) and the above definition that

$$(\widetilde{\mathbf{X}}'(\mathbf{w}), \widetilde{\mathbf{S}}'(\mathbf{w})) = (\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w})) \quad \forall \mathbf{w} \in (\mathcal{U} \oplus \mathcal{V}) \cap (\mathbb{R}_{++} \oplus \mathbb{D}^n_{++}). \tag{2.15}$$

Since $(\widetilde{\mathbf{X}}'(\widehat{\mathbf{w}}), \widetilde{\mathbf{S}}'(\widehat{\mathbf{w}}))$ coincides with $(\widetilde{\mathbf{X}}(\widehat{\mathbf{w}}), \widetilde{\mathbf{S}}(\widehat{\mathbf{w}}))$, it then follows from the continuity of (2.14) that the map $\nu \in \mathbb{R}_+ \mapsto (\widetilde{\mathbf{X}}(\nu, \widehat{\mathbf{D}}), \widetilde{\mathbf{S}}(\nu, \widehat{\mathbf{D}}))$ is continuous at 0, whence continuous. By applying this argument to each $\widehat{\mathbf{w}} = (0, \widehat{\mathbf{D}}) \in \{0\} \oplus \mathcal{V}$, we deduce the continuity of $\nu \in \mathbb{R}_+ \mapsto (\widetilde{\mathbf{X}}(\nu, \mathbf{D}), \widetilde{\mathbf{S}}(\nu, \mathbf{D}))$ for each $\mathbf{D} \in \mathcal{V}$. Together with (2.15) and the continuity of (2.14), we have $(\widetilde{\mathbf{X}}'(\mathbf{w}), \widetilde{\mathbf{S}}'(\mathbf{w})) = (\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ for each $\mathbf{w} \in \{0\} \oplus \mathcal{V}$, whence

$$(\widetilde{\mathbf{X}}'(\mathbf{w}), \widetilde{\mathbf{S}}'(\mathbf{w})) = (\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w})) \quad \forall \mathbf{w} \in (\mathcal{U} \oplus \mathcal{V}) \cap (\mathbb{R}_{+} \oplus \mathbb{D}^n_{++}).$$

Consequently, the analyticity of $\mathbf{w} \in \mathbb{R}_+ \oplus \mathbb{D}^n_{++} \mapsto (\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ at $\widehat{\mathbf{w}}$ follows from that of $\mathbf{w} \in (\mathcal{U} \oplus \mathcal{V}) \cap (\mathbb{R}_+ \oplus \mathbb{D}^n_{++}) \mapsto (\widetilde{\mathbf{X}}'(\mathbf{w}), \widetilde{\mathbf{S}}'(\mathbf{w}))$ at $\widehat{\mathbf{w}}$. □

2.2. **Extended Weighted Centers for SDP.** Since $\mathbf{L}_{\mathbf{X}}^T\mathbf{S}\mathbf{L}_{\mathbf{X}} \notin \mathbb{D}_{++}^n$ in general, the notion of Cholesky weighted centers does not "fill up" the primal-dual strictly feasible region. It was shown by the author [4] that the notion of Cholesky weighted centers can be extended so that the extended weighted centers "fill up" the primal-dual strictly feasible region. These extended weighted centers are solutions to

$$\begin{gathered} \boldsymbol{\mathcal{A}}\mathbf{X} = \mathbf{b}, \quad \boldsymbol{\mathcal{C}}\mathbf{S} = \mathbf{d}, \\ \mathbf{L}_{\mathbf{P}^T\mathbf{X}\mathbf{P}}^T \mathbf{P}^{-1}\mathbf{S}\mathbf{P}^{-T}\mathbf{L}_{\mathbf{P}^T\mathbf{X}\mathbf{P}} = \nu\mathbf{D}, \quad \mathbf{X}, \mathbf{S} \in \mathbb{S}_{++}^n, \end{gathered}$$

or equivalently,

$$\begin{gathered} \boldsymbol{\mathcal{A}}\mathbf{X} = \mathbf{b}, \quad \boldsymbol{\mathcal{C}}\mathbf{S} = \mathbf{d}, \\ \langle\!\langle \mathbf{P}^{-1}\mathbf{S}\mathbf{X}\mathbf{P} \rangle\!\rangle_H = \nu\mathbf{D}, \quad \mathbf{X}, \mathbf{S} \in \mathbb{S}_{++}^n, \end{gathered} \tag{2.16}$$

where $(\nu, \mathbf{D}, \mathbf{P}) \in \mathbb{R}_+ \oplus \mathbb{D}_{++}^n \oplus \mathbb{M}_\star^n$. Let $(\mathbf{X}(\nu, \mathbf{D}, \mathbf{P}), \mathbf{S}(\nu, \mathbf{D}, \mathbf{P}))$ denote the unique pair of solutions to (2.16), which we called the *extended Cholesky weighted centers determined by* $(\nu, \mathbf{D}, \mathbf{P})$, and let $(\mathbf{X}(0, \mathbf{D}, \mathbf{P}), \mathbf{S}(0, \mathbf{D}, \mathbf{P}))$ denote the limit of the primal-dual *extended Cholesky weighted central path* $\{(\mathbf{X}(\nu, \mathbf{D}, \mathbf{P}), \mathbf{S}(\nu, \mathbf{D}, \mathbf{P})) : \nu > 0\}$.

For the purpose of this paper here, we fix some arbitrary pair of strictly complementary solutions $(\mathbf{X}^*, \mathbf{S}^*)$, and some arbitrary $\widehat{\mathbf{Q}} \in \mathbb{O}^n$ that simultaneously diagonalizes $\mathbf{X}^*$ and $\mathbf{S}^*$ so that $\widehat{\mathbf{Q}}^T\mathbf{X}^*\widehat{\mathbf{Q}}$ is a diagonal matrix with leading nonzero diagonal entries, and use the subset of extended Cholesky weighted centers

$$\{(\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w})) : \mathbf{w} \in \mathbb{R}_{++} \oplus \mathbb{D}_{++}^n \oplus \widehat{\mathbf{Q}}\mathbb{L}_\star^n\},$$

where $\widehat{\mathbf{Q}}\mathbb{L}_\star^n$ denotes the set $\{\widehat{\mathbf{Q}}\mathbf{L} : \mathbf{L} \in \mathbb{L}_\star^n\}$.

Note that $\widehat{\mathbf{Q}}$ is chosen so that there exists some $k \in \{0, \ldots, n\}$ satisfying

$$\begin{gathered} \forall\ (\mathbf{X}, \mathbf{S}) \in relint(\mathcal{O}_p) \oplus relint(\mathcal{O}_d), \\ (\widehat{\mathbf{Q}}^T\mathbf{X}\widehat{\mathbf{Q}})_{BB} \in \mathbb{S}_{++}^k, \quad (\widehat{\mathbf{Q}}^T\mathbf{S}\widehat{\mathbf{Q}})_{NN} \in \mathbb{S}_{++}^{n-k}, \\ (\widehat{\mathbf{Q}}^T\mathbf{X}\widehat{\mathbf{Q}})_{NB} = (\widehat{\mathbf{Q}}^T\mathbf{S}\widehat{\mathbf{Q}})_{NB} = \mathbf{0}, \\ (\widehat{\mathbf{Q}}^T\mathbf{X}\widehat{\mathbf{Q}})_{NN} = \mathbf{0} \quad \text{and} \quad (\widehat{\mathbf{Q}}^T\mathbf{S}\widehat{\mathbf{Q}})_{BB} = \mathbf{0}, \end{gathered} \tag{2.17}$$

where $B = \{1, \ldots, k\}$ and $N = \{k+1, \ldots, n\}$.

The following lemma shows that the restriction to this subset does not reduce the ability to "fill up" the primal-dual strictly feasible region.

Lemma 10. *Suppose that* $\widehat{\mathbf{Q}}$ *is as defined above. For each* $(\mathbf{X}, \mathbf{S}) \in \mathcal{F}_p^\circ \oplus \mathcal{F}_d^\circ$*, there exist* $(\mathbf{D}, \widehat{\mathbf{Q}}\mathbf{L}) \in \mathbb{D}_{++}^n \oplus \widehat{\mathbf{Q}}\mathbb{L}_\star^n$ *such that*

$$\langle\!\langle \mathbf{L}^{-1}\widehat{\mathbf{Q}}^T\mathbf{S}\mathbf{X}\widehat{\mathbf{Q}}\mathbf{L} \rangle\!\rangle_H = \mathbf{D}.$$

Proof. For simplicity of notation, let $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{S}}$ denote, respectively, $\widehat{\mathbf{Q}}^T\mathbf{X}\widehat{\mathbf{Q}}$ and $\widehat{\mathbf{Q}}^T\mathbf{S}\widehat{\mathbf{Q}}$. Let $\mathbf{P}$ be a matrix whose columns are eigenvectors of the product $\widetilde{\mathbf{S}}\widetilde{\mathbf{X}}$ and let $\mathbf{D}$ be the diagonal matrix with the corresponding eigenvalues on its diagonal. Since $\mathbf{P}$ is invertible, it has an *LU*-decomposition $\mathbf{P}\mathbf{Q} = \mathbf{L}\mathbf{U}$, where $\mathbf{Q} \in \mathbb{O}^n$ is a permutation matrix, $\mathbf{L} \in \mathbb{L}_\star^n$ and $\mathbf{U} \in \mathbb{U}^n$. Thus it follows from $\mathbf{P}^{-1}\widetilde{\mathbf{S}}\widetilde{\mathbf{X}}\mathbf{P} = \mathbf{D}$ that $\mathbf{L}^{-1}\widetilde{\mathbf{S}}\widetilde{\mathbf{X}}\mathbf{L} = \mathbf{U}(\mathbf{Q}\mathbf{D}\mathbf{Q}^T)\mathbf{U}^{-1} \in \mathbb{U}^n$, whence $\langle\!\langle \mathbf{L}^{-1}\widetilde{\mathbf{S}}\widetilde{\mathbf{X}}\mathbf{L} \rangle\!\rangle_H = \mathbf{Q}\mathbf{D}\mathbf{Q}^T$. □

By following the argument in Section 2.1, we have

Theorem 11. *Suppose that* $\widehat{\mathbf{Q}}$ *is as defined above. If* (2.1) *has strictly complementary solutions, then the map*

$$\mathbf{w} \in \mathbb{R}_+ \oplus \mathbb{D}^n_{++} \oplus \widehat{\mathbf{Q}}\mathbb{L}^n_\star \mapsto (\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$$

is analytic.

Proof. We shall show that $\mathbf{w} \mapsto (\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$ is analytic at each $\widehat{\mathbf{w}} = (\widehat{\nu}, \widehat{\mathbf{D}}, \widehat{\mathbf{Q}}\widehat{\mathbf{L}}) \in \mathbb{R}_+ \oplus \mathbb{D}^n_{++} \oplus \widehat{\mathbf{Q}}\mathbb{L}^n_\star$.

For each $\widehat{\mathbf{w}} = (\widehat{\nu}, \widehat{\mathbf{D}}, \widehat{\mathbf{Q}}\widehat{\mathbf{L}}) \in \mathbb{R}_+ \oplus \mathbb{D}^n_{++} \oplus \widehat{\mathbf{Q}}\mathbb{L}^n_\star$ the pair $(\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$ solves

$$\boldsymbol{\mathcal{F}}(\mathbf{X}, \mathbf{S}, \widehat{\nu}, \widehat{\mathbf{D}}, \widehat{\mathbf{Q}}\widehat{\mathbf{L}}) = (\mathbf{0}, \mathbf{0}, \mathbf{0}),$$

where $\boldsymbol{\mathcal{F}} : \mathbb{S}^n \oplus \mathbb{S}^n \oplus \mathbb{R} \oplus \mathbb{D}^n \oplus \widehat{\mathbf{Q}}\mathbb{L}^n_\star \to \mathbb{R}^m \oplus \mathbb{R}^{\bar{n}^2-m} \oplus \mathbb{S}^n$ is the analytic map

$$(\mathbf{X}, \mathbf{S}, \nu, \mathbf{D}, \widehat{\mathbf{Q}}\mathbf{L}) \mapsto (\boldsymbol{\mathcal{A}}\mathbf{X} - \mathbf{b}, \boldsymbol{\mathcal{C}}\mathbf{S} - \mathbf{d}, \langle\!\langle \mathbf{L}^{-1}\widehat{\mathbf{Q}}^T\mathbf{S}\mathbf{X}\widehat{\mathbf{Q}}\mathbf{L}\rangle\!\rangle_H - \nu\mathbf{D}),$$

When $\widehat{\nu} > 0$, it follows from Lemma 1, by taking

$$\widehat{\boldsymbol{\mathcal{A}}} : \mathbf{X} \mapsto \boldsymbol{\mathcal{A}}(\widehat{\mathbf{Q}}\widehat{\mathbf{L}}^{-T}\mathbf{X}\widehat{\mathbf{L}}^{-1}\widehat{\mathbf{Q}}^T), \quad \widehat{\boldsymbol{\mathcal{C}}} : \mathbf{S} \mapsto \boldsymbol{\mathcal{C}}(\widehat{\mathbf{Q}}\widehat{\mathbf{L}}\mathbf{S}\widehat{\mathbf{L}}^T\widehat{\mathbf{Q}}^T),$$
$$\widehat{\mathbf{X}} = \widehat{\mathbf{L}}^T\widehat{\mathbf{Q}}^T\widehat{\mathbf{X}}(\widehat{\mathbf{w}})\widehat{\mathbf{Q}}\widehat{\mathbf{L}} \quad \text{and} \quad \widehat{\mathbf{S}} = \widehat{\mathbf{L}}^{-1}\widehat{\mathbf{Q}}^T\widehat{\mathbf{S}}(\widehat{\mathbf{w}})\widehat{\mathbf{Q}}\widehat{\mathbf{L}}^{-T}$$

in the lemma, that the Jacobian $\frac{\partial}{\partial(\mathbf{X},\mathbf{S})}\boldsymbol{\mathcal{F}}$ at $(\mathbf{X}(\widehat{\mathbf{w}}), \mathbf{S}(\widehat{\mathbf{w}}), \widehat{\mathbf{w}})$ is nonsingular. Hence we deduce the analyticity of $\mathbf{w} \mapsto (\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$ at $\widehat{\mathbf{w}}$ from the Implicit Function Theorem, as we did in Theorem 2.

For $\widehat{\nu} = 0$, we define $(\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ for $\mathbf{w} \in \mathbb{R}_{++} \oplus \mathbb{D}^n_{++} \oplus \widehat{\mathbf{Q}}\mathbb{L}^n_\star$ by

$$\begin{aligned}
\widetilde{\mathbf{X}}(\mathbf{w})_{BB} &= (\widehat{\mathbf{Q}}^T\mathbf{X}(\mathbf{w})\widehat{\mathbf{Q}})_{BB}, & \widetilde{\mathbf{S}}(\mathbf{w})_{BB} &= \nu^{-1}(\widehat{\mathbf{Q}}^T\mathbf{S}(\mathbf{w})\widehat{\mathbf{Q}})_{BB},\\
\widetilde{\mathbf{X}}(\mathbf{w})_{NB} &= \nu^{-1}(\widehat{\mathbf{Q}}^T\mathbf{X}(\mathbf{w})\widehat{\mathbf{Q}})_{NB}, & \widetilde{\mathbf{S}}(\mathbf{w})_{NB} &= \nu^{-1}(\widehat{\mathbf{Q}}^T\mathbf{S}(\mathbf{w})\widehat{\mathbf{Q}})_{NB},\\
\widetilde{\mathbf{X}}(\mathbf{w})_{NN} &= \nu^{-1}(\widehat{\mathbf{Q}}^T\mathbf{X}(\mathbf{w})\widehat{\mathbf{Q}})_{NN}, & \widetilde{\mathbf{S}}(\mathbf{w})_{NN} &= (\widehat{\mathbf{Q}}^T\mathbf{S}(\mathbf{w})\widehat{\mathbf{Q}})_{NN},
\end{aligned}$$

and for $\mathbf{w} \in \{0\} \oplus \mathbb{D}^n_{++} \oplus \widehat{\mathbf{Q}}\mathbb{L}^n_\star$ by

$$\widetilde{\mathbf{X}}(\mathbf{w})_{BB} = (\widehat{\mathbf{Q}}^T\mathbf{X}(\mathbf{w})\widehat{\mathbf{Q}})_{BB}, \quad \widetilde{\mathbf{S}}(\mathbf{w})_{NN} = (\widehat{\mathbf{Q}}^T\mathbf{S}(\mathbf{w})\widehat{\mathbf{Q}})_{NN},$$

and leave the definitions of $\widetilde{\mathbf{X}}(\mathbf{w})_{NN}$, $\widetilde{\mathbf{X}}(\mathbf{w})_{NB}$, $\widetilde{\mathbf{S}}(\mathbf{w})_{BB}$ and $\widetilde{\mathbf{S}}(\mathbf{w})_{NB}$ for $\mathbf{w} \in \{0\}\oplus\mathbb{D}^n_{++}\oplus\widehat{\mathbf{Q}}\mathbb{L}^n_\star$ to later. It follows from Lemma 4 and (2.17) that regardless of how $\widetilde{\mathbf{X}}(\mathbf{w})_{NN}$, $\widetilde{\mathbf{X}}(\mathbf{w})_{NB}$, $\widetilde{\mathbf{S}}(\mathbf{w})_{BB}$ and $\widetilde{\mathbf{S}}(\mathbf{w})_{NB}$ are defined for $\mathbf{w} \in \{0\} \oplus \mathbb{D}^n_{++} \oplus \widehat{\mathbf{Q}}\mathbb{L}^n_\star$, we have

$$\begin{aligned}
\widetilde{\mathbf{X}}(\mathbf{w})_{BB} &= (\widehat{\mathbf{Q}}^T\mathbf{X}(\mathbf{w})\widehat{\mathbf{Q}})_{BB}, & \nu\widetilde{\mathbf{S}}(\mathbf{w})_{BB} &= (\widehat{\mathbf{Q}}^T\mathbf{S}(\mathbf{w})\widehat{\mathbf{Q}})_{BB},\\
\nu\widetilde{\mathbf{X}}(\mathbf{w})_{NB} &= (\widehat{\mathbf{Q}}^T\mathbf{X}(\mathbf{w})\widehat{\mathbf{Q}})_{NB}, & \nu\widetilde{\mathbf{S}}(\mathbf{w})_{NB} &= (\widehat{\mathbf{Q}}^T\mathbf{S}(\mathbf{w})\widehat{\mathbf{Q}})_{NB},\\
\nu\widetilde{\mathbf{X}}(\mathbf{w})_{NN} &= (\widehat{\mathbf{Q}}^T\mathbf{X}(\mathbf{w})\widehat{\mathbf{Q}})_{NN}, & \widetilde{\mathbf{S}}(\mathbf{w})_{NN} &= (\widehat{\mathbf{Q}}^T\mathbf{S}(\mathbf{w})\widehat{\mathbf{Q}})_{NN},
\end{aligned}$$

for each $\mathbf{w} \in \mathbb{R}_+\oplus\mathbb{D}^n_{++}\oplus\widehat{\mathbf{Q}}\mathbb{L}^n_\star$. Consequently the analyticity of $\mathbf{w} \mapsto (\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ at $\widehat{\mathbf{w}}$ will imply the same for $\mathbf{w} \mapsto (\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$.

For each $\mathbf{w} = (\nu, \mathbf{D}, \widehat{\mathbf{Q}}\mathbf{L}) \in \mathbb{R}_{++} \oplus \mathbb{D}^n_{++} \oplus \widehat{\mathbf{Q}}\mathbb{L}^n_\star$, $(\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ satisfies the following equations

$$\begin{aligned}
&(2.18a) && \mathbf{0} = \boldsymbol{\mathcal{A}}_{11}(\mathbf{X}_{BB} - \mathbf{X}(0,\mathbf{D},\widehat{\mathbf{Q}})_{BB}) + \nu\boldsymbol{\mathcal{A}}_{12}\mathbf{X}_{NB} + \nu\boldsymbol{\mathcal{A}}_{13}\mathbf{X}_{NN},\\
&(2.18b) && \mathbf{0} = \boldsymbol{\mathcal{A}}_{22}\mathbf{X}_{NB} + \boldsymbol{\mathcal{A}}_{23}\mathbf{X}_{NN},\\
&(2.18c) && \mathbf{0} = \boldsymbol{\mathcal{A}}_{33}\mathbf{X}_{NN},\\
&(2.18d) && \mathbf{0} = \boldsymbol{\mathcal{C}}_{11}\mathbf{S}_{BB},\\
&(2.18e) && \mathbf{0} = \boldsymbol{\mathcal{C}}_{21}\mathbf{S}_{BB} + \boldsymbol{\mathcal{C}}_{22}\mathbf{S}_{NB},\\
&(2.18f) && \mathbf{0} = \nu\boldsymbol{\mathcal{C}}_{31}\mathbf{S}_{BB} + \nu\boldsymbol{\mathcal{C}}_{32}\mathbf{S}_{NB} + \boldsymbol{\mathcal{C}}_{33}(\mathbf{S}_{NN} - \mathbf{S}(0,\mathbf{D},\widehat{\mathbf{Q}})_{NN}),\\
&(2.18g) && \mathbf{D}_{BB} = \langle\!\langle \mathbf{L}_{BB}^{-1}\mathbf{S}_{BB}\mathbf{X}_{BB}\mathbf{L}_{BB}\rangle\!\rangle_H\\
& && \quad +\nu\langle\!\langle \mathbf{L}_{BB}^{-1}\mathbf{S}_{*B}^T\mathbf{X}_{N*}^T\mathbf{L}_{NB} + \mathbf{L}_{BB}^{-1}\mathbf{S}_{NB}^T\mathbf{X}_{NB}\mathbf{L}_{BB}\rangle\!\rangle_H,\\
&(2.18h) && \mathbf{D}_{NN} = \langle\!\langle \mathbf{L}_{NN}^{-1}\mathbf{S}_{NN}\mathbf{X}_{NN}\mathbf{L}_{NN}\rangle\!\rangle_H\\
& && \quad +\nu\langle\!\langle (\mathbf{L}^{-1})_{NB}\mathbf{S}_{*B}^T\mathbf{X}_{N*}^T\mathbf{L}_{NN} + \mathbf{L}_{NN}^{-1}\mathbf{S}_{NB}\mathbf{X}_{NB}^T\mathbf{L}_{NN}\rangle\!\rangle_H,\\
&(2.18i) && \mathbf{0} = \mathbf{L}_{NN}^{-1}\mathbf{S}_{N*}\mathbf{X}_{*B}\mathbf{L}_{BB} + \mathbf{L}_{NN}^{-1}\mathbf{S}_{NN}\mathbf{X}_{NN}\mathbf{L}_{NB}\\
& && \quad +(\mathbf{L}^{-1})_{NB}\mathbf{S}_{BB}\mathbf{X}_{BB}\mathbf{L}_{BB} + \nu(\mathbf{L}^{-1})_{NB}\mathbf{S}_{*B}^T\mathbf{X}_{N*}^T\mathbf{L}_{NB}\\
& && \quad +\nu\big(\mathbf{L}_{NN}^{-1}\mathbf{S}_{NB}\mathbf{X}_{NB}^T\mathbf{L}_{NB} + (\mathbf{L}^{-1})_{NB}\mathbf{S}_{NB}^T\mathbf{X}_{NB}\mathbf{L}_{BB}\big).
\end{aligned}$$

Note that we have used $B = \{1, \ldots, k\}$ and $N = \{k+1, \ldots, n\}$ in deriving (2.18i).

For each $\mathbf{w} = (0, \mathbf{D}, \widehat{\mathbf{Q}}\mathbf{L}) \in \{0\} \oplus \mathbb{D}^n_{++} \oplus \widehat{\mathbf{Q}}\mathbb{L}^n_\star$, (2.18a) and (2.18f) are clearly satisfied by $(\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$. The equations (2.18g) and (2.18h) are satisfied if and only if

$$\widetilde{\mathbf{S}}(\mathbf{w})_{BB} = \mathbf{L}_{BB}\mathbf{L}^{-T}_{\mathbf{L}_{BB}^T(\widehat{\mathbf{Q}}^T\mathbf{X}(\mathbf{w})\widehat{\mathbf{Q}})_{BB}\mathbf{L}_{BB}}\mathbf{D}_{BB}\mathbf{L}^{-1}_{\mathbf{L}_{BB}^T(\widehat{\mathbf{Q}}^T\mathbf{X}(\mathbf{w})\widehat{\mathbf{Q}})_{BB}\mathbf{L}_{BB}}\mathbf{L}_{BB}^T$$

and

$$\widetilde{\mathbf{X}}(\mathbf{w})_{NN} = \mathbf{L}_{NN}^{-T}\mathbf{U}^{-T}_{\mathbf{L}_{NN}^{-1}(\widehat{\mathbf{Q}}^T\mathbf{S}(\mathbf{w})\widehat{\mathbf{Q}})_{NN}\mathbf{L}_{NN}^{-T}}\mathbf{D}_{NN}\mathbf{U}^{-1}_{\mathbf{L}_{NN}^{-1}(\widehat{\mathbf{Q}}^T\mathbf{S}(\mathbf{w})\widehat{\mathbf{Q}})_{NN}\mathbf{L}_{NN}^{-T}}\mathbf{L}_{NN}^{-1},$$

which we shall use as definitions of $\widetilde{\mathbf{S}}(\mathbf{w})_{BB}$ and $\widetilde{\mathbf{X}}(\mathbf{w})_{BB}$. Once again, we conclude from Theorem 5 that (2.18c) and (2.18d) hold with these definitions. It then follows from the same argument as before, using Lemma 7, that there is precisely one definition for the pair $(\widetilde{\mathbf{X}}(\mathbf{w})_{NB}, \widetilde{\mathbf{S}}(\mathbf{w})_{NB})$ so that $(\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ satisfies (2.18b), (2.18e) and (2.18i). Finally we conclude the nonsingularity of the system of equations (2.18) at $(\widetilde{\mathbf{X}}(\widehat{\mathbf{w}}), \widetilde{\mathbf{S}}(\widehat{\mathbf{w}}), \widehat{\mathbf{w}})$ using an argument similar to the proof of Lemma 8. Consequently we deduce the analyticity of $\mathbf{w} \mapsto (\widetilde{\mathbf{X}}(\mathbf{w}), \widetilde{\mathbf{S}}(\mathbf{w}))$ at $\widehat{\mathbf{w}}$ from the Implicit Function Theorem, as we did in Theorem 9. □

We end this section with a remark: although the statement of Theorem 11 is more general than that of Theorem 9, the former does not imply the later in general. The reason is that the Cholesky weighted centers discussed in Theorem 9 may not be extended Cholesky weighted centers. In fact, the Cholesky weighted centers are extended Cholesky weighted centers if and only if the orthogonal matrix $\widehat{\mathbf{Q}}$ in Theorem 11 has the LU-decomposition

$$\widehat{\mathbf{Q}}^T = \widehat{\mathbf{L}}\widehat{\mathbf{U}},$$

where $\widehat{\mathbf{L}} \in \mathbb{L}^n_\star$ and $\widehat{\mathbf{U}} \in \mathbb{U}^n$. In this case we can deduce Theorem 9 from Theorem 11 by fixing the parameter $\mathbf{L}$ in the theorem to $\widehat{\mathbf{L}}$. Thus, while the assumption $B =$

$\{1, \ldots, k\}$ simplifies the notation (as is evident when we compare (2.18i) with (2.13i) and (2.13j)), we cannot, in general, make this stronger assumption in the proof of Theorem 9.

The importance of Theorem 9 is illustrated in its application to homogeneous cone programming, which is the topic of the next section.

2.3. **Application to Homogeneous Cone Programming.** In this section, we consider the following primal-dual pair of homogeneous cone programming (HCP) problems:

$$\inf_{\mathbf{x}}\{\widehat{\mathbf{s}}^T\mathbf{x} : \mathbf{x} \in \mathcal{L}+\widehat{\mathbf{x}},\ \mathbf{x} \in cl(K)\} \quad \text{and} \quad \inf_{\mathbf{s}}\{\widehat{\mathbf{x}}^T\mathbf{s} : \mathbf{s} \in \mathcal{L}^\perp+\widehat{\mathbf{s}},\ \mathbf{s} \in cl(K^*)\}, \tag{2.19}$$

where $K \in \mathbb{R}^d$ is a homogeneous cone (i.e., a pointed, open, convex cone whose group of automorphisms acts transitively on it), $K^* := \{\mathbf{s} : \mathbf{x}^T\mathbf{s} > 0,\ \forall \mathbf{x} \in K\}$ is its dual cone, $(\widehat{\mathbf{x}}, \widehat{\mathbf{s}}) \in K \oplus K^*$ and $\mathcal{L} \subseteq \mathbb{R}^d$ is a linear subspace.

It was shown by the author [3] that all homogeneous cones are SDP-representable, i.e., for each homogeneous cone K, there exists an injective linear map $\boldsymbol{\mathcal{M}} : \mathbb{R}^d \to \mathbb{S}^n$ such that $\mathbf{x} \in K$ if and only if $\boldsymbol{\mathcal{M}}(\mathbf{x}) \in \mathbb{S}^n_{++}$. Thus, the primal HCP problem in (2.19) can be reformulated as the primal SDP problem

$$\inf_{\mathbf{X}}\{(\boldsymbol{\mathcal{M}}^{-1})^H(\widehat{\mathbf{s}}) \bullet \mathbf{X} : \mathbf{X} \in \boldsymbol{\mathcal{M}}(\mathcal{L}) + \boldsymbol{\mathcal{M}}(\widehat{\mathbf{x}}),\ \mathbf{X} \in \mathbb{S}^n_+\}.$$

Furthermore, it was shown by the author and Tunçel [5] that HCP problems inherit strict complementarity from the corresponding SDP formulations; i.e., a HCP problem has strictly complementary solutions if and only if any SDP reformulation has such solutions.

It was further established by the author [4, Section 4.3] that under a suitable choice of the representation $\boldsymbol{\mathcal{M}}$, the central path defined by the only known optimal self-concordant barrier for K coincides with a Cholesky weighted central path for the representing SDP problem.

Consequently the analyticity of Cholesky weighted central paths for SDP translates directly to the analyticity of central paths for HCP. This result is formally stated as

Theorem 12. *If the primal-dual HCP problems* (2.19) *have strictly complementary solutions, then the map*

$$\mu \in \mathbb{R}_+ \mapsto (\mathbf{x}(\mu), \mathbf{s}(\mu))$$

is analytic, where $(\mathbf{x}(\mu), \mathbf{s}(\mu))$ *denotes the pair of primal-dual solutions on the central path defined by the only known optimal barrier for* K *that satisfies* $\mathbf{x}(\mu)^T\mathbf{s}(\mu) = \mu$ *for* $\mu > 0$*, and* $(\mathbf{x}(0), \mathbf{s}(0))$ *denotes* $\lim_{\mu\downarrow 0}(\mathbf{x}(\mu), \mathbf{s}(\mu))$.

3. Error bounds for SDP

All primal-dual path-following algorithms generate pairs of primal-dual strictly feasible solutions $\{(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))\}_{t=1}^\infty$ in a certain neighborhood around the central path. In this section, we derive, under the assumption of strict complementarity, a necessary and sufficient condition for such iterates to converge at a rate of the same order as that of their duality gaps; i.e., the following Lipschitzian-type bound holds:

$$\max\{dist(\widehat{\mathbf{X}}(t), \mathcal{O}_p), dist(\widehat{\mathbf{S}}(t), \mathcal{O}_d)\} = O(\widehat{\mathbf{X}}(t) \bullet \widehat{\mathbf{S}}(t)), \tag{LB}$$

where $dist(\mathbf{X}, \mathcal{T})$ denotes the value $\inf\{\|\mathbf{X} - \mathbf{V}\|_F : \mathbf{V} \in \mathcal{T}\}$ for each $\mathbf{X} \in \mathbb{S}^n$ and each $\mathcal{T} \subseteq \mathbb{S}^n$. Such Lipschitzian bound played important roles in the analysis of superlinear convergence in [11] and [15].

Henceforth, we assume that the sequence $\{(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))\}_{t=1}^{\infty}$ of primal-dual solutions for (2.1) satisfies $\lim_{t\to\infty} \widehat{\mathbf{X}}(t) \bullet \widehat{\mathbf{S}}(t) = 0$ and the condition

$$(\rho :=) \inf_{t=1,2,\ldots} \{\lambda_{\min}(\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)) / \operatorname{tr}(\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t))\} > 0. \tag{3.1}$$

Condition (3.1) is equivalent to the primal-dual iterates lying within the wide neighborhood of the central path

$$\{(\mathbf{X}, \mathbf{S}) \in \mathcal{F}_p \oplus \mathcal{F}_d : \mathbf{X} \bullet \mathbf{S}/n - \lambda_{\min}(\mathbf{X}\mathbf{S}) \le \tau \mathbf{X} \bullet \mathbf{S}/n\},$$

where $\tau = 1 - n\rho$.

For the sake of clarity, we shall make the following assumption.

Assumption 13. *The primal and dual optimal faces $\mathcal{O}_p$ and $\mathcal{O}_d$ satisfy* (2.5) *with*

$$B = \{1, \ldots, k\} \quad \text{and} \quad N = \{k+1, \ldots, n\}$$

for some $k \in \{0, \ldots, n\}$.

We remark that the above assumption is without any loss of generality—by picking a pair of strictly complementary solutions $(\mathbf{X}^*, \mathbf{S}^*)$ and an orthogonal matrix $\mathbf{Q} \in \mathbb{O}^n$ that simultaneously diagonalizes $\mathbf{X}^*$ and $\mathbf{S}^*$ so that $\mathbf{Q}^T\mathbf{X}^*\mathbf{Q}$ is a diagonal matrix with leading nonzero diagonal entries, and transforming the primal and dual variables respectively via $\mathbf{X} \mapsto \mathbf{Q}^T\mathbf{X}\mathbf{Q}$ and $\mathbf{S} \mapsto \mathbf{Q}^T\mathbf{S}\mathbf{Q}$, we observe that Assumption 13 holds, and Condition (3.1) and the Lipschitzian bound (LB) are invariant under this transformation.

The following example shows that Condition (3.1) alone is insufficient to guarantee the Lipschitzian bound (LB).

Example 14. *Consider the primal-dual SDP problems*

$$\begin{aligned} &\inf\ x_3 \quad \text{s.t. } x_1 = 1,\ x_2 = 0,\ \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \in \mathbb{S}^2_+, \\ &\inf\ s_1 \quad \text{s.t. } s_3 = 1,\ \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \in \mathbb{S}^2_+, \end{aligned}$$

which have unique solutions $\mathbf{X}^* = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\mathbf{S}^* = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ respectively. Fix an arbitrary $\rho \in (0, 1/2)$ and consider the sequence of primal-dual strictly feasible solutions $\{(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))\}_{t=1}^{\infty}$, where

$$\widehat{\mathbf{X}}(t) = \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix} \quad \text{and} \quad \widehat{\mathbf{S}}(t) = \begin{bmatrix} t^{-1} & (1-2\rho)t^{-1/2} \\ (1-2\rho)t^{-1/2} & 1 \end{bmatrix}.$$

Since the eigenvalues of

$$\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t) = \begin{bmatrix} t^{-1} & (1-2\rho)t^{-3/2} \\ (1-2\rho)t^{-1/2} & t^{-1} \end{bmatrix}$$

are $2(1-\rho)t^{-1}$ and $2\rho t^{-1}$, this sequence satisfies Condition (3.1). *However,*

$$dist(\widehat{\mathbf{S}}(t), \mathcal{O}_d) = \|\widehat{\mathbf{S}}(t) - \mathbf{S}^*\|_F = \Theta(t^{-1/2})$$

converges to zero at a rate slower than the duality gap $\widehat{\mathbf{X}}(t) \bullet \widehat{\mathbf{S}}(t) = 2t^{-1}$. *Since* $\rho \in (0, 1/2)$ *is arbitrary, this phenomenon occurs no matter how small (but positive) the radius of the neighborhood.*

3.1. **Sufficient Conditions for Lipschitzian Bound.** In this section, we consider two sufficient conditions for (LB).

One way to guarantee the Lipschitzian bound is to place each pair $(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))$ on some weighted central path, and use the analyticity of the weighted central path to conclude that the iterates are Lipschitzian over a compact set of weights.

This approach was successfully used by Monteiro and Lu [13, Theorem 5.3] (under the assumption of strict complementarity) to conclude (LB) for infeasible iterates that lie in the neighborhood

$$\mathcal{N}(\gamma) := \{(\mathbf{X}, \mathbf{S}) : \|(\mathbf{SX} + \mathbf{XS})/2 - \nu \mathbf{I}\|_2 \le \gamma\nu \text{ for some } \nu > 0\}$$

for some $\gamma < 1$, where $\|\cdot\|_2$ denotes the operator 2-norm $\mathbf{X} \in \mathbb{S}^n \mapsto \max_{\mathbf{v}}\{|\mathbf{v}^T\mathbf{X}\mathbf{v}| : \mathbf{v} \in \mathbb{R}^n,\ \mathbf{v}^T\mathbf{v} = 1\}$. While their result, when applied to feasible solutions, captures iterates that are arbitrarily close to the boundary of the primal-dual feasible region, it precludes a measurable portion of the primal-dual strictly feasible region (see, e.g., [23]).

In contrast, the subset of extended Cholesky weighted centers considered in Section 2.2 coincides with the primal-dual strictly feasible region. Applying the same approach to this collection of extended Cholesky weighted centers yields the following parallel result.

Theorem 15. *Suppose that the pair of primal-dual SDP problems* (2.1) *satisfies Assumption 13 and has strictly complementary solutions, and the sequence of primal-dual strictly feasible solutions* $\{(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))\}_{t=1}^{\infty}$ *of* (2.1) *has duality gaps* $\widehat{\nu}(t) := \widehat{\mathbf{X}}(t) \bullet \widehat{\mathbf{S}}(t)$ *converging to zero, and satisfies Condition* (3.1). *If there exists a compact subset* $\mathcal{L}_\star \subseteq \mathbb{L}^n_\star$ *such that for each t there exists* $\widehat{\mathbf{L}}(t) \in \mathcal{L}_\star$ *and* $\widehat{\mathbf{D}}(t) \in \mathbb{D}^n_{++}$ *with*

$$\widehat{\mathbf{X}}(t) = \mathbf{X}(\widehat{\nu}(t), \widehat{\mathbf{D}}(t), \widehat{\mathbf{L}}(t)) \quad \textit{and} \quad \widehat{\mathbf{S}}(t) = \mathbf{S}(\widehat{\nu}(t), \widehat{\mathbf{D}}(t), \widehat{\mathbf{L}}(t)),$$

then the Lipschitzian bound (LB) *holds.*

Proof. Since the diagonal entries of $\widehat{\mathbf{D}}(t)$ are the eigenvalues of $\widehat{\nu}(t)^{-1}\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)$, it follows from Condition (3.1) that $\{\widehat{\mathbf{D}}(t)\}_{t=1}^{\infty}$ lies in the compact set $\mathcal{D} := \{\mathbf{D} \in \mathbb{D}^n : \mathbf{D}_{ii} \in [\rho, 1]\}$. Moreover it follows from $\widehat{\nu}(t) \to 0$ that $\{\widehat{\nu}(t)\}_{t=1}^{\infty}$ is bounded above, whence lies in some compact set $\mathcal{I} := [0, \nu_{\max}]$. Consequently, under the hypothesis of the theorem, $\{(\widehat{\nu}(t), \widehat{\mathbf{D}}(t), \widehat{\mathbf{L}}(t))\}_{t=1}^{\infty}$ is in the compact set $\mathcal{I} \oplus \mathcal{D} \oplus \mathcal{L}_\star$. By Theorem 11, the map

$$\mathbf{w} \mapsto (\mathbf{X}(\mathbf{w}), \mathbf{S}(\mathbf{w}))$$

is analytic, whence Lipschitz over the compact set $\mathcal{I} \oplus \mathcal{D} \oplus \mathcal{L}_\star$ with a certain Lipschitz constant, say K. Thus

$$dist(\widehat{\mathbf{X}}(t), \mathcal{O}_p) \le \|\widehat{\mathbf{X}}(t) - \mathbf{X}(0, \widehat{\mathbf{D}}(t), \widehat{\mathbf{L}}(t))\|_F \le K|\widehat{\nu}(t) - 0| = O(\widehat{\mathbf{X}}(t) \bullet \widehat{\mathbf{S}}(t)),$$

and similarly $dist(\widehat{\mathbf{S}}(t), \mathcal{O}_d) = O(\widehat{\mathbf{X}}(t) \bullet \widehat{\mathbf{S}}(t))$. □

Although both Theorem 15 and [13, Theorem 5.3] give sufficient conditions for the Lipschitzian bound to hold, [13, Theorem 5.3] precludes solutions that fall outside the neighborhood $\mathcal{N}(1)$, while Theorem 15 applies to all strictly feasible solutions.

The following example shows that the two sufficient conditions mentioned above are incomparable, and neither is necessary for (LB).

Example 16. *Consider the primal-dual SDP problems from Example 14*

$$\begin{aligned} &\inf\ x_3 \quad \text{s.t. } x_1 = 1,\ x_2 = 0,\ \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \in \mathbb{S}^2_+, \\ &\inf\ s_1 \quad \text{s.t. } s_3 = 1,\ \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \in \mathbb{S}^2_+, \end{aligned}$$

which have unique solutions $\mathbf{X}^* = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ *and* $\mathbf{S}^* = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. *Consider the sequence of primal-dual strictly feasible solutions* $\{(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))\}_{t=1}^{\infty}$, *where*

$$\widehat{\mathbf{X}}(t) = \begin{bmatrix} 1 & 0 \\ 0 & \alpha t^{-1} \end{bmatrix} \quad \textit{and} \quad \widehat{\mathbf{S}}(t) = \begin{bmatrix} \beta t^{-1} & t^{-1} \\ t^{-1} & 1 \end{bmatrix}$$

for some $\alpha > 0$ *and some* $\beta > 1$. *Clearly the sequence satisfies the Lipschitzian bound.*

Note that

$$\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t) = \begin{bmatrix} \beta t^{-1} & \alpha t^{-2} \\ t^{-1} & \alpha t^{-1} \end{bmatrix}$$

has eigenvectors $\mathbf{v}^1(t) := [\beta - \alpha - r(t), 2]^T$ *and* $\mathbf{v}^2(t) := [\beta - \alpha + r(t), 2]^T$ *corresponding to the distinct eigenvalues* $\lambda_1(t) = (\beta + \alpha - r(t))/(2t)$ *and* $\lambda_2(t) = (\beta + \alpha + r(t))/(2t)$ *respectively, where* $r(t) = \sqrt{(\beta - \alpha)^2 + 4\alpha t^{-1}}$, *and has symmetric part*

$$\frac{1}{2}(\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t) + \widehat{\mathbf{X}}(t)\widehat{\mathbf{S}}(t)) = \frac{1}{2t}\begin{bmatrix} 2\beta & \alpha t^{-1} + 1 \\ \alpha t^{-1} + 1 & 2\alpha \end{bmatrix}.$$

For the case $\alpha = \beta > 1$, *we have* $r(t) = 2\sqrt{\alpha t^{-1}} \downarrow 0$, *and hence* $\mathbf{v}^1(t), \mathbf{v}^2(t) \to [0, 2]^T$. *For any* $\widehat{\mathbf{D}}(t) \in \mathbb{D}^n_{++}$ *and* $\widehat{\mathbf{L}}(t) \in \mathbb{L}^n_\star$ *satisfying*

$$\widehat{\mathbf{X}}(t) = \mathbf{X}(\widehat{\nu}(t), \widehat{\mathbf{D}}(t), \widehat{\mathbf{L}}(t)) \quad \textit{and} \quad \widehat{\mathbf{S}}(t) = \mathbf{S}(\widehat{\nu}(t), \widehat{\mathbf{D}}(t), \widehat{\mathbf{L}}(t)),$$

it follows from

$$\widehat{\mathbf{L}}(t)^{-1}\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)\widehat{\mathbf{L}}(t) \in \mathbb{U}^2$$

that $\widehat{\mathbf{L}}(t)_{*1}$ *is an eigenvector of* $\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)$, *whence a non-zero multiple of either* $\mathbf{v}^1(t)$ *or* $\mathbf{v}^2(t)$. *In either case, we conclude that*

$$\|\widehat{\mathbf{L}}(t)\|_F \|\widehat{\mathbf{L}}(t)^{-1}\|_F \geq |\widehat{\mathbf{L}}(t)_{21}||(\widehat{\mathbf{L}}(t)^{-1})_{11}| = |\widehat{\mathbf{L}}(t)_{21}||\widehat{\mathbf{L}}(t)_{11}^{-1}| \to \infty,$$

and thus the sequence $\{(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))\}_{t=1}^{\infty}$ *does not satisfy the hypothesis of Theorem 15.*

On the other hand, for $\nu = \alpha t^{-1}$,

$$\begin{aligned} \|\tfrac{1}{2}(\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t) + \widehat{\mathbf{X}}(t)\widehat{\mathbf{S}}(t)) - \nu \mathbf{I}\|_2 &= \left\| \begin{bmatrix} 0 & (\alpha t^{-2} + t^{-1})/2 \\ (\alpha t^{-2} + t^{-1})/2 & 0 \end{bmatrix} \right\|_2 \\ &= \frac{1}{2t}(2\alpha - 1) = \alpha t^{-1}(1 - (2\alpha)^{-1}), \end{aligned}$$

so that $(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t)) \in \mathcal{N}(\gamma)$ *with* $\gamma = 1 - (2\alpha)^{-1} < 1$ *satisfies the hypothesis of* [13, Theorem 5.3].

For the case $\alpha \le 1/(4\beta)$, *we have*

$$\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t) = \frac{1}{2t}\begin{bmatrix}\beta-\alpha+r(t) & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}\beta+\alpha+r(t) & \frac{2\alpha t^{-1}}{\beta-\alpha+r(t)}\\ 0 & \beta+\alpha-r(t)\end{bmatrix}\begin{bmatrix}\beta-\alpha+r(t) & 0\\ 2 & 1\end{bmatrix}^{-1},$$

so that

$$\widehat{\mathbf{X}}(t) = \mathbf{X}(\widehat{\nu}(t), \widehat{\mathbf{D}}(t), \widehat{\mathbf{L}}(t)) \quad \text{and} \quad \widehat{\mathbf{S}}(t) = \mathbf{S}(\widehat{\nu}(t), \widehat{\mathbf{D}}(t), \widehat{\mathbf{L}}(t))$$

with $\widehat{\nu}(t) = \widehat{\mathbf{X}}(t) \bullet \widehat{\mathbf{S}}(t) = t^{-1}(\alpha+\beta)$,

$$\mathbf{D}(t) = \frac{1}{2(\alpha+\beta)}\begin{bmatrix}\beta+\alpha+r(t) & 0\\ 0 & \beta+\alpha-r(t)\end{bmatrix} \quad \text{and} \quad \mathbf{L}(t) = \begin{bmatrix}\beta-\alpha+r(t) & 0\\ 2 & 1\end{bmatrix}.$$

Since $\beta-\alpha+r(t) \downarrow 2(\beta-\alpha) > 0$, *it follows that* $\{\mathbf{L}(t) : t = 1, 2, \ldots\}$ *is a compact subset of* $\mathbb{L}^n_\star$, *whence the sequence* $\{(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))\}_{t=1}^\infty$ *satisfies the hypothesis of Theorem 15.*

On the other hand,

$$t^2 \det(\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t) + \widehat{\mathbf{X}}(t)\widehat{\mathbf{S}}(t)) = 4\alpha\beta - (1+\alpha t^{-1})^2 \le 1 - (1+\alpha t^{-1})^2 < 0$$

implies $\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t) + \widehat{\mathbf{X}}(t)\widehat{\mathbf{S}}(t) \notin \mathbb{S}^n_{++}$, *whence the sequence* $\{(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))\}_{t=1}^\infty$ *does not satisfy the hypothesis of* [13, Theorem 5.3].

3.2. **Two Necessary and Sufficient Conditions.** In Example 14 where the Lipschitzian bound fails, the product $\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)$ becomes increasingly singular as $t \to \infty$ even though the normalized eigenvalues of the product are bounded away from zero. The next theorem shows that this is precisely the reason that (LB) fails.

For simplicity of notation, we use $\widehat{\nu}(t)$ to denote the duality gap $\widehat{\mathbf{X}}(t) \bullet \widehat{\mathbf{S}}(t)$ and $\widehat{\mathbf{Z}}(t)$ to denote the scaled product $\widehat{\nu}(t)^{-1}\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)$.

Theorem 17. *Suppose that the primal-dual SDP problems* (2.1) *have strictly complementary solutions, and the sequence of primal-dual strictly feasible solutions* $\{(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))\}_{t=1}^\infty$ *of* (2.1) *has duality gaps converging to zero, and satisfies Condition* (3.1). *Then the Lipschitzian bound* (LB) *holds only if*

$$\kappa(\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)) = O(1), \tag{C1}$$

where $\kappa(\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t))$ *denotes the condition number* $\|\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)\|_F \|(\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t))^{-1}\|_F$ *of the product* $\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)$.

Proof. Suppose, on the contrary, (3.1) and (LB) hold but (C1) fails. By considering a subsequence of $\{(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))\}_{t=1}^\infty$ if necessary, we may assume, without loss of generality,

$$\kappa(\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)) \to \infty \text{ as } t \to \infty.$$

Since the transformation used to deduce Assumption 13 does not affect (C1), we may make this assumption without any loss of generality. Consider the symmetric matrices $\widetilde{\mathbf{X}}(t), \widetilde{\mathbf{S}}(t) \in \mathbb{S}^n_{++}$ satisfying, for $t = 1, 2, \ldots$,

$$\begin{aligned}
\widetilde{\mathbf{X}}(t)_{BB} &= \widehat{\mathbf{X}}(t)_{BB}, & \widetilde{\mathbf{S}}(t)_{BB} &= \widehat{\nu}(t)^{-1}\widehat{\mathbf{S}}(t)_{BB},\\
\widetilde{\mathbf{X}}(t)_{NB} &= \widehat{\nu}(t)^{-1}\widehat{\mathbf{X}}(t)_{NB}, & \widetilde{\mathbf{S}}(t)_{NB} &= \widehat{\nu}(t)^{-1}\widehat{\mathbf{S}}(t)_{NB},\\
\widetilde{\mathbf{X}}(t)_{NN} &= \widehat{\nu}(t)^{-1}\widehat{\mathbf{X}}(t)_{NN}, & \widetilde{\mathbf{S}}(t)_{NN} &= \widehat{\mathbf{S}}(t)_{NN}.
\end{aligned}$$

Since the Lipschitzian bound (LB) and (2.5) of Assumption 13 hold, and $\mathcal{O}_p$ and $\mathcal{O}_d$ are bounded as a consequence of Slater's condition, it follows that $\{(\widetilde{\mathbf{X}}(t), \widetilde{\mathbf{S}}(t))\}_{t=1}^{\infty}$ is a bounded sequence. By considering a subsequence of $\{(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))\}_{t=1}^{\infty}$ if necessary, we may assume, without loss of generality, that $\{(\widetilde{\mathbf{X}}(t), \widetilde{\mathbf{S}}(t))\}_{t=1}^{\infty}$ converges to, say, $(\widetilde{\mathbf{X}}(\infty), \widetilde{\mathbf{S}}(\infty))$. For each $t \in \{1, 2, \dots\}$,

$$\widehat{\mathbf{Z}}(t) = \begin{bmatrix} \widetilde{\mathbf{S}}(t)_{BB}\widetilde{\mathbf{X}}(t)_{BB} + \widehat{\nu}(t)\widetilde{\mathbf{S}}(t)_{NB}^T\widetilde{\mathbf{X}}(t)_{NB}, & \widehat{\nu}(t)\widetilde{\mathbf{S}}(t)_{BB}\widetilde{\mathbf{X}}(t)_{NB}^T + \widehat{\nu}(t)\widetilde{\mathbf{S}}(t)_{NB}^T\widetilde{\mathbf{X}}(t)_{NN} \\ \widetilde{\mathbf{S}}(t)_{NB}\widetilde{\mathbf{X}}(t)_{BB} + \widetilde{\mathbf{S}}(t)_{NN}\widetilde{\mathbf{X}}(t)_{NB}, & \widetilde{\mathbf{S}}(t)_{NN}\widetilde{\mathbf{X}}(t)_{NN} + \widehat{\nu}(t)\widetilde{\mathbf{S}}(t)_{NB}\widetilde{\mathbf{X}}(t)_{NB}^T \end{bmatrix},$$

so that

$$\widehat{\mathbf{Z}}(t) \to \begin{bmatrix} \widetilde{\mathbf{S}}(\infty)_{BB}\widetilde{\mathbf{X}}(\infty)_{BB} & \mathbf{0} \\ \widetilde{\mathbf{S}}(\infty)_{NB}\widetilde{\mathbf{X}}(\infty)_{BB} + \widetilde{\mathbf{S}}(\infty)_{NN}\widetilde{\mathbf{X}}(\infty)_{NB} & \widetilde{\mathbf{S}}(\infty)_{NN}\widetilde{\mathbf{X}}(\infty)_{NN} \end{bmatrix} =: \widehat{\mathbf{Z}}(\infty).$$

Since (from (3.1)) $\lambda_{\min}(\widehat{\nu}(t)^{-1}\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)) \geq \rho$ for all t, it follows that $\lambda_{\min}(\widehat{\mathbf{Z}}(\infty)) \geq \rho > 0$. Thus $\widehat{\mathbf{Z}}(\infty)$ is nonsingular. Consequently $\widehat{\mathbf{Z}}(t)$, whence $\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t) = \widehat{\nu}(t)\widehat{\mathbf{Z}}(t)$, has bounded condition number. □

We shall show that under strict complementarity and Condition (3.1), Condition (C1) is also sufficient for the Lipschitzian bound (LB) to hold. At the same time, we derive another necessary and sufficient condition.

Lemma 18. *It holds*

$$\|\widehat{\mathbf{X}}(t)_{BB}\|_F = O(1) \quad \textit{and} \quad \|\widehat{\mathbf{S}}(t)_{NN}\|_F = O(1), \tag{3.2}$$

$$\|\widehat{\mathbf{X}}(t)_{NN}\|_F = O(\widehat{\nu}(t)) \quad \textit{and} \quad \|\widehat{\mathbf{S}}(t)_{BB}\|_F = O(\widehat{\nu}(t)), \ \textit{and} \tag{3.3}$$

$$\|\widehat{\mathbf{X}}(t)_{NB}\|_F = O(\sqrt{\widehat{\nu}(t)}) \quad \textit{and} \quad \|\widehat{\mathbf{S}}(t)_{NB}\|_F = O(\sqrt{\widehat{\nu}(t)}). \tag{3.4}$$

Proof. For (3.2) and (3.3), see proofs of [15, Lemma 3.1] and [15, Lemma 3.2] respectively. We now prove the first equation of (3.4). The second equation is similarly proved. Since $\widehat{\mathbf{X}}(t) \in \mathbb{S}_{++}^n$, it follows that $\widehat{\mathbf{X}}(t)_{BB} - \widehat{\mathbf{X}}(t)_{NB}^T\widehat{\mathbf{X}}(t)_{NN}^{-1}\widehat{\mathbf{X}}(t)_{NB} \in \mathbb{S}_{++}^k$. Thus

$$\operatorname{tr}\widehat{\mathbf{X}}(t)_{BB} \geq \operatorname{tr}(\widehat{\mathbf{X}}(t)_{NB}^T\widehat{\mathbf{X}}(t)_{NN}^{-1}\widehat{\mathbf{X}}(t)_{NB}) \geq \lambda_{\max}(\widehat{\mathbf{X}}(t)_{NN})^{-1}\|\widehat{\mathbf{X}}(t)_{NB}\|_F^2,$$

from which it follows that

$$\|\widehat{\mathbf{X}}(t)_{NB}\|_F^2 \leq \lambda_{\max}(\widehat{\mathbf{X}}(t)_{NN})\operatorname{tr}\widehat{\mathbf{X}}(t)_{BB} = O(\widehat{\nu}(t)).$$

□

Lemma 19 (c.f. Lemma 12 of [4])**.** *Under Condition* (3.1) *and the assumption* $\lim_{t\to\infty}\widehat{\mathbf{X}}(t) \bullet \widehat{\mathbf{S}}(t) = 0$*, if* $(\widehat{\mathbf{X}}(\infty), \widehat{\mathbf{S}}(\infty))$ *is a limit point of the sequence* $\{(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))\}_{t=1}^{\infty}$*, then* $(\widehat{\mathbf{X}}(\infty), \widehat{\mathbf{S}}(\infty)) \in relint(\mathcal{O}_p) \oplus relint(\mathcal{O}_d)$*.*

Proof. Since $\mathcal{O}_p$ and $\mathcal{O}_d$ satisfy (2.5), it suffices to show $rank(\widehat{\mathbf{X}}(\infty)) = k$ and $rank(\widehat{\mathbf{S}}(\infty)) = n - k$. We shall prove the former and remark that the later can be similarly proven.

Let $\{\widehat{\mathbf{X}}(t_\ell)\}_{\ell=1}^{\infty}$ be a subsequence converging to $\widehat{\mathbf{X}}(\infty)$. Since $\{\widehat{\mathbf{X}}(t_\ell)\}$ is bounded, so is $\{\mathbf{L}_{\widehat{\mathbf{X}}(t_\ell)}\}$. Thus by choosing a subsequence of $\{\widehat{\mathbf{X}}(t_\ell)\}$ if necessary, we may assume $\mathbf{L}_{\widehat{\mathbf{X}}(t_\ell)} \to \widehat{\mathbf{L}}$. Let $\tilde{\mathbf{X}} \in relint(\mathcal{O}_p)$ and $\tilde{\mathbf{S}} \in \mathcal{O}_d$ be fixed but arbitrary. Let $\widehat{\mathbf{W}}(t)$ denote $\widehat{\nu}(t)^{-1}\mathbf{L}_{\widehat{\mathbf{X}}(t)}^T\widehat{\mathbf{S}}(t)\mathbf{L}_{\widehat{\mathbf{X}}(t)}$. It follows from Condition (3.1) that $\lambda_{\min}(\widehat{\mathbf{W}}(t)) \geq$

$\rho > 0$. From $(\widehat{\mathbf{X}}(t_\ell) - \tilde{\mathbf{X}}) \bullet (\widehat{\mathbf{S}}(t_\ell) - \tilde{\mathbf{S}}) = 0$, $\widehat{\mathbf{X}}(t_\ell) \bullet \widehat{\mathbf{S}}(t_\ell) = \widehat{\nu}(t_\ell)$ and $\tilde{\mathbf{X}} \bullet \tilde{\mathbf{S}} = 0$, we deduce that $\widehat{\mathbf{X}}(t_\ell) \bullet \tilde{\mathbf{S}} + \widehat{\mathbf{S}}(t_\ell) \bullet \tilde{\mathbf{X}} = \widehat{\nu}(t_\ell)$. Therefore

$$\begin{aligned} \widehat{\nu}(t_\ell) &\geq \widehat{\mathbf{S}}(t_\ell) \bullet \tilde{\mathbf{X}} = \widehat{\nu}(t_\ell)\,\mathrm{tr}(\widehat{\mathbf{W}}(t_\ell)\mathbf{L}^{-1}_{\widehat{\mathbf{X}}(t_\ell)}\tilde{\mathbf{X}}\mathbf{L}^{-T}_{\widehat{\mathbf{X}}(t_\ell)}) \\ &\geq \rho\widehat{\nu}(t_\ell)\,\mathrm{tr}(\mathbf{L}^{-1}_{\widehat{\mathbf{X}}(t_\ell)}\tilde{\mathbf{X}}\mathbf{L}^{-T}_{\widehat{\mathbf{X}}(t_\ell)}) \\ &\geq \rho\widehat{\nu}(t_\ell)\textstyle\sum_{i=1}^k (\mathbf{L}_{\widehat{\mathbf{X}}(t_\ell)})^{-2}_{ii}(\mathbf{L}_{\tilde{\mathbf{X}}})^2_{ii}, \end{aligned}$$

from which it follows that $(\mathbf{L}_{\widehat{\mathbf{X}}(t_\ell)})_{ii} \geq \sqrt{\rho}(\mathbf{L}_{\tilde{\mathbf{X}}})_{ii}$ for each $i \in \{1, \ldots, k\}$. Since $\tilde{\mathbf{X}} \in relint(\mathcal{O}_p)$, it follows from Assumption 13 that $(\mathbf{L}_{\tilde{\mathbf{X}}})_{ii} > 0$ for each $i \in \{1, \ldots, k\}$. Thus for all $i \in \{1, \ldots, k\}$,

$$\widehat{\mathbf{L}}_{ii} = \lim_{\ell \to \infty} (\mathbf{L}_{\widehat{\mathbf{X}}(t_\ell)})_{ii} \geq \sqrt{\rho}(\mathbf{L}_{\tilde{\mathbf{X}}})_{ii} > 0.$$

This implies that $rank(\widehat{\mathbf{L}}) \geq k$, and hence $rank(\widehat{\mathbf{X}}(\infty)) = k$. □

Theorem 20. *Suppose that the primal-dual SDP problems* (2.1) *have strictly complementary solutions, and the sequence of primal-dual strictly feasible solutions* $\{(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))\}_{t=1}^{\infty}$ *of* (2.1) *has duality gaps converging to zero, and satisfies Condition* (3.1)*. Then the Lipschitzian bound* (LB) *holds if*

$$\text{(C2)} \qquad \|\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)\|_F = O(\widehat{\mathbf{X}}(t) \bullet \widehat{\mathbf{S}}(t)).$$

Proof. As before, we remark that Assumption 13 can be made without any loss of generality.

Suppose, on the contrary, (C2) holds but (LB) fails. Then there exists a subsequence $\{(\widehat{\mathbf{X}}(t_\ell), \widehat{\mathbf{S}}(t_\ell))\}_{\ell=1}^{\infty}$ such that

$$\text{(3.5)} \qquad \widehat{\nu}(t_\ell)^{-1} \max\{dist(\widehat{\mathbf{X}}(t_\ell), \mathcal{O}_p), dist(\widehat{\mathbf{S}}(t_\ell), \mathcal{O}_d)\} \to \infty.$$

By choosing a subsequence if necessary, we may assume that $\{(\widehat{\mathbf{X}}(t_\ell), \widehat{\mathbf{S}}(t_\ell))\}_{\ell=1}^{\infty}$ converges to, say, $(\widehat{\mathbf{X}}(\infty), \widehat{\mathbf{S}}(\infty))$. It follows from Lemma 19 and Assumption 13 that $\widehat{\mathbf{X}}(\infty)_{BB} \in \mathbb{S}^k_{++}$ and $\widehat{\mathbf{S}}(\infty)_{NN} \in \mathbb{S}^{n-k}_{++}$, whence $\|\widehat{\mathbf{X}}(t_\ell)^{-1}_{BB}\|_F, \|\widehat{\mathbf{S}}(t_\ell)^{-1}_{NN}\|_F = O(1)$. Thus we deduce from Condition (C2), that

$$\text{(3.6)} \qquad \begin{aligned} \|\widehat{\mathbf{S}}(t_\ell)_{NB}\|_F &= \|\widehat{\nu}(t_\ell)\widehat{\mathbf{X}}(t_\ell)^{-1}_{BB}\widehat{\mathbf{Z}}(t_\ell)^T_{NB} - \widehat{\mathbf{X}}(t_\ell)^{-1}_{BB}\widehat{\mathbf{X}}(t_\ell)^T_{NB}\widehat{\mathbf{S}}(t_\ell)_{NN}\|_F \\ &= O(\widehat{\nu}(t_\ell) + \|\widehat{\mathbf{X}}(t_\ell)_{NB}\|_F) \end{aligned}$$

and

$$\text{(3.7)} \qquad \begin{aligned} &\widehat{\mathbf{S}}(t_\ell)_{NB} \bullet \widehat{\mathbf{X}}(t_\ell)_{NB} \\ &= \mathrm{tr}\,\widehat{\nu}(t_\ell)\widehat{\mathbf{X}}(t_\ell)_{NB}\widehat{\mathbf{X}}(t_\ell)^{-1}_{BB}\widehat{\mathbf{Z}}(t_\ell)^T_{NB} - \mathrm{tr}\,\widehat{\mathbf{X}}(t_\ell)_{NB}\widehat{\mathbf{X}}(t_\ell)^{-1}_{BB}\widehat{\mathbf{X}}(t_\ell)^T_{NB}\widehat{\mathbf{S}}(t_\ell)_{NN} \\ &= O(\|\widehat{\mathbf{X}}(t_\ell)_{NB}\|_F\widehat{\nu}(t_\ell)) - \Theta(\|\widehat{\mathbf{X}}(t_\ell)_{NB}\|^2_F). \end{aligned}$$

Consider the system of linear matrix equations

$$\begin{aligned} \boldsymbol{\mathcal{A}}(\mathbf{X} - \widehat{\mathbf{X}}(\infty)) = \mathbf{0}, \quad \mathbf{X}_{NB} = \mathbf{0}, \quad \mathbf{X}_{NN} = \mathbf{0}, \quad \mathbf{X}_{BB} \in \mathbb{S}^k_+, \\ \boldsymbol{\mathcal{C}}(\mathbf{S} - \widehat{\mathbf{S}}(\infty)) = \mathbf{0}, \quad \mathbf{S}_{NB} = \mathbf{0}, \quad \mathbf{S}_{BB} = \mathbf{0}, \quad \mathbf{S}_{NN} \in \mathbb{S}^{n-k}_+. \end{aligned}$$

Since the system has Slater points $(\widehat{\mathbf{X}}(\infty), \widehat{\mathbf{S}}(\infty))$, it follows from [22, Lemma 2.3] that there exist solutions $(\tilde{\mathbf{X}}(t_\ell), \tilde{\mathbf{S}}(t_\ell))$ to the above system such that

$$\begin{aligned}
&\sqrt{\|\widehat{\mathbf{X}}(t_\ell) - \tilde{\mathbf{X}}(t_\ell)\|_F^2 + \|\widehat{\mathbf{S}}(t_\ell) - \tilde{\mathbf{S}}(t_\ell)\|_F^2} \\
&= O(\|\widehat{\mathbf{X}}(t_\ell)_{NB}\|_F + \|\widehat{\mathbf{X}}(t_\ell)_{NN}\|_F + \|\widehat{\mathbf{S}}(t_\ell)_{NB}\|_F + \|\widehat{\mathbf{S}}(t_\ell)_{BB}\|_F) \\
&= O(\|\widehat{\mathbf{X}}(t_\ell)_{NB}\|_F + \widehat{\nu}(t_\ell)),
\end{aligned} \tag{3.8}$$

where we have used (3.3) and (3.6) in the second equality. Thus we deduce using $(\widehat{\mathbf{X}}(t_\ell) - \tilde{\mathbf{X}}(t_\ell)) \bullet (\widehat{\mathbf{S}}(t_\ell) - \tilde{\mathbf{S}}(t_\ell)) = 0$ and (3.3) that

$$\begin{aligned}
&-2\widehat{\mathbf{X}}(t_\ell)_{NB} \bullet \widehat{\mathbf{S}}(t_\ell)_{NB} \\
&= (\widehat{\mathbf{X}}(t_\ell)_{BB} - \tilde{\mathbf{X}}(t_\ell)_{BB}) \bullet \widehat{\mathbf{S}}(t_\ell)_{BB} + \widehat{\mathbf{X}}(t_\ell)_{NN} \bullet (\widehat{\mathbf{S}}(t_\ell)_{NN} - \tilde{\mathbf{S}}(t_\ell)_{NN}) \\
&= O(\widehat{\nu}(t_\ell)(\|\widehat{\mathbf{X}}(t_\ell)_{BB} - \tilde{\mathbf{X}}(t_\ell)_{BB}\|_F + \|\widehat{\mathbf{S}}(t_\ell)_{NN} - \tilde{\mathbf{S}}(t_\ell)_{NN}\|_F)) \\
&= O(\widehat{\nu}(t_\ell)\|\widehat{\mathbf{X}}(t_\ell)_{NB}\|_F + \widehat{\nu}(t_\ell)^2),
\end{aligned}$$

and hence, using (3.7),

$$\|\widehat{\mathbf{X}}(t_\ell)_{NB}\|_F^2 = O(\widehat{\nu}(t_\ell)\|\widehat{\mathbf{X}}(t_\ell)_{NB}\|_F + \widehat{\nu}(t_\ell)^2),$$

which can happen only if

$$\|\widehat{\mathbf{X}}(t_\ell)_{NB}\|_F = O(\widehat{\nu}(t_\ell)).$$

Consequently it follows from (3.8) that

$$dist(\widehat{\mathbf{X}}(t_\ell), \mathcal{O}_p) \le \|\widehat{\mathbf{X}}(t_\ell) - \tilde{\mathbf{X}}(t_\ell)\|_F = O(\|\widehat{\mathbf{X}}(t_\ell)_{NB}\|_F + \widehat{\nu}(t_\ell)) = O(\widehat{\nu}(t_\ell))$$

and

$$dist(\widehat{\mathbf{S}}(t_\ell), \mathcal{O}_d) \le \|\widehat{\mathbf{S}}(t_\ell) - \tilde{\mathbf{S}}(t_\ell)\|_F = O(\|\widehat{\mathbf{X}}(t_\ell)_{NB}\|_F + \widehat{\nu}(t_\ell)) = O(\widehat{\nu}(t_\ell)),$$

contradicting (3.5). □

Corollary 21. *Suppose that the primal-dual SDP problems* (2.1) *have strictly complementary solutions, and the sequence of primal-dual strictly feasible solutions* $\{(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))\}_{t=1}^\infty$ *of* (2.1) *has duality gaps converging to zero, and satisfies Condition* (3.1)*. Then the following statements are equivalent:*

$$\max\{dist(\widehat{\mathbf{X}}(t), \mathcal{O}_p), dist(\widehat{\mathbf{S}}(t), \mathcal{O}_d)\} = O(\widehat{\mathbf{X}}(t) \bullet \widehat{\mathbf{S}}(t)); \tag{LB}$$

$$\kappa(\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)) = O(1); \tag{C1}$$

$$\|\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)\|_F = O(\widehat{\mathbf{X}}(t) \bullet \widehat{\mathbf{S}}(t)). \tag{C2}$$

Proof. Theorem 17 states that the Lipschitzian bound (LB) implies Condition (C1) and Theorem 20 states that Condition (C2) implies the Lipschitzian bound (LB). Thus it suffices to show that Condition (C1) implies Condition (C2).

It follows from Lemma 18 that

$$\widehat{\nu}(t)\begin{bmatrix}\|\widehat{\mathbf{Z}}(t)_{BB}\|_F & \|\widehat{\mathbf{Z}}(t)_{BN}\|_F \\ \|\widehat{\mathbf{Z}}(t)_{NB}\|_F & \|\widehat{\mathbf{Z}}(t)_{NN}\|_F\end{bmatrix} = \begin{bmatrix}O(\widehat{\nu}(t)) & O(\widehat{\nu}(t)^{3/2}) \\ O(\widehat{\nu}(t)^{1/2}) & O(\widehat{\nu}(t))\end{bmatrix}.$$

Thus Condition (C2) holds if $\|\widehat{\mathbf{Z}}(t)_{NB}\|_F = O(1)$. A related inequality that can be deduced from Condition (C1) is

$$\|\widehat{\mathbf{Z}}(t)_{NB}\|_F \|(\widehat{\mathbf{Z}}(t)^{-1})_{NB}\|_F \le \|\widehat{\mathbf{Z}}(t)\|_F \|\widehat{\mathbf{Z}}(t)^{-1}\|_F = \kappa(\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)) = O(1). \tag{3.9}$$

Using the block-matrix inverse formulae

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}^{-1} = \begin{bmatrix}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}, & -(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1}\\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}, & \mathbf{D}^{-1}+\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1}\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1}, & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\\ -(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1}, & (\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\end{bmatrix},$$

which hold whenever $\mathbf{A}$, $\mathbf{D}$, and $\left[\begin{smallmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{smallmatrix}\right]$ are invertible, we derive

$$\widehat{\mathbf{X}}(t)^{-1} = \begin{bmatrix}\widehat{\mathbf{X}}(t)_{BB}^{-1}+\widehat{\mathbf{X}}(t)_{BB}^{-1}\widehat{\mathbf{X}}(t)_{NB}^T\widehat{\mathbf{M}}(t)^{-1}\widehat{\mathbf{X}}(t)_{NB}\widehat{\mathbf{X}}(t)_{BB}^{-1}, & -\widehat{\mathbf{X}}(t)_{BB}^{-1}\widehat{\mathbf{X}}(t)_{NB}^T\widehat{\mathbf{M}}(t)^{-1}\\ -\widehat{\mathbf{M}}(t)^{-1}\widehat{\mathbf{X}}(t)_{NB}\widehat{\mathbf{X}}(t)_{BB}^{-1}, & \widehat{\mathbf{M}}(t)^{-1}\end{bmatrix}$$

where $\widehat{\mathbf{M}}(t) = \widehat{\mathbf{X}}(t)_{NN} - \widehat{\mathbf{X}}(t)_{NB}\widehat{\mathbf{X}}(t)_{BB}^{-1}\widehat{\mathbf{X}}(t)_{NB}^T$, and

$$\widehat{\mathbf{S}}(t)^{-1} = \begin{bmatrix}\widehat{\mathbf{N}}(t)^{-1}, & -\widehat{\mathbf{N}}(t)^{-1}\widehat{\mathbf{S}}(t)_{NB}^T\widehat{\mathbf{S}}(t)_{NN}^{-1}\\ -\widehat{\mathbf{S}}(t)_{NN}^{-1}\widehat{\mathbf{S}}(t)_{NB}\widehat{\mathbf{N}}(t)^{-1}, & \widehat{\mathbf{S}}(t)_{NN}^{-1}+\widehat{\mathbf{S}}(t)_{NN}^{-1}\widehat{\mathbf{S}}(t)_{NB}\widehat{\mathbf{N}}(t)^{-1}\widehat{\mathbf{S}}(t)_{NB}^T\widehat{\mathbf{S}}(t)_{NN}^{-1}\end{bmatrix}$$

where $\widehat{\mathbf{N}}(t) = \widehat{\mathbf{S}}(t)_{BB} - \widehat{\mathbf{S}}(t)_{NB}^T\widehat{\mathbf{S}}(t)_{NN}^{-1}\widehat{\mathbf{S}}(t)_{NB}$. Thus

$$\begin{aligned}
&\widehat{\nu}(t)^{-1}(\widehat{\mathbf{Z}}(t)^{-1})_{NB}\\
&= (\widetilde{\mathbf{X}}(t)^{-1})_{NB}^T(\widetilde{\mathbf{S}}(t)^{-1})_{BB} + (\widetilde{\mathbf{X}}(t)^{-1})_{NN}(\widetilde{\mathbf{S}}(t)^{-1})_{NB}^T\\
&= -\widehat{\mathbf{M}}(t)^{-1}\widehat{\mathbf{X}}(t)_{NB}\widehat{\mathbf{X}}(t)_{BB}^{-1}\widehat{\mathbf{N}}(t)^{-1} - \widehat{\mathbf{M}}(t)^{-1}\widehat{\mathbf{S}}(t)_{NN}^{-1}\widehat{\mathbf{S}}(t)_{NB}\widehat{\mathbf{N}}(t)^{-1}\\
&= -\widehat{\mathbf{M}}(t)^{-1}\widehat{\mathbf{S}}(t)_{NN}^{-1}(\widehat{\mathbf{S}}(t)_{NN}\widehat{\mathbf{X}}(t)_{NB} + \widehat{\mathbf{S}}(t)_{NB}\widehat{\mathbf{X}}(t)_{BB})\widehat{\mathbf{X}}(t)_{BB}^{-1}\widehat{\mathbf{N}}(t)^{-1}\\
&= -\widehat{\nu}(t)\widehat{\mathbf{M}}(t)^{-1}\widehat{\mathbf{S}}(t)_{NN}^{-1}\widehat{\mathbf{Z}}(t)_{NB}\widehat{\mathbf{X}}(t)_{BB}^{-1}\widehat{\mathbf{N}}(t)^{-1}.
\end{aligned} \tag{3.10}$$

The matrices $\widehat{\mathbf{X}}(t)$ and $\widehat{\mathbf{S}}(t)$ are positive definite, whence so are the respective Schur complements $\widehat{\mathbf{M}}(t)$ and $\widehat{\mathbf{N}}(t)$ of $\widehat{\mathbf{X}}(t)_{BB}$ and $\widehat{\mathbf{S}}(t)_{NN}$. We then deduce from (3.3) that

$$\begin{aligned}
\|\widehat{\mathbf{M}}(t)\|_F^2 &= \widehat{\mathbf{M}}(t) \bullet (\widehat{\mathbf{X}}(t)_{NN} - \widehat{\mathbf{X}}(t)_{NB}\widehat{\mathbf{X}}(t)_{BB}^{-1}\widehat{\mathbf{X}}(t)_{NB}^T)\\
&= \widehat{\mathbf{M}}(t) \bullet \widehat{\mathbf{X}}(t)_{NN} - \widehat{\mathbf{M}}(t) \bullet \widehat{\mathbf{X}}(t)_{NB}\widehat{\mathbf{X}}(t)_{BB}^{-1}\widehat{\mathbf{X}}(t)_{NB}^T\\
&\le \widehat{\mathbf{M}}(t) \bullet \widehat{\mathbf{X}}(t)_{NN}\\
&= (\widehat{\mathbf{X}}(t)_{NN} - \widehat{\mathbf{X}}(t)_{NB}\widehat{\mathbf{X}}(t)_{BB}^{-1}\widehat{\mathbf{X}}(t)_{NB}^T) \bullet \widehat{\mathbf{X}}(t)_{NN}\\
&= \widehat{\mathbf{X}}(t)_{NN} \bullet \widehat{\mathbf{X}}(t)_{NN} - \widehat{\mathbf{X}}(t)_{NB}\widehat{\mathbf{X}}(t)_{BB}^{-1}\widehat{\mathbf{X}}(t)_{NB}^T \bullet \widehat{\mathbf{X}}(t)_{NN}\\
&\le \|\widehat{\mathbf{X}}(t)_{NN}\|_F^2 = O(\widehat{\nu}(t)^2)
\end{aligned}$$

and, similarly,

$$\|\widehat{\mathbf{N}}(t)\|_F \le \|\widehat{\mathbf{S}}(t)_{BB}\|_F = O(\widehat{\nu}(t)).$$

Together with (3.2) and the sub-multiplicity of the Frobenius norm, it then follows from (3.10) that

$$\begin{aligned}
\|\widehat{\mathbf{Z}}(t)_{NB}\|_F &\le \widehat{\nu}(t)^{-2}\|\widehat{\mathbf{S}}(t)_{NN}\|_F\|\widehat{\mathbf{M}}(t)\|_F\|(\widehat{\mathbf{Z}}(t)^{-1})_{NB}\|_F\|\widehat{\mathbf{N}}(t)\|_F\|\widehat{\mathbf{X}}(t)_{BB}\|_F\\
&= O(\|(\widehat{\mathbf{Z}}(t)^{-1})_{NB}\|_F).
\end{aligned}$$

Consequently if Condition (C1) holds, then (3.9) follows, whence

$$\|\widehat{\mathbf{Z}}(t)_{NB}\|_F = O\big(\|(\widehat{\mathbf{Z}}(t)^{-1})_{NB}\|_F^{1/2}\|\widehat{\mathbf{Z}}(t)_{NB}\|_F^{1/2}\big) = O(1),$$

which proves Condition (C2). □

4. Conclusion

In this paper, two sets of results are presented.

In the first part, the analyticity of certain weighted centers for semidefinite programming problems is proved for the classes of

(1) Cholesky weighted centers defined by the centrality equation $\mathbf{L}_{\mathbf{X}}^T \mathbf{S} \mathbf{L}_{\mathbf{X}} = \nu \mathbf{D}$ as functions the parameters $(\nu, \mathbf{D}) \in \mathbb{R}_+ \oplus \mathbb{D}_{++}^n$, and
(2) extended Cholesky weighted centers defined by the centrality equation

$$\mathbf{L}_{\widehat{\mathbf{Q}}^T \mathbf{L}^T \mathbf{X} \mathbf{L} \widehat{\mathbf{Q}}}^T \widehat{\mathbf{Q}}^T \mathbf{L}^{-1} \mathbf{S} \mathbf{L}^{-T} \widehat{\mathbf{Q}} \mathbf{L}_{\widehat{\mathbf{Q}}^T \mathbf{L}^T \mathbf{X} \mathbf{L} \widehat{\mathbf{Q}}} = \nu \mathbf{D}$$

as functions of the parameters $(\nu, \mathbf{D}, \mathbf{L}) \in \mathbb{R}_+ \oplus \mathbb{D}_{++}^n \oplus \widehat{\mathbf{Q}} \mathbb{L}_\star^n$,

under the assumption of strict complementarity (and of course, Slater's condition). It is mentioned at the end of Section 2.2 that while the second result is more general than the first, it does not, in general, imply the former. Moreover the former result is important as it is used in Section 2.3 to deduce the analyticity of central paths of homogeneous cone programming problems.

The second part investigates the Lipschitz-type bound

$$\max\{dist(\widehat{\mathbf{X}}(t), \mathcal{O}_p), dist(\widehat{\mathbf{S}}(t), \mathcal{O}_d)\} = O(\widehat{\mathbf{X}}(t) \bullet \widehat{\mathbf{S}}(t))$$

for iterates $(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))$ that stay within some wide neighborhood of the primal-dual central path of an semidefinite programming problem. This bound played an important role in the analysis of superlinear convergence in some literature. It is shown in this paper that under strict complementarity,

(1) the analyticity of the extended Cholesky weighted centers provides a sufficient condition for the Lipschitzian bound, and
(2) the Lipschitzian bound is equivalent to either of the following statements:
    (a) the condition number of $\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)$ is bounded;
    (b) every entry of the product $\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)$ grows at a rate that is at most linear in the trace of the same product.

The first result allows us to enforce the Lipschitzian bound in a primal-dual interior-point algorithm, provided we can control the matrix $\mathbf{L}(t)$ used in declaring the pair $(\widehat{\mathbf{X}}(t), \widehat{\mathbf{S}}(t))$ as extended Cholesky weighted centers—the sequence $\{\mathbf{L}(t)\}$ should not have a singular accumulation point. It would seem impossible to control $\mathbf{L}(t)$ without imposing some checks on the product $\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)$. Indeed the second result clearly points out the necessity to keep the product $\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)$ in check. The only well-known central path neighborhood for which this can be accomplished is the natural neighborhood associated with the AHO direction, but in this case we lose polynomial complexity as discussed in the introductory section. It is shown in this paper, via a simple example, that any neighborhood that relies solely on the eigenvalues of the product $\widehat{\mathbf{S}}(t)\widehat{\mathbf{X}}(t)$ is insufficient to guarantee a Lipschitzian bound.

Thus it remains open the question of existence of a polynomial-time primal-dual interior-point algorithm for SDP, that converges locally at a superlinear rate under the sole assumption of strict complementarity.

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