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A Width-Recursive Depth-First Tree Search Approach for the Design of Discrete Coefficient Perfect Reconstruction Lattice Filter Bank

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Abstract—The lattice structure two-channel orthogonal filter bank structurally guarantees the perfect reconstruction (PR) property. Thus, it is eminently suitable for hardware realization even under severe coefficient quantization condition. Nevertheless, its frequency response is still adversely affected by coefficient quantization. In this paper, a novel recursive-in-width depth-first tree search technique is presented for the design of lattice structure PR orthogonal filter banks subject to discrete coefficient value constraint. A frequency-response deterioration measure is developed to serve as a branching criterion. At any node, the coefficient which will cause the largest deterioration in the frequency response of the filter when quantized is selected for branching. The improvement in the frequency response ripple magnitude achieved by our algorithm over that by simple rounding of coefficient values differs widely from example to example ranging from a fraction of a decibel to over 10 dB.

Index Terms—Discrete coefficient, lattice filter, orthogonal filter bank, perfect reconstruction (PR), sensitivity measure, signed power-of-two, width-recursive depth-first tree search.

I. INTRODUCTION

A TWO-CHANNEL filter bank [1]–[11] is shown in Fig. 1(a), where $H_N^0(z)$ and $H_N^1(z)$ are the low-pass and high-pass analysis filters, respectively, and $G_N^0(z)$ and $G_N^1(z)$ are the low-pass and high-pass synthesis filters, respectively. These filters are chosen such that an input signal $x(n)$ can be decomposed into its subband components, decimated and then reconstructed from the decimated subband components with little or no distortion. Such system have been used in various signal processing applications including audio, image and video compression [4]–[6].

The implementation of a perfect reconstruction filter bank using the tap delay line structure suffers from the disadvantage that the perfect reconstruction property is affected by coefficient quantization. A lattice analysis/synthesis system, which structurally ensures perfect reconstruction, was introduced in [3]. An important virtue of the lattice structure filter bank is that the perfect reconstruction property is preserved even under severe coefficient quantization. Since the perfect reconstruction property is structurally ensured, it is only necessary to consider

Fig. 1. (a) A typical two channel filter bank. (b) The analysis bank of a lattice PR orthogonal filter bank.

the frequency response when the coefficient values are optimized in the discrete value space. Many optimization techniques [12], [13] have been proposed for the design of discrete coefficient finite-impulse response (FIR) filters. In the design of a transversal FIR filter subject to a given frequency-response requirement, mixed integer linear programming (MILP) [12] is the only known method which can guarantee global optimality in the minimax sense. Unfortunately, MILP cannot be used to design lattice filter because the objective function is not a linear function of the lattice filter's coefficient values. An efficient method incorporating the unconstrained weighted least squares technique into a tree search algorithm for the design of high-order discrete coefficient FIR filters was introduced in [13]. Several techniques [9], [10] for the design of lattice filter bank with discrete coefficient values have also been reported in the literature. The method used in [9] is one involving a search in the discrete space in the vicinity of the optimum continuous coefficient space whereas that used in [10] is a simulation of the evolution process of natural selection. The method used in [21] is a successive reoptimization method.

In this paper, we present a novel recursive-in-width depth-first tree search technique for the design of perfect reconstruction lattice structure filter bank with discrete coefficient values. In our technique, the width of the tree being searched is initially set to one and subsequently increased by one at a time until an arbitrary predefined value is reached. The sequence in which the coefficients are selected for quantization is based on the frequency-response deterioration measure associated with the coefficients; the coefficient that will cause the largest deterioration

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in the frequency response of the filter is quantized first and the coefficient that will cause the least deterioration is quantized last.

This paper is divided into 8 sections. A brief description of the perfect reconstruction filter bank for the purpose of defining the notations used in this paper is presented in Section II. Presented in Section III is the coefficient sensitivity analysis for the perfect reconstruction lattice filter. A frequency-response deterioration measure is developed in Section IV. The frequency-response deterioration measure associated with a coefficient is defined as the product of the sensitivity of that coefficient and the grid spacing of the discrete space in the neighborhood of that coefficient value. At any node, among those coefficients which have not been quantized, the coefficient with the largest frequency-response deterioration measure is selected for quantization. Section V describes the recursive-in-width depth-first tree search algorithm. In this algorithm, after a coefficient is selected, it is assigned several discrete values immediately larger than or smaller than its continuous value. To each assigned discrete value, the other coefficient values which have not been constrained to discrete values are reoptimized. The reoptimization of the unquantized coefficients after the quantization of each of the coefficient values is also presented in Section V. Examples illustrating the results of our technique are shown in Section VI. In Section VII, a statistical analysis of quantization errors for the lattice structure orthogonal filter bank is presented.

II. PERFECT RECONSTRUCTION LATTICE ORTHOGONAL FILTER BANK

Consider the two-channel filter bank in Fig. 1(a). Let the length of each of the analysis and synthesis filters be $2N$. The relationship between the input signal $X(z)$ and the reconstructed signal $\hat{X}(z)$ is given by

$$\begin{aligned} \hat{X}(z) = & \frac{1}{2} [H_N^0(z)G_N^0(z) + H_N^1(z)G_N^1(z)] X(z) \\ & + \frac{1}{2} [H_N^0(-z)G_N^0(z) + H_N^1(-z)G_N^1(z)] X(-z). \end{aligned} \quad (1)$$

In a perfect reconstruction (PR) orthogonal filter bank, these filters are chosen as

$$\begin{aligned} G_N^0(z) &= 2H_N^1(-z) \\ G_N^1(z) &= -2H_N^0(-z) \\ H_N^1(z) &= -z^{-2N+1}H_N^0(-z^{-1}). \end{aligned} \quad (2)$$

Substituting (2) into (1), the overall transfer function is given by

$$H(z) = z^{-2N+1} [H_N^0(z)H_N^0(z^{-1}) + H_N^0(-z)H_N^0(-z^{-1})]. \quad (3)$$

For perfect reconstruction

$$H_N^0(z)H_N^0(z^{-1}) + H_N^0(-z)H_N^0(-z^{-1}) = 1 \quad (4)$$

it implies the power complementary property [3]

$$|H_N^0(e^{j\omega})|^2 + |H_N^0(e^{j(\omega-\pi)})|^2 = 1. \quad (5)$$

In [3], a lattice, such as the one shown in Fig. 1(b), was proposed for the design and implementation of PR orthogonal filter banks. The filters obtained using the lattice are structurally

ensured to satisfy (5). Consider a $(2N - 1)$ th-order lattice structure (with N coefficients) implementing the analysis bank shown in Fig. 1(b). Let the z -transform transfer functions of the low-pass and high-pass channels be $H_N^0(z)$ and $H_N^1(z)$, respectively. Thus, we have

$$\begin{aligned} \begin{bmatrix} H_N^0(z) \\ H_N^1(z) \end{bmatrix} &= \beta_N A(\alpha_{N-1}) \Lambda A(\alpha_{N-2}) \Lambda \cdots \Lambda A(\alpha_k) \Lambda \cdots \Lambda A(\alpha_0) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \end{aligned} \quad (6)$$

where

$$\beta_N^2 = \frac{1}{2} \prod_{k=0}^{N-1} \frac{1}{1 + \alpha_k^2} \quad (7)$$

$$A(\alpha_k) = \begin{bmatrix} 1 & -\alpha_k \\ \alpha_k & 1 \end{bmatrix} \quad (8)$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & z^{-2} \end{bmatrix}. \quad (9)$$

In Fig. 1(b), β_N appears as a scaling amplifier at the input. This is purely for the convenience of simplifying Fig. 1(b). In actual implementation, β_N may be factored and the factors distributed in between the lattice stages to optimize for roundoff noise performance.

A weighted least squares objective function

$$f = \sum_{\omega \in [\omega_s, \pi]} B(\omega) |H_N^0(e^{j\omega})|^2 \quad (10)$$

is used as the criterion to optimize the PR orthogonal filter bank. In (10), $B(\omega)$ is the error weighting function and ω_s is the stopband edge. The stopband edge satisfies the constrain $\pi/2 < \omega_s < \pi$. The quasi-Newton method with Davidon-Fletcher-Powell update of the approximate Hessian matrix [15] is used to solve the minimization problem. The weighting function $B(\omega)$ is updated using the Lim-Lee-Chen-Yang algorithm [16] until an equiripple PR orthogonal bank is achieved. The Lim-Lee-Chen-Yang algorithm which updates the weighting function using the envelope of the ripple magnitude converges many times faster than Lawson's algorithm [17] which updates the weighting function using the magnitude of the deviation.

III. COEFFICIENT SENSITIVITY ANALYSIS

The sensitivities of $H_N^0(z)$ and $H_N^1(z)$ with respect to the coefficient α_k denoted by $P_N^k(z)$ and $Q_N^k(z)$, respectively, are given by

$$\begin{aligned} \begin{bmatrix} P_N^k(z) \\ Q_N^k(z) \end{bmatrix} &= \begin{bmatrix} \frac{\partial H_N^0(z)}{\partial \alpha_k} \\ \frac{\partial H_N^1(z)}{\partial \alpha_k} \end{bmatrix} \\ &= \beta_N A(\alpha_{N-1}) \Lambda A(\alpha_{N-2}) \Lambda \cdots \Lambda A(\alpha_{k+1}) \Lambda \frac{1}{1 + \alpha_k^2} \\ &\quad \times A(\alpha_k) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Lambda A(\alpha_{k-1}) \Lambda \cdots \Lambda A(\alpha_0) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \end{aligned} \quad (11)$$

TABLE I
COEFFICIENT VALUES AND STOPBAND COEFFICIENT SENSITIVITIES OF A
27-TH ORDER PR ORTHOGONAL FILTER BANK

k	α_k	$\frac{1}{1+\alpha_k^2}$	Peak sensitivity	Average sensitivity
0	-4.5786172	0.0455296	0.0455296	0.0364138
1	1.4968931	0.3085763	0.3085762	0.2580223
2	-0.8626808	0.5733228	0.5733225	0.5053473
3	0.5783206	0.7493697	0.7493694	0.6894843
4	-0.4110491	0.8554602	0.8554599	0.8118847
5	0.2977277	0.9185758	0.9185754	0.8904949
6	-0.2145744	0.9559844	0.9559839	0.9397274
7	0.1510030	0.9777064	0.9777060	0.9693368
8	-0.1019100	0.9897211	0.9897206	0.9859883
9	0.0646631	0.9958361	0.9958356	0.9944542
10	-0.0376196	0.9985863	0.9985863	0.9981892
11	0.0193689	0.9996250	0.9996248	0.9995457
12	-0.0083315	0.9999306	0.9999301	0.9999220
13	0.0026672	0.9999929	0.9999929	0.9999926

and can be rewritten as

$$\begin{aligned}
 & \begin{bmatrix} P_N^k(z) \\ Q_N^k(z) \end{bmatrix} \\
 &= \frac{\beta_N}{\beta_k} \frac{1}{1+\alpha_k^2} A(\alpha_{N-1}) \Lambda \cdots \Lambda A(\alpha_{k+1}) \Lambda A(\alpha_k) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
 & \quad \times \Lambda \begin{bmatrix} H_k^0(z) \\ H_k^1(z) \end{bmatrix} \\
 &= \frac{\beta_N}{\beta_k} \frac{1}{1+\alpha_k^2} A(\alpha_{N-1}) \Lambda \cdots \Lambda A(\alpha_{k+1}) \\
 & \quad \times \Lambda A(\alpha_k) \begin{bmatrix} -z^{-2} H_k^1(z) \\ H_k^0(z) \end{bmatrix}. \quad (12)
 \end{aligned}$$

From (12), the following can be obtained

$$P_N^k(z) \widetilde{P_N^k(z)} + Q_N^k(z) \widetilde{Q_N^k(z)} = \frac{1}{(1+\alpha_k^2)^2} < 1, \quad (13)$$

for $\alpha_k \neq 0$

where $\widetilde{P_N^k(z)}$ and $\widetilde{Q_N^k(z)}$ denote the conjugate of $P_N^k(z)$ and $Q_N^k(z)$, respectively. The proof for (13) is shown in Appendix A. From (12) and (13), it can be shown that

$$|P_{N,k}(z)| = \left| \frac{\partial H_{N,k}(z)}{\partial \alpha_k} \right| \leq \frac{1}{1+\alpha_k^2} < 1, \quad k \geq 0. \quad (14)$$

The coefficient sensitivity $|P_{N,k}(e^{j\omega})|$ is thus bounded by $1/(1+\alpha_k^2)$; in general, it is a function of frequency. To give an idea on the relative values of (a) the peak absolute value of $P_{N,k}(e^{j\omega})$, (b) the average of the absolute value of $P_{N,k}(e^{j\omega})$ in the stopband, (c) α_k , and (d) $1/(1+\alpha_k^2)$, we tabulate in Table I these quantities for a particular 27th-order filter bank. It is interesting to note from Table I that the coefficient sensitivity increases with increasing k .

IV. FREQUENCY RESPONSE DETERIORATION MEASURE

The quantization of a coefficient α_k causes the value of α_k to be shifted by a small amount $\Delta\alpha_k$ from its continuous value. The frequency-response deviation caused by the quantization of α_k is represented by $\Delta H_{N^0,k}^0(e^{j\omega})$, where $\Delta H_{N^0,k}^0(e^{j\omega})$ is given by

$$\Delta H_{N^0,k}^0(e^{j\omega}) = \frac{\partial H_{N^0,k}^0(e^{j\omega})}{\partial \alpha_k} \times \Delta\alpha_k = P_N^k(e^{j\omega}) \times \Delta\alpha_k \quad (15)$$

provided that $\Delta\alpha_k$ is small.

The quantization of a coefficient with a higher sensitivity and a larger value of $|\Delta\alpha_k|$ causes a larger deviation to the frequency response. Thus, the product of coefficient sensitivity and quantization step size is an important measure on the frequency-response deviation caused by quantizing a coefficient. A frequency-response deterioration measure, S_N^k , given by

$$S_N^k(z) = P_N^k(z) \times G_k \quad (16)$$

is defined for the purpose of estimating the effect on the frequency response of the filter as a result of quantizing α_k . In (16), G_k is the grid spacing defined as the distance between the upper discrete level and the lower discrete level of a continuous coefficient α_k . The grid spacing is an indication of the quantization step size if the coefficient is actually quantized. A coefficient value changes from iteration to iteration as the optimization process proceeds. Thus, the grid spacing for a coefficient value changes from iteration to iteration in the case of a nonuniformly distributed coefficient space such as the power-of-two coefficient space.

The coefficient with the largest frequency-response deterioration measure is selected to be quantized first. After the quantization of each coefficient, all the other coefficients are then reoptimized to partially compensate for the frequency-response deterioration due to the quantization of the selected coefficient. Since the coefficient with the largest frequency-response deterioration measure will cause the largest frequency-response deviation, selecting it to be the first coefficient to be quantized will have the advantage that there are many coefficient values which can be reoptimized to compensate for the frequency-response deterioration caused by its quantization. On the contrary, if the coefficient with the largest frequency-response deterioration measure is quantized after all other coefficients are quantized, its effect cannot be compensated for by adjusting other coefficient values.

If the discrete coefficient grid is evenly distributed such as in the case of the integer grid where all coefficient values must be an integer after multiplying by a constant, the grid spacings are equal for all coefficients. In this case, $P_N^k(z)$ may be used instead of $S_N^k(z)$ for selecting a coefficient for quantization since the value of G_k for all k are equal.

V. RECURSIVE-IN-WIDTH DEPTH-FIRST TREE SEARCH

Our technique starts with the design of the optimum continuous coefficient value minimax PR orthogonal filter bank using the weighted least squares algorithm described in Section II. After the continuous optimum solution is obtained, a coefficient α_k is selected for quantization. The method of selecting α_k has been discussed in Section IV. A straightforward method for assigning a discrete value to α_k is to round it to its nearest discrete value. As the optimum value for α_k may be at a considerable distance from the infinite precision solution, it is necessary to assign several discrete values (in the vicinity of its continuous optimum value) to α_k . For each discrete value assigned to α_k , the remaining unquantized coefficients are reoptimized to partially compensate for the frequency-response deterioration due to the quantization of α_k . A tree search algorithm is then produced.

Fig. 2. An example of a branch and bound tree.

Before embarking on describing our novel tree search algorithm, we shall briefly describe two existing tree search algorithms. Our new algorithm is developed based on these two algorithms.

- 1) In the branch and bound depth first search algorithm [20], after the continuous optimum solution is obtained, a coefficient α_k is selected for branching. Suppose that α_2 is selected and that the integer space is the desired discrete coefficient space. Suppose also that the continuous optimum value of α_2 is 3.4. Two subproblem P_1 and P_2 are created by imposing the bounds $\alpha_2 \leq 3$ and $\alpha_2 \geq 4$, respectively. See Fig. 2, problem P_2 is stored and P_1 is solved. Another coefficient (say α_1) is then selected for partitioning into two subproblems P_3 and P_4 by imposing bounds on the selected coefficient (say $\alpha_1 \leq 0$ and $\alpha_1 \geq 1$, respectively). P_4 is stored and P_3 is solved. P_3 is then further partitioned into P_5 and P_6 . In a similar way, P_6 is stored and P_5 is solved. Suppose that P_5 yields a discrete solution. P_5 is then fathomed and P_6 is solved. In Fig. 2, a line underneath a node indicates that no further exploration from that node can be profitable. Such a node is said to be fathomed. If P_6 yields a discrete solution, P_6 is fathomed and the algorithm backtracks to P_3 and switches to solve P_4 . The branching, backtracking and searching process continues until all the nodes are fathomed. The algorithm searches the tree in a depth-first manner and earns its name “depth-first” search.
- 2) A tree search technique which can yield a good suboptimal solution quickly was described in [13] for the design of filters subject to discrete coefficient constraint. In that technique, after obtaining the continuous coefficient value design, P_0 , a coefficient α_k is selected and L discrete values are assigned to α_k . See Fig. 3. This produces L optimization problems—an optimization problem for each discrete value of α_k . Fig. 3 shows the case where L is 3. After these L problems are solved, another coefficient is selected for quantization. Thus, each of the L problems produces L further optimization problems. Hence, there

Fig. 3. An example of the hybrid of breadth-first and depth-first tree structure for the case where $L = 3$.
 Fig. 5. Tree search illustration for the case where $N = 4$ and $L = 3$.

step of the algorithm. This will be a problem if there is insufficient computing resources to complete the execution of the algorithm. The optimization algorithm for optimizing linear phase FIR filters to meet a given frequency-response requirement does not require large computing resources. Thus, choosing a fairly large value of L is not a problem in the case of [13]. In the design of lattice perfect reconstruction filter bank, the optimization algorithm requires long computer time. The type of tree search algorithm used in [13] is obviously not suitable. A suitable tree search technique should be one which will produce a good suboptimal solution within a reasonable time and will produce improved solutions as more time elapsed; that implies some form of depth-first search strategy. Taking the particular nature of the problem into consideration, in this paper, a recursive-in-width depth-first tree search algorithm is developed.

Our new algorithm is developed from that described in [13]. It starts with $L = 1$. At each node of the tree, the coefficient selected for quantization is the one with the largest performance deterioration measure discussed in Section IV. When the predefined maximum tree width is larger than 1, the above solution with $L = 1$ becomes the first suboptimal discrete solution.

Since the weighting function $B(\omega)$ in (10) is updated after every iteration as the optimization process proceeds, the objective function value f is not a good indicator of the optimality condition. In our algorithm, the minimum weighted attenuation in the stopband of the analysis filter is used as a criterion for evaluating the quality of a solution.

After obtaining the first suboptimal discrete solution (i.e., node P_A in Fig. 4), the width of the tree is incremented by one. The search is backtracked to P_B and branched into P_C by

fixing the last continuous coefficient value to its next nearest discrete value (It has been fixed at its nearest discrete value at P_A). Another discrete solution is obtained. If this solution is better than the previous one, it replaces the previous one as the best known discrete solution; otherwise, it is discarded. The search then backtracks to P_D and switches to search along P_E and P_F as shown in Fig. 4. The process of backtracking, switching, and searching forward is repeated until all nodes which has the possibility of yielding a better discrete solution than the best currently known discrete solution are searched. The search along a given path is terminated whenever the minimum weighted stopband attenuation is smaller than that of the best know discrete solution. For $L = 2$, the tree looks very much like a branch and bound depth first search tree with the exception that, in our case, the branch length to a discrete solution is equal to the number of the discrete variables whereas, in the case of the branch and bound depth first search, the branch lengths to a discrete solution are usually larger than the number of discrete variables.

The width of the tree is increased recursively by one at a time until a predefined tree width, L , is reached. An example of the tree for $N = 4$ and $L = 3$ is shown in Fig. 5. Our new tree search strategy has the following advantages. First, it quickly yields a suboptimal discrete solution; second, it covers a large search space if the necessary computing resources is available.

VI. DESIGN EXAMPLE

We shall choose the design of a 27th-order (14 coefficients) filter bank as an example to illustrate our technique. The low-pass filter's stopband edge is at 0.64π and its stopband frequency response is equiripple.

Fig. 6. Frequency-response plots for the analysis low-pass filters. Each coefficient of the discrete coefficient design has a precision of 8 bits after the binary point.

Fig. 7. Frequency-response plots for the analysis low-pass filters. Each coefficient of the discrete coefficient design is represented using two signed power-of-two terms.

The frequency responses of the low-pass filters for the (1) continuous coefficient optimum design, (2) discrete space design obtained by our algorithm and (3) discrete space design obtained by simple rounding of coefficient values are shown in Figs. 6 and 7. In Fig. 6, each coefficient value is allocated with 8 bits after the binary point, i.e., it is an integer if multiplied by 2^8 . The minimum stopband attenuations for the three designs in Fig. 6 are 60.17, 53.80, and 45.99 dB, respectively.

In Fig. 7, each coefficient value is represented as a sum of 2 signed power-of-two (SPT) terms; the smallest power-of-two term is 2^{-10} . The minimum stopband attenuations for the three designs in Fig. 7 are 60.17, 45.45, and 25.38 dB, respectively. The discrete coefficient values are listed in Table II.

From Figs. 6 and 7, it is obvious that the stopband attenuation of the discrete space design obtained by using our algorithm is significantly superior to those obtained by simple rounding of coefficient values.

The stopband attenuation obtained improves with increasing tree width associated with increasing computing cost. The stopband attenuation obtained for each tree width and its computing time are plotted in Fig. 8 for filters with fix point coefficient values and plotted in Fig. 9 for filters with SPT term coefficient

TABLE II
DISCRETE COEFFICIENT VALUES OF A 27-TH ORDER PR ORTHOGONAL FILTER BANK, $\omega_s = 0.64\pi$

k	finite wordlength coefficients	SPT coefficients
0	-4278×2^{-8}	$-2^{+2} - 2^{-2}$
1	882×2^{-8}	$2^{+0} + 2^{-2}$
2	-373×2^{-8}	$-2^{-1} - 2^{-7}$
3	216×2^{-8}	$-2^{-4} - 2^{-10}$
4	-143×2^{-8}	$2^{+0} - 2^{-3}$
5	100×2^{-8}	$-2^{+0} - 2^{-8}$
6	-71×2^{-8}	$2^{-6} + 2^{-10}$
7	50×2^{-8}	$2^{+0} - 2^{-6}$
8	-34×2^{-8}	$-2^{+0} - 2^{-7}$
9	22×2^{-8}	$2^{-2} + 2^{-5}$
10	-13×2^{-8}	$2^{-6} - 2^{-9}$
11	7×2^{-8}	$-2^{-4} + 2^{-7}$
12	-3×2^{-8}	$2^{-5} - 2^{-8}$
13	1×2^{-8}	$-2^{-7} + 2^{-9}$

Fig. 8. The stopband attenuation and computing cost versus the tree width for finite coefficient wordlength design.

Fig. 9. The stopband attenuation and computing cost versus the tree width for an example where each coefficient value is represented by a sum of 2 SPT terms.

values. It can be seen from Fig. 8 that the stopband attenuation improved as L increased from 1 to 2 but remained unchanged after that. Its computing time increases with the tree width, L . In Fig. 9, the stopband attenuation improved until $L = 7$ and its computing time is approximately proportional to L^2 .

Our technique does not necessarily result in the optimal solution. However, it does provide an efficient and reasonably good

Fig. 10. The minimum stopband attenuations of the low-pass filters—*a*: minimum stopband attenuation of continuous coefficient design; *b*: minimum stopband attenuation of SPT coefficient designs by using our algorithm; and *c*: minimum stopband attenuation of discrete coefficient designs by simple coefficient rounding technique.

solution to the problem. In order to show the relationship between the filter length and the performance of a filter and that between the filter length and the computer time required, a set of PR orthogonal filter banks are designed. The stopband edge of the filter banks' low-pass filters is 0.54π . The filter length, N , ranges from 4 to 40. Each coefficient value of the discrete coefficient design is represented as a sum of 2 SPT terms; the smallest SPT term is 2^{-12} . The minimum stopband attenuations of the low-pass filters for: 1) the continuous coefficient optimum design; 2) the discrete coefficient design obtained by simple rounding of coefficient values; and 3) the discrete coefficient design by using our algorithm are shown in Fig. 10. It can be seen from Fig. 10 that, for the same filter specifications, the minimum stopband attenuation of the continuous designs when expressed in dB is proportional to the filter length N . For discrete coefficient designs, the minimum stopband attenuation is very close to the continuous coefficient design for small values of N . When N exceeds a certain limit, the peak stopband gain of the discrete coefficient design deviates away from the continuous coefficient design and finally remains fairly constant despite increasing filter length. As can be seen from Fig. 10, the peak stopband gain of the discrete coefficient filter designed using our method is significantly smaller than that of the rounded coefficient design. The computing time required by our algorithm for tree width equals to 2 for various filter length is plotted in Fig. 11. As can be seen from Fig. 11, the computing time increases exponentially with respect to filter length N .

There are several schools of thoughts for the distribution of the SPT terms to the coefficients. Some authors design filters where each coefficient is allocated with the same number of SPT terms. Some authors design filters where each coefficient is allocated with different number of SPT terms but the total number of SPT terms for the entire filter is fixed. It has been demonstrated in [14] that filters with different number of SPT terms allocated to each coefficient have significantly better frequency-response performance than filters with the same number of total SPT terms but with all coefficients allocated with the same number of SPT terms. We feel that both schools of thoughts have their own respective merits; it depends on the hardware platform used to

Fig. 11. The computing time of a set of discrete coefficient designs by using our algorithm when the tree width is equal to 2.

TABLE III
COEFFICIENT VALUES OF A DESIGN WHOSE SPECIFICATION IS SPECIFIED IN APPENDIX B

k	Continuous coefficients	SPT coefficient. Each coefficient has on average two SPT terms.	SPT coefficient. Each coefficient has two SPT terms
0	-2.6619195	$-2^{+2}+2^{+0}+2^{-2}+2^{-4}$	$-2^{+2}+2^{+0}$
1	0.8784588	$2^{+0}-2^{-3}-2^{-6}$	$2^{+0}+2^{-8}$
2	-0.5167097	$-2^{-1}-2^{-8}$	$-2^{-1}-2^{-4}$
3	0.3580536	$2^{-1}-2^{-3}-2^{-5}$	$2^{-1}-2^{-3}$
4	-0.2670765	$-2^{-2}-2^{-7}$	$-2^{-2}-2^{-5}$
5	0.2072396	$2^{-2}-2^{-4}+2^{-7}$	$2^{-2}-2^{-5}$
6	-0.1640125	$-2^{-3}-2^{-5}$	$-2^{-3}-2^{-5}$
7	0.1310766	2^{-3}	$2^{-3}+2^{-7}$
8	-0.1049166	$-2^{-3}+2^{-5}$	$-2^{-3}+2^{-6}$
9	0.0835565	$2^{-4}+2^{-6}$	$2^{-4}+2^{-6}$
10	-0.0659682	$-2^{-4}+2^{-10}$	$-2^{-4}-2^{-8}$
11	0.0510935	$2^{-4}-2^{-6}$	$2^{-4}-2^{-6}$
12	-0.0388140	-2^{-5}	$-2^{-5}-2^{-8}$
13	0.0286118	2^{-5}	$2^{-5}-2^{-10}$
14	-0.0201751	-2^{-6}	$-2^{-6}-2^{-9}$
15	0.0228699	2^{-6}	$2^{-6}+2^{-8}$

implement the filters. Our discrete space optimization technique is suitable for optimizing both these two cases. If it is desired to optimize filters with different number of SPT terms for each coefficient, the SPT terms must be preallocated by using some other SPT term allocation techniques such as the one reported in [14].

References [9] and [10] reported two different techniques for optimizing lattice perfect reconstruction filter banks with different number of SPT terms allocated to each coefficient. For the purpose of comparison, the specifications corresponding to the example tabulated in [9, Table I] (reproduced in Appendix B) was used. The infinite precision optimum solution has a peak stopband gain of -30.7 dB and the coefficient values are tabulated in column two of Table III. Our technique produces a design with a peak stopband gain of -29.9 dB. The coefficient values for our design were tabulated in column three of Table III. [9] and [10] reported solutions with 29.0 dB attenuation in the stopband. The superiority of our technique is evident. If a further constrain that all the coefficients must have the same number of SPT terms is imposed, our technique produced a design with peak stopband ripple of -29.1 dB. The coefficients are tabulated in column four of Table III.

VII. STATISTICAL EFFECT OF COEFFICIENT QUANTIZATION

A statistical analysis on the effect of coefficient quantization on the frequency response is presented in this section. The rounding of an infinite precision coefficient value to its nearest discrete value may be modeled as adding an error term to the coefficient value. Suppose that the error term associated with the rounding of α_k is $\Delta\alpha_k$. Let the frequency-response error of $H_{N,0}(z)$ due to the rounding of all of its coefficients be denoted by $E_{N,0}(e^{j\omega})$. For small $|\Delta\alpha_k|$, $E_{N,0}(e^{j\omega})$ may be approximated as

$$E_{N,0}(e^{j\omega}) = \sum_{k=0}^{N-1} P_{N,k}(e^{j\omega}) \times \Delta\alpha_k. \quad (17)$$

It is reasonable to assume that $\Delta\alpha_k$, $k = 0, 1, \dots, N-1$, are mutually independent. If each coefficient value is represented by a finite wordlength value with J bits after the binary point, then $\Delta\alpha_k$ is uniformly distributed between $-Q/2$ and $Q/2$, and thus has zero mean and variance $Q^2/12$ where Q is the coefficient quantization step size and is equal to 2^{-J} . The statistical model in [18] and [19] is used in our analysis. From (14), (17), and the statistical property of $\Delta\alpha_k$, it can be shown that, for coefficient rounding, $E_{N,0}(e^{j\omega})$ has zero mean and variance $\sigma_{E_{N,0}}^2$ given by

$$\begin{aligned} \sigma_{E_{N,0}}^2 &= \overline{E_{N,0}(e^{j\omega})^2} = \sum_{k=0}^{N-1} P_{N,k}^2(e^{j\omega}) \times \overline{\Delta\alpha_k^2} \\ &< \sum_{k=0}^{N-1} \overline{\Delta\alpha_k^2} = \frac{Q^2}{12} N. \end{aligned} \quad (18)$$

Define

$$\sigma_E = \sqrt{\frac{Q^2 N}{12}} = \frac{Q}{2} \sqrt{\frac{N}{3}}. \quad (19)$$

It can be seen from (17) that $E_{N,0}(e^{j\omega})$ consists of a summation of N terms. With the operation of the central limit theorem, the distribution of $E_{N,0}(e^{j\omega})$ for any given ω approaches Gaussian when N is large. Thus, for large N , $|E_{N,0}(e^{j\omega})|$ is less than $2\sigma_E$ with 98% chance. Hence,

$$|E_{N,0}(e^{j\omega})| \lesssim 2\sigma_E, \quad (20)$$

where $\lesssim$ denotes “is 98% chance less than or equal to.”

Let D^* and D denote the minimum stopband attenuation in dB of the rounded coefficient filter and the infinite precision coefficient filter, respectively. It can be shown that

$$D - D^* \lesssim 20 \log \left(1 + 10^{D/20} 2^{-J} \sqrt{\frac{N}{3}} \right) \quad (21)$$

where $\log$ denotes the base 10 logarithm operator.

Fig. 12 shows a $(D - D^*)$ versus N plot for several examples of lattice PR orthogonal filter banks with $J = 7$, $D = 30$ dB and N ranging from 4 to 40. In Fig. 12, the filters obtained using simple coefficient rounding is denoted using “o” and those obtained using our new technique is denoted using “+”. The function $20 \log \left(1 + 10^{D/20} 2^{-J} \sqrt{N/3} \right)$ is also plotted (solid curve) in Fig. 12. It can be seen from Fig. 12

Fig. 12. Comparison between simulation results and the predicted statistical bound.

that none of the examples have $(D - D^*)$ value larger than $20 \log \left(1 + 10^{D/20} 2^{-J} \sqrt{N/3} \right)$ and that the $(D - D^*)$ values for those designed using our technique is significantly smaller than the $(D - D^*)$ values for those obtained using simple coefficient rounding.

VIII. CONCLUSION

A recursive-in-width tree search algorithm is introduced for optimizing perfect reconstruction lattice filter bank. In this new tree search algorithm, a frequency-response deterioration measure is introduced for the selection of a coefficient for branching at each node. The frequency responses of the filters obtained using our algorithm are significantly superior to those obtained by simple rounding of the coefficient values and those designed using the techniques reported in [9] and [10]. The computing cost for the design of high-order discrete coefficient PR orthogonal filter bank is generally very high. However, good suboptimal results may be obtained with affordable computing cost by searching the tree within a limited width range.

APPENDIX A PROOF FOR (13)

Proof

We shall prove this by mathematical induction. From (12), for $n = k + 1$, $P_{k+1}^k(z)$ and $Q_{k+1}^k(z)$ are given by

$$\begin{aligned} \begin{bmatrix} P_{k+1}^k(z) \\ Q_{k+1}^k(z) \end{bmatrix} &= \frac{\beta_{k+1}}{\beta_k} \frac{1}{1 + \alpha_k^2} A(\alpha_k) \begin{bmatrix} -z^{-2} H_k^1(z) \\ H_k^0(z) \end{bmatrix} \\ &= \frac{1}{\sqrt{1 + \alpha_k^2}} \cdot \frac{1}{1 + \alpha_k^2} \begin{bmatrix} 1 & -\alpha_k \\ \alpha_k & 1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} -z^{-2} H_k^1(z) \\ H_k^0(z) \end{bmatrix} \\ &= \frac{1}{\sqrt{(1 + \alpha_k)^3}} \begin{bmatrix} -z^{-2} H_k^1(z) - \alpha_k H_k^0(z) \\ -z^{-2} \alpha_k H_k^1(z) + H_k^0(z) \end{bmatrix}. \end{aligned} \quad (22)$$

Hence

$$\begin{aligned}
 & P_n^k(z)P_n^k(z) + Q_n^k(z)Q_n^k(z) \\
 &= \frac{1}{(1 + \alpha_k^2)^3} \\
 & \times \left[(-z^{-2}H_k^1(z) - \alpha_k H_k^0(z)) \cdot (-z^2\widetilde{H_k^1(z)} - \alpha_k \widetilde{H_k^0(z)}) \right. \\
 & \quad \left. + (-z^{-2}\alpha_k H_k^1(z) + H_k^0(z)) (-z^2\alpha_k \widetilde{H_k^1(z)} + \widetilde{H_k^0(z)}) \right] \\
 &= \frac{1 + \alpha_k^2}{(1 + \alpha_k^2)^3} \left[H_k^0(z)\widetilde{H_k^0(z)} + H_k^1(z)\widetilde{H_k^1(z)} \right]. \quad (23)
 \end{aligned}$$

Since H_k^0 and H_k^1 satisfy the power-complement image condition [3]

$$H_k^0(z)\widetilde{H_k^0(z)} + H_k^1(z)\widetilde{H_k^1(z)} = 1. \quad (24)$$

Hence, (13) holds for $n = k + 1$.

Suppose that (13) is true for $n = m$. Thus

$$\begin{aligned}
 [P_m^k(z)Q_m^k(z)] &= \frac{\beta_m}{\beta_k} \frac{1}{1 + \alpha_k^2} A(\alpha_{m-1})\Lambda \cdots \Lambda A(\alpha_{k+1}) \\
 & \quad \times \Lambda A(\alpha_k) \begin{bmatrix} -z^{-2}H_k^1(z) \\ H_k^0(z) \end{bmatrix} \quad (25)
 \end{aligned}$$

and

$$P_m^k(z)P_m^k(z) + Q_m^k(z)Q_m^k(z) = \frac{1}{(1 + \alpha_k^2)^2} < 1, \quad \text{for } \alpha_k \neq 0. \quad (26)$$

Then, for $n = m + 1$

$$\begin{aligned}
 \begin{bmatrix} P_{m+1}^k(z) \\ Q_{m+1}^k(z) \end{bmatrix} &= \frac{\beta_{m+1}}{\beta_k(1 + \alpha_k^2)} A(\alpha_m)\Lambda \cdots \Lambda A(\alpha_k) \begin{bmatrix} -z^{-2}H_k^1(z) \\ H_k^0(z) \end{bmatrix} \\
 &= \frac{1}{\sqrt{1 + \alpha_m^2}} A(\alpha_m)\Lambda \begin{bmatrix} P_m^k(z) \\ Q_m^k(z) \end{bmatrix} \\
 &= \frac{1}{\sqrt{1 + \alpha_m^2}} \begin{bmatrix} P_m^k(z) - \alpha_m z^{-2}Q_m^k(z) \\ \alpha_m P_m^k(z) + z^{-2}Q_m^k(z) \end{bmatrix}. \quad (27)
 \end{aligned}$$

We obtain

$$\begin{aligned}
 & P_{m+1}^k(z)P_{m+1}^k(z) + Q_{m+1}^k(z)Q_{m+1}^k(z) \\
 &= \frac{1}{1 + \alpha_k^2} \\
 & \times \left[(P_m^k(z) - \alpha_m z^{-2}Q_m^k(z)) \cdot (P_m^k(z) - \alpha_m z^2\widetilde{Q_m^k(z)}) \right. \\
 & \quad \left. + (\alpha_m P_m^k(z) + z^{-2}Q_m^k(z)) (\alpha_m \widetilde{P_m^k(z)} + z^{-2}\widetilde{Q_m^k(z)}) \right] \\
 &= P_m^k(z)P_m^k(z) + Q_m^k(z)Q_m^k(z). \quad (28)
 \end{aligned}$$

Hence, if (13) is true for $n = m$, it is also true for $n = m + 1$. Since (13) is true for $n = k + 1$, it is true for all integer n , $k + 1 \leq n \leq N$. ■

APPENDIX B

SPECIFICATIONS CORRESPONDING TO THE EXAMPLE TABULATED IN [9, Table I]

A 31th-order filter bank with stopband edge at 0.56π , the stopband frequency response is equiripple. A total number of 64 SPT terms are allocated to all the coefficients.

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