On primes in quadratic progressions

Baier, Stephan; Zhao, Liangyi

2008

Baier, S., & Zhao, L. (2008). On primes in quadratic progressions. International Journal of Number Theory.

<https://hdl.handle.net/10356/91766>

<https://doi.org/10.1142/S1793042109002523>

ON PRIMES IN QUADRATIC PROGRESSIONS

STEPHAN BAIER AND LIANGYI ZHAO

ABSTRACT. We verify the Hardy-Littlewood conjecture on primes in quadratic progressions on average. The results in the present paper significantly improve those of a previous paper of the authors [3].

Mathematics Subject Classification (2000): 11L07, 11L20, 11L40, 11N13, 11N32, 11N37

Keywords: primes in quadratic progressions, primes represented by polynomials

1. INTRODUCTION

It was due to Dirichlet that any linear polynomial represents infinitely many primes provided the coefficients are co-prime. Though long been conjectured, analogous statements are not known for any polynomial of higher degree. G. H. Hardy and J. E. Littlewood [14] conjectured that

$$(1.1) \quad \sum_{n \leq x} \Lambda(n^2 + k) \sim \mathfrak{S}(k)x,$$

where Λ is the von Mangoldt function and $\mathfrak{S}(k)$ is a constant that depends only on k , as defined in (2.2). Their conjecture is in an equivalent but different form as in (1.1). In fact, their conjecture is more general than (1.1) as it concerns the representation of primes by any quadratic polynomial that may conceivably represent infinitely many primes.

It is most note-worthy that an upper bound of the order of magnitude predicted by (1.1) was proved by A. Granville and R. A. Mollin in [13] unconditionally uniform in the family of quadratic polynomials, and uniform in x under the Riemann hypothesis for a certain Dirichlet L -function. Furthermore, it is shown unconditionally in [13] that for large R and N with $R^\varepsilon < N < \sqrt{R}$,

$$\#\{n \leq N : n^2 + n + A \in \mathbb{P}\} \asymp L \left(1, \left(\frac{1-4A}{\cdot}\right)\right)^{-1} \frac{N}{\log N}$$

holds for at least a positive proportion of integers A in the range $R < A < 2R$. They also proved in [13] that an asymptotic formula for the number of prime values of $f(x)$, with f belonging to certain families of quadratic polynomials, holds for x in some ranges under the assumption of the existence of a Siegel zero for the Dirichlet L -function.

One may also find several results on approximations to the problem of detecting primes of the form $n^2 + 1$ in the literature. For example, Ankeny [1] and Kubilius [22] showed independently that under the Riemann hypothesis for Hecke L -functions there exist infinitely many primes of the form $p = m^2 + n^2$ with $n < c \log p$, where c is some positive constant. Using sieve methods, Harman and Lewis [15] showed unconditionally that there exist infinitely many primes of the above form with $n \leq p^{0.119}$.

It was established by C. Hooley [18] that if D is not a perfect square then the greatest prime factor of $n^2 - D$ exceeds n^θ infinitely often if $\theta < \theta_0 = 1.1001 \dots$. J.-M. Deshouillers and H. Iwaniec [9] improved this to the effect that $n^2 + 1$ has infinitely often a prime factor greater than $n^{\theta_0 - \varepsilon}$, where $\theta_0 = 1.202 \dots$ satisfies $2 - \theta_0 - 2 \log(2 - \theta_0) = \frac{5}{4}$. The last-mentioned result can also be generalized to $n^2 - D$ by Hooley's arguments.

Moreover, H. Iwaniec [21] showed that there are infinitely many integers n such that $n^2 + 1$ is the product of at most two primes. The result improves a previous one of P. Kuhn [23] that $n^2 + 1$ is the product of at most three primes for infinitely many integers n and can be extended to any irreducible polynomial $an^2 + bn + c$ with $a > 0$ and c odd.

The results mentioned in the last two paragraphs were based on sieve methods. It is also noteworthy that J. B. Friedlander and H. Iwaniec [11], using results on half-dimensional sieve of H. Iwaniec [20], obtained lower bounds for the number of integers with no small prime divisors represented by a quadratic polynomial.

It is easy to see that $n^2 + 1$ represents an infinitude of primes if and only if there are infinitely many primes p such that the fractional part of $\sqrt{p}$ is very small, namely $< 1/\sqrt{p}$. Balog, Harman and the first-named author [4, 5, 16] dealt with the following related question. Given a positive real λ and a real number θ , for what positive numbers τ can one prove that there exist infinitely many primes p for which the inequality

$$\{p^\lambda - \theta\} < p^{-\tau}$$

is satisfied? This problem in turn is related to estimating the number of primes of the form $[n^c]$, where $c > 1$ is fixed and n runs over the positive integers, the so-called Pyateckii-Šapiro primes [29].

Another approximation to the $n^2 + 1$ problem was given by the authors [2]. It was proved unconditionally that for all $\varepsilon > 0$, there exist infinitely many primes of the form $p = am^2 + 1$ such that $a \leq p^{5/9+\varepsilon}$, and noted that the last majorant can be taken to be $a \leq p^{1/2+\varepsilon}$ under the assumption of the generalized Riemann hypothesis (GRH) for Dirichlet L -functions or a generalization of a conjecture of the second-named author in [31] on large sieve for square moduli. It was also noted in [2] that under the Elliott-Halberstam [10] conjecture for square moduli, one can show an infinitude of primes $p = am^2 + 1$ with $a \leq p^\varepsilon$.

It is noteworthy that certain cases of the asymptotics (1.1) would follow from the part of another unsolved conjecture due to S. Lang and H. Trotter [24] regarding elliptic curves. See for example [24] for the details. Conjectures similar to (1.1) also exist for polynomials of higher degree. Hypothesis H of A. Schinzel and W. Sierpiński [30] gives that if f is an irreducible polynomial with integer coefficients that is not congruent to zero modulo any prime, then $f(n)$ is prime for infinitely many integers n . P. T. Bateman and R. A. Horn [6] gave the more explicit version, with asymptotic formula, of the last-mentioned conjecture.

We use the following standard notations and conventions in number theory throughout paper.

$f = O(g)$ means $|f| \leq cg$ for some unspecified constant $c > 0$ which may not be the same at each occurrence. $f \ll g$ means $f = O(g)$.

Following the general convention, we use ε to denote a small positive constant which may not be the same at each occurrence.

2. STATEMENTS OF THE RESULTS

The asymptotic formula in (1.1) is studied on average by the authors in [3] and it is established that (1.1) holds true for almost all $k \leq K$ with $x^2(\log x)^{-A} \leq K \leq x^2$ for any $A > 0$ and noted under the assumption of GRH for Dirichlet L -functions that the afore-mentioned range for K may be taken to be the wider range of $x^{2-\delta} \leq K \leq x^2$ for some $\delta > 0$. In this present paper, we aim to improve the theorem in [3] and prove that (1.1) holds for almost all natural numbers $k \leq K$ if $x^{1+\varepsilon} \leq K \leq x^2/2$. More precisely, we have the following in this paper.

Theorem 1. *Suppose that $z \geq 3$. Given $B > 0$, we have, for $z^{1/2+\varepsilon} \leq K \leq z/2$,*

$$(2.1) \quad \sum_{1 \leq k \leq K} \left| \sum_{z < n^2 + k \leq 2z} \Lambda(n^2 + k) - \mathfrak{S}(k) \sum_{z < n^2 + k \leq 2z} 1 \right|^2 \ll \frac{Kz}{(\log z)^B},$$

where

$$(2.2) \quad \mathfrak{S}(k) = \prod_{p>2} \left(1 - \frac{\left(\frac{-k}{p}\right)}{p-1} \right)$$

with $\left(\frac{-k}{p}\right)$ being the Legendre symbol.

From Theorem 1, we deduce the following corollary.

Corollary. *Given $A, B > 0$ and $\mathfrak{S}(k)$ as defined in the theorem, we have, for $z^{1/2+\varepsilon} \leq K \leq z/2$, that*

$$(2.3) \quad \sum_{z < n^2+k \leq 2z} \Lambda(n^2+k) = \mathfrak{S}(k) \sum_{z < n^2+k \leq 2z} 1 + O\left(\frac{\sqrt{z}}{(\log z)^B}\right)$$

holds for all natural numbers k not exceeding K with at most $O(K(\log z)^{-A})$ exceptions.

We further note here that in [3] that k is set to run over only the square-free numbers. This unfortunate restriction is also removed in the present paper. Moreover, it can be easily shown, as done in section 1 of [3] that $\mathfrak{S}(k)$ converges and

$$\mathfrak{S}(k) \gg \frac{1}{\log k} \gg \frac{1}{\log K} \gg \frac{1}{\log z}.$$

The above inequality shows that the main terms in (2.1) and (2.3) are indeed dominating for the k 's under consideration if $B > 1$ and that we truly have an “almost all” result.

Actually, we shall prove the following sharpened version of Theorem 1 for short segments of quadratic progressions on average.

Theorem 2. *Suppose that $z \geq 3$, $z^{2/3+\varepsilon} \leq \Delta \leq z^{1-\varepsilon}$ and $z^{1/2+\varepsilon} \leq K \leq z/2$. Then, given $B > 0$, we have*

$$(2.4) \quad \int_z^{2z} \sum_{1 \leq k \leq K} \left| \sum_{t < n^2+k \leq t+\Delta} \Lambda(n^2+k) - \mathfrak{S}(k) \sum_{t < n^2+k \leq t+\Delta} 1 \right|^2 dt \ll \frac{\Delta^2 K}{(\log z)^B}.$$

We shall deduce Theorem 1 from Theorem 2 in section 11. Moreover, we note that under GRH, the Δ -range in Theorem 2 can be extended to $z^{1/2+\varepsilon} \leq \Delta \leq z^{1-\varepsilon}$. This is due to the fact that under GRH, Lemmas 4 and 5 hold for $\delta \geq z^\varepsilon$, and Lemma 6 for $\delta \geq t^{1/2+\varepsilon}$. It is note-worthy that for $\Delta = z^{1/2+\varepsilon}$ the segments of quadratic progressions under consideration are extremely short; that is, they contain only $O(z^\varepsilon)$ elements.

Theorem 2 can be interpreted as saying that the asymptotics

$$\sum_{t < n^2+k \leq t+\Delta} \Lambda(n^2+k) \sim \mathfrak{S}(k) \sum_{t < n^2+k \leq t+\Delta} 1$$

holds for almost all k and t in the indicated ranges.

Unlike [3], we do not use the circle method in the present paper. Here our approach is a variant of the dispersion method of J. V. Linnik [25], similar to that used by H. Mikawa in the study of the twin prime problem in [26]. Expanding the modulus square in (2.4), we will get in (4.2) three terms $U(t)$, $V(t)$ and $W(t)$ of which we must estimate.

The cross term $V(t)$ will involve both the von Mangoldt function and the singular series $\mathfrak{S}(k)$. The singular series is then split into two parts in (9.1). The first part is shown to be small using techniques similar to those used for the analogous terms in [3], which at the bottom invokes the large sieve for real characters of Heath-Brown [17]. The second part will give raise to a main term using a result on primes in arithmetic progressions in short intervals, Lemma 6.

$W(t)$, which will involve only the square of the singular series $\mathfrak{S}(k)$, is split into two parts once again in (10.2). One of the parts can be shown to be small using techniques from [3]. The other will again yield a main term using familiar estimates for character sums.

The treatment of $U(t)$, the sum which will involve only the von Mangoldt function, is the most complicated. $U(t)$ is again decomposed into two parts, one of which can easily be shown to be small. Transforming the other part of the sum, we arrive at certain congruence relations for the summands which are detected using character sums. The main term will, as usual, come from the principle characters, after some transformations. The contribution of the non-principle characters is once again split into two parts. The first is disposed with the classical large sieve and the second with second moment estimates for certain character sums twisted with the von Mangoldt function, Lemma 5. Further splittings are needed in the estimates of the last-mentioned two parts to remove the dependency of certain parameters on others.

When combined, we shall discover that the three main terms mentioned above cancel out, giving us the desired result. The restrictions on the sizes of K and Δ are needed at various places in sections 6 - 9. It would be highly desirable to have the results in which $K = o(\sqrt{z})$ since in that situation the quadratic progressions under consideration would be completely disjoint.

3. PRELIMINARIES

In this section, we enumerate the lemmas needed in the proofs of the theorems. First, we shall use the large sieve inequality for Dirichlet characters.

Lemma 1 (Large Sieve). *Let $\{a_n\}$ be a sequence of complex numbers. Suppose that $M \in \mathbb{Z}$, $N, Q \in \mathbb{N}$. Then we have*

$$\sum_{Q \leq q \leq 2Q} \frac{1}{\varphi(q)} \sum_{\chi \bmod q}^* \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 \ll \left(Q + \frac{N}{Q} \right) \sum_{n=M+1}^{M+N} |a_n|^2,$$

where $\sum^*$ henceforth denotes the sum over primitive characters to the specified modulus.

Proof. See for example [8], [12], [27] or [28] for the proof. $\square$

We shall also need the following version of the large sieve for single moduli q .

Lemma 2. *Let $\{a_n\}$ be a sequence of complex numbers. Suppose that $M \in \mathbb{Z}$, $N, q \in \mathbb{N}$. Then we have*

$$\sum_{\chi \bmod q}^* \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 \leq (q + N) \sum_{n=M+1}^{M+N} |a_n|^2.$$

Proof. See for example [8], [12], [27] or [28] for the proof. $\square$

We shall also use the well-known estimate of Polya-Vinogradov for character sums.

Lemma 3 (Polya-Vinogradov). *For any non-principal character $\chi \pmod{q}$ we have*

$$\left| \sum_{M < n \leq M+N} \chi(n) \right| \leq 6\sqrt{q} \log q.$$

Proof. This is quoted from [19] and is Theorem 12.5 there. $\square$

Furthermore, we shall use the following mean-square estimate for the von-Mangoldt function in short intervals.

Lemma 4. *Let $z \geq 3$, $z^{1/6+\varepsilon} \leq \delta \leq z$ and $0 < M \leq \delta$. Then, for any given $C > 0$, we have*

$$\int_z^{2z} \left| \sum_{t < n \leq t+M} \Lambda(n) - M \right|^2 dt \ll \frac{z\delta^2}{(\log z)^C}.$$

Proof. See Chapter 10 of [19] for the proof of this lemma. $\square$

We shall also need the following modified version of Lemma 4 for character sums with Λ -coefficients.

Lemma 5. *Let $A, C > 0$ be given. Suppose that $z \geq 3$, $z^{1/6+\varepsilon} \leq \delta \leq z$, $0 < M \leq \delta$ and $2 \leq q \leq (\log z)^A$. Then we have, for any non-principal Dirichlet character χ modulo q ,*

$$\int_z^{2z} \left| \sum_{t < n \leq t+M} \Lambda(n) \chi(n) \right|^2 dt \ll \frac{z\delta^2}{(\log z)^C}.$$

Proof. The proof goes along the same lines as Lemma 4. $\square$

Furthermore, we shall use the following generalization to short intervals of the Siegel-Walfisz theorem on primes in arithmetic progressions.

Lemma 6. *Let $A, C > 0$ be given. Suppose that $t \geq 3$, $t^{7/12+\varepsilon} \leq \delta \leq t$, $1 \leq l \leq (\log t)^A$ and $(a, l) = 1$. Then*

$$\sum_{\substack{t < n \leq t+\delta \\ n \equiv a \pmod{l}}} \Lambda(n) = \frac{\delta}{\varphi(l)} + O\left(\frac{\delta}{(\log t)^C}\right).$$

Proof. For $l = 1$, this is Theorem 10.5 of [19] with a better error term (saving of an arbitrary power of logarithm) which can be obtained from Vinogradov's widening of the classical zero-free region of the Riemann zeta-functions. The proof for moduli $l \leq (\log t)^A$ goes along the same lines by using zero density estimates and a similar zero-free region for Dirichlet L -functions. To obtain the desired zero-free region one again uses Vinogradov's method together with Siegel's bound for exceptional zeros. See [19] for the details. $\square$

In fact, we shall need Lemma 6 only for the range $t^{2/3+\varepsilon} \leq \delta \leq t$. We shall also use the following lemma on the average of $q/\varphi(4q)$.

Lemma 7. *For $x \geq 1$ we have*

$$\sum_{q \leq x} \frac{q}{\varphi(4q)} = \frac{x}{2} \prod_{p > 2} \left(1 + \frac{1}{p(p-1)}\right) + O(\log x).$$

Proof. The proof is similar as that of (5.36) in Lemma 5.4.2. in [7]. $\square$

Finally, we shall use the following lemma on the Legendre-symbol.

Lemma 8. *For any square-free number l , we have*

$$\sum_{\substack{a \pmod{l} \\ \gcd(a, l) = 1}} \sum_{m \pmod{l}} \left(\frac{m^2 - a}{l}\right) = \mu(l) \varphi(l).$$

Proof. By the virtue of multiplicativity, it suffices to prove the lemma for primes $l = p$. In this case, we have

$$\sum_{\substack{a \pmod{p} \\ \gcd(a, p) = 1}} \sum_{m \pmod{p}} \left(\frac{m^2 - a}{p}\right) = \sum_{m \pmod{p}} \sum_{a \pmod{p}} \left(\frac{m^2 - a}{p}\right) - \sum_{m \pmod{p}} \left(\frac{m^2}{p}\right) = -(p-1)$$

by the orthogonality relations for Dirichlet characters. This completes the proof. $\square$

4. PREPARATION OF THE TERMS

Throughout the sequel, we assume that $z \geq 3$, $z^{1/2+\varepsilon} \leq K \leq z/2$ and $z^{2/3+\varepsilon} \leq \Delta \leq z^{1-\varepsilon}$. We set

$$(4.1) \quad L := (\log z)^C,$$

where C is a large positive constant. The variables k, m, n denote natural numbers.

We first rewrite the integrand in (2.4). Expanding the square, we obtain

$$(4.2) \quad \sum_{1 \leq k \leq K} \left| \sum_{t < n^2 + k \leq t + \Delta} \Lambda(n^2 + k) - \mathfrak{S}(k) \sum_{t < n^2 + k \leq t + \Delta} 1 \right|^2 = U(t) - 2V(t) + W(t),$$

where

$$(4.3) \quad U(t) = \sum_{1 \leq k \leq K} \sum_{\substack{n_1, n_2 \\ t < n_1^2 + k, n_2^2 + k \leq t + \Delta}} \Lambda(n_1^2 + k) \Lambda(n_2^2 + k),$$

$$(4.4) \quad V(t) = \sum_{1 \leq k \leq K} \mathfrak{S}(k) \sum_{t < n_1^2 + k \leq t + \Delta} 1 \sum_{t < n_2^2 + k \leq t + \Delta} \Lambda(n_2^2 + k)$$

and

$$(4.5) \quad W(t) = \sum_{1 \leq k \leq K} \mathfrak{S}^2(k) \sum_{\substack{n_1, n_2 \\ t < n_1^2 + k, n_2^2 + k \leq t + \Delta}} 1.$$

As mentioned in the introduction, we shall develop asymptotic formulas for $U(t)$, $V(t)$ and $W(t)$, and the main terms will cancel out, giving the desired result.

5. DECOMPOSITION OF $U(t)$

We aim to derive an upper bound of correct order of magnitude for

$$\int_z^{2z} U(t) dt.$$

We note that the average order of $U(t)$ may be expected to be

$$\sim \frac{\Delta^2 K}{z}.$$

Therefore, on average, the quantity

$$E := \frac{\Delta^2 K}{z \log^B z}, \text{ with } B > 0,$$

should be small compared to $U(t)$. In the sequel, we will make frequent use of the quantity E to bound error terms.

We now decompose $U(t)$ into two parts

$$(5.1) \quad U(t) = \sum_{1 \leq k \leq K} \sum_{\substack{n_1, n_2 \\ t < n_1^2 + k, n_2^2 + k \leq t + \Delta \\ |n_1^2 - n_2^2| \leq \Delta/L}} \Lambda(n_1^2 + k) \Lambda(n_2^2 + k) + 2 \sum_{1 \leq k \leq K} \sum_{\substack{n_1, n_2 \\ t < n_1^2 + k, n_2^2 + k \leq t + \Delta \\ n_1^2 - n_2^2 > \Delta/L}} \Lambda(n_1^2 + k) \Lambda(n_2^2 + k),$$

where L is defined in (4.1). The first sum on the right-hand side of (5.1) is easily seen to be $O(E)$. Re-writing $m_i = n_i^2 + k$ with $i = 1$ and 2 , the second sum on the right-hand side of (5.1) is

$$\tilde{U}(t) = 2 \sum_{\substack{t < m_1, m_2 \leq t + \Delta \\ m_1 - m_2 > \Delta/L}} \Lambda(m_1) \Lambda(m_2) \sum_{\substack{n_1, n_2 \\ m_1 - m_2 = n_1^2 - n_2^2 \\ 1 \leq m_1 - n_1^2 \leq K}} 1.$$

It suffices to consider only the case when both m_1 and m_2 are odd at the cost of a small error of size $\ll E$. Now we set

$$q = (n_1 - n_2)/2 \quad \text{and} \quad r = (n_1 + n_2)/2.$$

In the case that m_1 and m_2 are both odd, $m_1 - m_2$ is even and hence the condition

$$m_1 - m_2 = n_1^2 - n_2^2$$

implies that n_1 and n_2 are of the same parity. Therefore q and r are integers in this case. Moreover, the condition $m_1 - m_2 = n_1^2 - n_2^2$ is equivalent to

$$m_1 - m_2 = 4qr.$$

Now $\tilde{U}(t)$ becomes

$$\tilde{U}(t) = 2 \sum_{\substack{t < m_1, m_2 \leq t + \Delta \\ m_1 - m_2 > \Delta/L}} \Lambda(m_1) \Lambda(m_2) \sum_{\substack{q, r \in \mathbb{N} \\ m_1 - m_2 = 4qr \\ 0 < m_1 - (q+r)^2 \leq K \\ |q| < |r|}} 1 + O(E).$$

We note that

$$\sqrt{z} \leq \sqrt{m_1 - K} + \sqrt{m_2 - K} \leq 2r = n_1 + n_2 \leq \sqrt{m_1} + \sqrt{m_2} \leq 4\sqrt{z}$$

if z is sufficiently large. Therefore, the variable q satisfies the condition

$$D_1 \leq q = \frac{m_1 - m_2}{4r} \leq D_2,$$

where

$$D_1 := \frac{\Delta}{8L\sqrt{z}}, \quad D_2 := \frac{\Delta}{2\sqrt{z}}.$$

Moreover, if $m_1 > m_2$, the condition

$$0 < m_1 - (q + r)^2 = m_1 - \left(q + \frac{m_1 - m_2}{4q}\right)^2 \leq K$$

holds if and only if

$$m_1 - 4q(\sqrt{m_1} - q) < m_2 \leq m_1 - 4q(\sqrt{m_1 - K} - q).$$

Now we set

$$\mathcal{I}(t, m, q) = \left(m - 4q(\sqrt{m} - q), m - 4q(\sqrt{m - K} - q)\right] \cap (t, t + \Delta].$$

Then $\tilde{U}(t)$ is majorized by

$$(5.2) \quad 2 \sum_{D_1 \leq q \leq D_2} \sum_{t < m_1 \leq t + \Delta} \Lambda(m_1) \sum_{\substack{m_2 \in \mathcal{I}(t, m_1, q) \\ m_2 \equiv m_1 \pmod{4q}}} \Lambda(m_2) + O(E).$$

Due to the presence of Λ , the contribution of m_1 and m_2 in (5.2) that are not prime to $4q$ is small, $O(E)$ with an absolute implied constant. For the m_1 and m_2 that are prime to $4q$, we use Dirichlet characters to detect the congruence relation in (5.2), and this part becomes

$$(5.3) \quad 2 \sum_{D_1 \leq q \leq D_2} \frac{1}{\varphi(4q)} \sum_{\chi \pmod{4q}} \sum_{t < m_1 \leq t + \Delta} \Lambda(m_1) \chi(m_1) \sum_{m_2 \in \mathcal{I}(t, m_1, q)} \Lambda(m_2) \bar{\chi}(m_2).$$

The main term in (5.3) comes from the principal characters. Up to a small error of size $\ll E$, this main term amounts to

$$(5.4) \quad M(t) = 2 \sum_{D_1 \leq q \leq D_2} \frac{1}{\varphi(4q)} \sum_{t < m_1 \leq t + \Delta} \Lambda(m_1) \sum_{m_2 \in \mathcal{I}(t, m_1, q)} \Lambda(m_2).$$

We will deal with $M(t)$ later in section 8. Up to a small error of size $\ll E$, the remaining part of (5.3) can be rewritten in the form

$$2 \sum_{d \leq 2D_2} \sum_{\substack{\max\{2, 4D_1/d\} \leq q_1 \leq 4D_2/d \\ 4|q_1 d}} \frac{1}{\varphi(q_1 d)} \sum_{\chi \pmod{q_1}}^* \sum_{t < m_1 \leq t + \Delta} \Lambda(m_1) \chi(m_1) \sum_{m_2 \in \mathcal{I}(t, m_1, q_1 d/4)} \Lambda(m_2) \bar{\chi}(m_2) =: 2F(t),$$

say. We write

$$F(t) = \sum_{d \leq D_1/L} \cdots + \sum_{D_1/L < d \leq 2D_2} \cdots = F_1(t) + F_2(t), \text{ say.}$$

We note that the expression $F_2(t)$ involves only small moduli $q_1 \ll L^2$, whereas the moduli q_1 contained in $F_1(t)$ satisfy the inequality $q_1 \geq 4L$.

6. ESTIMATION OF $F_1(t)$

In this section, we shall show that $F_1(t)$ is an error term, *i.e.* $F_1(t) \ll E$. To separate the sums over m_1 and m_2 contained in $F_1(t)$, we split the ranges of summation for q_1 and m_1 into certain subintervals and then approximate the range $\mathcal{I}(t, m_1, q_1 d/4)$ of summation for m_2 suitably. More in particular, we split the summation interval $4D_1/d \leq q_1 \leq 4D_2/d$ into $O(\log z)$ dyadic intervals $(Q, 2Q]$ and then split the summation interval $t < m_1 \leq t + \Delta$ into $O(\Delta L/T)$ subintervals $(s, s + M]$ of length $M \leq T/L$, where

$$T := \frac{QdK}{\sqrt{z}}.$$

We note that the inequality $T/L \leq \Delta$ is always satisfied since $Qd \leq \Delta/\sqrt{z}$ and $K \leq z$. Now, when $Q < q_1 \leq 2Q$ and $s < m_1 \leq s + M$, we replace $\mathcal{I}(t, m_1, q_1 d/4)$ with $\mathcal{I}(t, s, q_1 d/4)$ in the range of summation of m_2 . The error $R(t)$ caused by this change turns out to be small, *i.e.* it is $\ll E$. In the following, we indicate how the latter can be proved, but we skip the details.

The error term $R(t)$ in question is a sum over d , q_1 , the primitive characters χ modulo q_1 , m_1 , and m_2 . Here the inner-most sum over m_2 ranges over small intervals of length $\ll T/L$ since from $|m_1 - s| \leq T/L$, $z^{1/2+\varepsilon} \leq K \leq z/2$, $\Delta \leq z^{1-\varepsilon}$ and $Qd \ll \Delta/\sqrt{z}$ it follows that

$$|\{m_1 - 4q_1 d(\sqrt{m_1} - q_1 d)\} - \{s - 4q_1 d(\sqrt{s} - q_1 d)\}| \ll T/L$$

and

$$|\{m_1 - 4q_1 d(\sqrt{m_1 - K} - q_1 d)\} - \{s - 4q_1 d(\sqrt{s - K} - q_1 d)\}| \ll T/L.$$

We note that, in contrast, the length of the interval

$$\left(m_1 - 4q_1 d(\sqrt{m_1} - q_1 d), m_1 - 4q_1 d(\sqrt{m_1 - K} - q_1 d)\right]$$

is $\gg T$. Now we estimate the sums in $R(t)$ trivially. After a short computation, we arrive at the desired bound $R(t) \ll E$.

The remaining task in this section is to establish an estimate for

$$(6.1) \quad \sum_{Q < q_1 \leq 2Q} \frac{1}{\varphi(q_1 d)} \left| \sum_{\chi \bmod q_1}^* \sum_{s < m_1 \leq s+M} \Lambda(m_1) \chi(m_1) \sum_{m_2 \in \mathcal{I}(t, s, q_1 d/4)} \Lambda(m_2) \bar{\chi}(m_2) \right|.$$

If we can show that (6.1) satisfies the non-trivial bound

$$(6.2) \quad \ll \frac{T^2 \log^2 z}{\varphi(d) L^{3/2}},$$

then it can now be easily deduced that

$$F_1(t) \ll E,$$

as desired. Using Cauchy's inequality and the inequality

$$\frac{1}{\varphi(q_1 d)} \leq \frac{1}{\varphi(q_1)} \cdot \frac{1}{\varphi(d)},$$

(6.1) is bounded by

$$(6.3) \quad \ll \frac{(S_1 S_2)^{1/2}}{\varphi(d)},$$

where

$$S_1 = \sum_{Q < q_1 \leq 2Q} \frac{1}{\varphi(q_1)} \sum_{\chi \bmod q_1}^* \left| \sum_{s < m_1 \leq s+M} \Lambda(m_1) \chi(m_1) \right|^2$$

and

$$S_2 = \sum_{Q < q_1 \leq 2Q} \frac{1}{\varphi(q_1)} \sum_{\chi \bmod q_1}^* \left| \sum_{m_2 \in \mathcal{I}(t, s, q_1 d/4)} \Lambda(m_2) \bar{\chi}(m_2) \right|^2.$$

Using the large sieve inequality, Lemma 1, we obtain

$$(6.4) \quad S_1 \ll \left(Q + \frac{T}{QL} \right) \frac{T \log^2 z}{L}.$$

From Lemma 2, we deduce that

$$(6.5) \quad S_2 \ll (Q + T) T \log^2 z.$$

We note that

$$(6.6) \quad L \leq Q \leq \frac{T}{L^2}.$$

The first inequality in (6.6) comes from the fact that the moduli q_1 in the expression $F_1(t)$ satisfy the inequality $q_1 \geq L$. The second inequality in (6.6) follows from the definition of T and $K \geq z^{1/2+\varepsilon}$. From (6.3), (6.4), (6.5) and (6.6), we obtain that (6.1) satisfies the majorant in (6.2), as desired.

7. TREATMENT OF $F_2(t)$

Next, we turn to the term $F_2(t)$. We recall that the expression $F_2(t)$ involves only small moduli q_1 . More in particular, we have that $q_1 \ll L^2$. Fix any q_1 satisfying this inequality. Similarly as in the estimation of $F_1(t)$, we split the summation interval $t < m_1 \leq t + \Delta$ into $O(\Delta L/T)$ subintervals $(s, s + M]$ of length $M \leq T/L$, where now we set

$$T := \frac{q_1 d K}{\sqrt{z}}.$$

As before, we replace $\mathcal{I}(t, m_1, q_1 d/4)$ with $\mathcal{I}(t, s, q_1 d/4)$ in the range of summation of m_2 at the cost of a small error whose total contribution to $F_2(t)$ is $\ll E$. Moreover, we put

$$s = t + \sigma.$$

We aim to show that $F_2(t)$ is small on average, *i.e.*

$$(7.1) \quad \int_z^{2z} F_2(t) dt \ll zE = \frac{\Delta^2 K}{\log^B z}.$$

To establish (7.1), it now suffices to prove a non-trivial estimate of the form

$$(7.2) \quad \int_z^{2z} \left| \sum_{t+\sigma < m_1 \leq t+\sigma+M} \Lambda(m_1) \chi(m_1) \sum_{m_2 \in \mathcal{I}(t, t+\sigma, q_1 d/4)} \Lambda(m_2) \bar{\chi}(m_2) \right| dt \ll \frac{z T^2 \log z}{L^2}$$

for any fixed σ with $0 \leq \sigma \leq \Delta$ and any primitive character χ with conductor q_1 . Using Cauchy's inequality, the left-hand side of (7.2) is bounded by

$$(7.3) \quad \ll (I_1 I_2)^{1/2},$$

where

$$I_1 = \int_z^{2z} \left| \sum_{t+\sigma < m_1 \leq t+\sigma+M} \Lambda(m_1) \chi(m_1) \right|^2 dt$$

and

$$I_2 = \int_z^{2z} \left| \sum_{m_2 \in \mathcal{I}(t, t+\sigma, q_1 d/4)} \Lambda(m_2) \bar{\chi}(m_2) \right|^2 dt.$$

Taking into account that

$$\frac{T}{L} = \frac{q_1 d K}{\sqrt{z} L} \geq \frac{D_1 K}{\sqrt{z} L} \gg \frac{\Delta K}{z L^2} \gg z^{1/6+\varepsilon},$$

we deduce from Lemma 5 that

$$(7.4) \quad I_1 \ll \frac{zT^2}{L^4 \log^2 z}.$$

Now it already suffices to estimate I_2 trivially by

$$(7.5) \quad I_2 \ll zT^2 \log^2 z.$$

Combining (7.3), (7.4) and (7.5), we obtain (7.2).

8. CONTRIBUTION OF THE MAIN TERM $M(t)$

Now we want to derive an asymptotic estimate for

$$\int_z^{2z} M(t) dt,$$

the integral of the main term $M(t)$, defined in (5.4). In a similar way as we established (7.1) in the previous section, it can be shown that

$$(8.1) \quad \int_z^{2z} (M(t) - \tilde{M}(t)) dt \ll zE = \frac{\Delta^2 K}{\log^B z}$$

with

$$\tilde{M}(t) = 2 \sum_{D_1 \leq q \leq D_2} \frac{1}{\varphi(4q)} \sum_{t < m_1 \leq t + \Delta} \sum_{m_2 \in \mathcal{I}(t, m_1, q)} 1,$$

where here we use Lemma 4 instead of Lemma 5. It thus suffices to establish an asymptotic estimate for $\tilde{M}(t)$. We shall show that

$$(8.2) \quad \tilde{M}(t) = \frac{\Delta^2 K}{4t} \prod_{p>2} \left(1 + \frac{1}{p(p-1)}\right) + O(E).$$

First, we change the order of summation, thus obtaining

$$\tilde{M}(t) = 2 \sum_{t < m_1 \leq t + \Delta} \sum_{D_1 \leq q \leq D_2} \frac{1}{\varphi(4q)} \sum_{m_2 \in \mathcal{I}(t, m_1, q)} 1.$$

Now we make two replacements, each at the cost of an error of size $\ll E$. First we replace the summation interval for q by $1 \leq q \leq (m_1 - t)/(4\sqrt{m_1})$, and second we replace the summation interval $\mathcal{I}(t, m_1, q)$ for m_2 by

$$(m_1 - 4q(\sqrt{m_1} - q), m_1 - 4q(\sqrt{m_1} - q) + 2qK/\sqrt{m_1}).$$

Thus, we obtain

$$\begin{aligned} \tilde{M}(t) &= 2 \sum_{t < m_1 \leq t + \Delta} \sum_{q \leq (m_1 - t)/(4\sqrt{m_1})} \frac{1}{\varphi(4q)} \cdot \frac{2qK}{\sqrt{m_1}} + O(E) \\ &= 4K \sum_{t < m_1 \leq t + \Delta} \frac{1}{\sqrt{m_1}} \sum_{q \leq (m_1 - t)/(4\sqrt{m_1})} \frac{q}{\varphi(4q)} + O(E). \end{aligned}$$

Using Lemma 7, the above is

$$\begin{aligned} &= \frac{K}{2} \prod_{p>2} \left(1 + \frac{1}{p(p-1)}\right) \sum_{t < m_1 \leq t + \Delta} \frac{m_1 - t}{m_1} + O(E) \\ &= \frac{\Delta^2 K}{4t} \prod_{p>2} \left(1 + \frac{1}{p(p-1)}\right) + O(E), \end{aligned}$$

which completes the proof of (8.2).

9. CONTRIBUTION OF $V(t)$

We now consider $V(t)$. Expanding the Euler product that defines $\mathfrak{S}(k)$, and approximating the sum

$$\sum_{t < n_1^2 + k \leq t + \Delta} 1$$

by $\Delta/(2\sqrt{t})$, we obtain

$$V(t) = \frac{\Delta}{2\sqrt{t}} \sum_{1 \leq k \leq K} \sum_{\substack{l=1 \\ 2 \nmid l}}^{\infty} \frac{\mu(l)}{\varphi(l)} \left(\frac{-k}{l} \right) \sum_{t < n^2 + k \leq t + \Delta} \Lambda(n^2 + k) + O(E).$$

The right-hand side of the above can be re-written as

$$(9.1) \quad \begin{aligned} & \frac{\Delta}{2\sqrt{t}} \sum_{1 \leq k \leq K} \sum_{\substack{l > L \\ 2 \nmid l}} \frac{\mu(l)}{\varphi(l)} \left(\frac{-k}{l} \right) \sum_{t < n^2 + k \leq t + \Delta} \Lambda(n^2 + k) \\ & + \frac{\Delta}{2\sqrt{t}} \sum_{\substack{l \leq L \\ 2 \nmid l}} \frac{\mu(l)}{\varphi(l)} \sum_{1 \leq k \leq K} \sum_{t < n^2 + k \leq t + \Delta} \Lambda(n^2 + k) \left(\frac{-k}{l} \right) + O(E). \end{aligned}$$

The first term in (9.1) is, by Cauchy's inequality,

$$\begin{aligned} & \ll \frac{\Delta}{\sqrt{z}} \left(\sum_k \left| \sum_n \Lambda(n^2 + k) \right|^2 \right)^{1/2} \times \left(\sum_k \left| \sum_{\substack{l > L \\ 2 \nmid l}} \frac{\mu(l)}{\varphi(l)} \left(\frac{-k}{l} \right) \right|^2 \right)^{1/2} \\ & \ll \frac{\Delta}{\sqrt{z}} \cdot \frac{\Delta \sqrt{K} \log z}{\sqrt{z}} \cdot \frac{\sqrt{K}}{(\log z)^{B+1}} = E, \end{aligned}$$

where we have used the same techniques as in section 5 of [3], with the following observation

$$\sum_k \left| \sum_{\substack{l > L \\ 2 \nmid l}} \frac{\mu(l)}{\varphi(l)} \left(\frac{-k}{l} \right) \right|^2 = \sum_{s \leq \sqrt{K}} \sum_{\substack{k \leq K/s^2 \\ \mu(k) \neq 0}} \left| \sum_{\substack{l > L \\ 2 \nmid l}} \frac{\mu(l)}{\varphi(l)} \left(\frac{-k}{l} \right) \right|^2.$$

We now deal with the inner double sum in the second term of (9.1). It is

$$(9.2) \quad \begin{aligned} & = \sum_{t < b \leq t + \Delta} \Lambda(b) \sum_{\substack{1 \leq b - n^2 \leq t + \Delta \\ n \leq \sqrt{b}}} \left(\frac{n^2 - b}{l} \right) \\ & = \sum_{a \bmod l} \sum_{m \bmod l} \left(\frac{m^2 - a}{l} \right) \sum_{\substack{t < b \leq t + \Delta \\ b \equiv a \bmod l}} \Lambda(b) \sum_{\substack{n \equiv m \bmod l \\ 1 \leq b - n^2 \leq K}} 1. \end{aligned}$$

Approximating the inner-most sum of (9.2) by $K/(2l\sqrt{t})$, and estimating the sum of $\Lambda(b)$ by Lemma 6 upon noting that $\Delta \geq z^{2/3+\varepsilon}$, we transform (9.2) into

$$= \frac{\Delta K}{2\varphi(l)l\sqrt{t}} \sum_{\substack{a \bmod l \\ (a,l)=1}} \sum_{m \bmod l} \left(\frac{m^2 - a}{l} \right) + O\left(\frac{\Delta K}{\sqrt{z}(\log z)^{B+1}} \right).$$

By Lemma 8, the above is

$$= \frac{\Delta K}{2\sqrt{t}} \cdot \frac{\mu(l)}{l} + O\left(\frac{\Delta K}{\sqrt{z}(\log z)^{B+1}} \right)$$

if l is square-free which can be assumed due to the presence of $\mu(l)$ in (9.1). Now, combining everything, we obtain

$$(9.3) \quad V(t) = \frac{\Delta^2 K}{4t} \sum_{\substack{l \leq L \\ 2 \nmid l}} \frac{\mu(l)^2}{l\varphi(l)} + O(E) = \frac{\Delta^2 K}{4t} \prod_{p>2} \left(1 + \frac{1}{p(p-1)}\right) + O(E).$$

10. CONTRIBUTION OF $W(t)$

We now consider $W(t)$. We may approximate $W(t)$ by

$$(10.1) \quad W(t) = \frac{\Delta^2}{4t} \sum_{1 \leq k \leq K} \mathfrak{S}^2(k) + O(E).$$

Also we have that

$$(10.2) \quad \mathfrak{S}(k) = \sum_{\substack{1 \leq l \leq L \\ 2 \nmid l}} \frac{\mu(l)}{\varphi(l)} \left(\frac{-k}{l}\right) + \sum_{\substack{l > L \\ 2 \nmid l}} \frac{\mu(l)}{\varphi(l)} \left(\frac{-k}{l}\right).$$

First we have

$$\sum_k \sum_{\substack{l > L \\ 2 \nmid l}} \frac{\mu(l)}{\varphi(l)} \left(\frac{-k}{l}\right) \ll \frac{K}{(\log z)^B},$$

again using the same techniques as in section 5 [3]. Expanding the sum over k of the square of the first term in (10.2), we get

$$(10.3) \quad \sum_{1 \leq k \leq K} \left| \sum_{\substack{1 \leq l \leq L \\ 2 \nmid l}} \frac{\mu(l)}{\varphi(l)} \left(\frac{-k}{l}\right) \right|^2 = \sum_{1 \leq l \leq L} \frac{\mu^2(l)}{\varphi^2(l)} \left(K \frac{\varphi(l)}{l} + O(l) \right) + \sum_{\substack{1 \leq l_1, l_2 \leq L \\ l_1 \neq l_2}} \frac{\mu(l_1)\mu(l_2)}{\varphi(l_1)\varphi(l_2)} \sum_{1 \leq k \leq K} \left(\frac{-k}{l_1 l_2}\right).$$

Using the inequality of Polya-Vinogradov, Lemma 3, we deduce that the second term on the right-hand side of (10.3) is bounded by

$$\ll \frac{K}{(\log z)^B}.$$

Completing the sum

$$\sum_{1 \leq l \leq L} \frac{\mu^2(l)}{l\varphi(l)}$$

contained in the first term on the right-hand side of (10.3), and combining everything, we arrive at the estimate

$$(10.4) \quad W(t) = \frac{\Delta^2 K}{4t} \prod_{p>2} \left(1 + \frac{1}{p(p-1)}\right) + O(E).$$

11. PROOFS OF THEOREMS 1 AND 2

We first prove Theorem 2.

Proof. (of Theorem 2) Combining the estimates derived in sections 5-8, we obtain the following bound

$$(11.1) \quad \int_z^{2z} U(t) dt \leq \Delta^2 K \cdot \frac{\log 2}{4} \prod_{p>2} \left(1 + \frac{1}{p(p-1)}\right) + O\left(\frac{\Delta^2 K}{\log^B z}\right).$$

Using (11.1), (9.3) and (10.4), and taking into account that $U(t) - 2V(t) + W(t)$ is positive by (4.2), we deduce that

$$\int_z^{2z} (U(t) - 2V(t) + W(t)) dt \ll \frac{\Delta^2 K}{(\log z)^B}.$$

From the above and (4.2), we obtain the desired estimate (2.4). $\square$

Now we derive Theorem 1 from Theorem 2.

Proof. (of Theorem 1) We assume that $x^{2/3+\varepsilon} \leq \Delta \leq x^{1-\varepsilon}$ and write

$$\begin{aligned}
 (11.2) \quad & \sum_{z < n^2+k \leq 2z} \Lambda(n^2+k) - \mathfrak{S}(k)\sqrt{z} \\
 &= \sum_{z < n^2+k \leq 2z} \Lambda(n^2+k) - \mathfrak{S}(k) \sum_{z < n^2+k \leq 2z} 1 + O\left(\frac{\sqrt{z}}{(\log z)^B}\right) \\
 &= \frac{1}{\Delta} \int_z^{2z} \left(\sum_{t < n^2+k \leq t+\Delta} \Lambda(n^2+k) - \mathfrak{S}(k) \sum_{t < n^2+k \leq t+\Delta} 1 \right) dt + O\left(\frac{\sqrt{z}}{(\log z)^B}\right),
 \end{aligned}$$

where we have used that $\mathfrak{S}(k) \ll \log 2k$ as was shown in section 1 of [3]. Using (11.2) and Cauchy's inequality, the left-hand side of (2.1) is majorized by

$$(11.3) \quad \ll \frac{z}{\Delta^2} \int_z^{2z} \sum_{1 \leq k \leq K} \left| \sum_{t < n^2+k \leq t+\Delta} \Lambda(n^2+k) - \mathfrak{S}(k) \sum_{t < n^2+k \leq t+\Delta} 1 \right|^2 dt.$$

Now (2.1) follows from Theorem 2 and (11.3). $\square$

Acknowledgements. This paper was written when the second-named author was supported by a post-doctoral fellowship at *Institutionen för Matematik, Kungliga Tekniska Högskolan* in Stockholm and a grant from the Göran Gustafsson Foundation. He wishes to thank these sources for their support. Furthermore, for the discussions with and encouragements given to him during this work, he owes a debt of gratitude to Prof. John B. Friedlander.

REFERENCES

- [1] N. C. Ankeny, *Representations of primes by quadratic forms*, Amer. J. Math. **74** (1952), no. 4, 913–919.
- [2] S. Baier and L. Zhao, *Bombieri-Vinogradov theorem for sparse sets of moduli*, Acta Arith. **125** (2006), no. 2, 187–201.
- [3] ———, *Primes in quadratic progressions on average*, Math. Ann. **338** (2007), no. 4, 963–982.
- [4] S. Baier, *On the p^λ problem*, Acta Arith. **113** (2004), 77–101.
- [5] A. Balog, *On the fractional part of p^θ* , Arch. Math. (Basel) **40** (1983), 434–440.
- [6] P. T. Bateman and R. A. Horn, *A heuristic asymptotic formula concerning the distribution of prime numbers*, Math. Comp. **16** (1962), 363–367.
- [7] J. Brüdern, *Einführung in die analytische Zahlentheorie*, Springer-Verlag, Berlin etc., 1995.
- [8] H. Davenport, *Multiplicative Number Theory*, Third Edition, Graduate Texts in Mathematics, vol. 74, Springer-Verlag, Barcelona, etc., 2000.
- [9] J.-M. Deshouillers and H. Iwaniec, *On the greatest prime factor of $n^2 + 1$* , Ann. Inst. Fourier (Grenoble) **32** (1982).
- [10] P. D. T. A. Elliott and H. Halberstam, *A conjecture in prime number theory*, 1970 Symposia Mathematica, 1968/69, pp. 59–72.
- [11] J. B. Friedlander and H. Iwaniec, *Quadratic polynomials and quadratic forms*, Acta Math. **141** (1978), no. 1-2, 1–15.
- [12] P. X. Gallagher, *The large sieve*, Mathematika **14** (1967), 14–20.
- [13] A. Granville and R. A. Mollin, *Rabinowitsch revisited*, Acta Arith. **96** (2000), no. 2, 139–153.
- [14] G. H. Hardy and J. E. Littlewood, *Some problems of 'partitio numerorum'; III: On the expression of a number as sum of primes*, Acta Math. **44** (1922), no. 3, 1–70.
- [15] G. Harman and P. Lewis, *Gaussian primes in narrow sectors*, Mathematika **48** (2001), 119–135.
- [16] G. Harman, *Fractional and integral parts of p^λ* , Acta Arith. **58** (1991), 141–152.
- [17] D. R. Heath-Brown, *A mean value estimate for real character sums*, Acta Arith. **72** (1995), no. 3, 235–275.
- [18] C. Hooley, *On the greatest prime factor of a quadratic polynomial*, Acta Math. **117** (1967), 281–299.
- [19] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, 2004.
- [20] H. Iwaniec, *The half dimensional sieve*, Acta Arith. **29** (1976), no. 1, 69–95.
- [21] ———, *Almost-primes represented by quadratic polynomials*, Invent. Math. **47** (1978), no. 2, 171–188.
- [22] J. P. Kubilius, *On a problem in the n -dimensional analytic theory of numbers*, Vilniaus Valst. Univ. Mokslo darbai Chem. Moksly, Ser **4** (1955), 5–43.
- [23] P. Kuhn, *Über die Primteiler eines Polynoms*, Proceedings of the International Congress of Mathematicians, 1954, pp. 35–37.
- [24] S. Lang and H. Trotter, *Frobenius distributions in GL_2 -extensions*, Lecture Notes in Mathematics, vol. 504, Springer-Verlag, Berlin, etc., 1976.
- [25] J. V. Linnik, *The Dispersion Method in Binary Additive Problems*, Translation of Mathematical Monographs, vol. 4, American Mathematical Society, Providence, 1963. Translated from Russian.

- [26] H. Mikawa, *On prime twins in arithmetic progressions*, Tsukuba J. Math. **16** (1992), no. 1, 377–387.
- [27] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Lecture Notes in Mathematics, vol. 227, Springer-Verlag, Barcelona, etc., 1971.
- [28] ———, *The analytic principles of large sieve*, Bull. Amer. Math. Soc. **84** (1978), no. 4, 547–567.
- [29] I. I. Pyateckiĭ-Sapiro, *On the distribution of prime numbers in sequences of the form $[f(n)]$* , Mat. Sbornik N.S. **75** (1953), no. 33, 559–566.
- [30] A. Schinzel and W. Sierpiński, *Sur certaines hypothèses concernant les nombres premiers*, Acta Arith. **4** (1958), 185–208. Errata, *ibid.* **5** (1959), 259.
- [31] L. Zhao, *Large sieve inequality for characters to square moduli*, Acta Arith. **112** (2004), no. 3, 297–308.

School of Engineering and Science, Jacobs University Bremen
P. O. Box 750561, Bremen 28725 Germany
Email: `s.baier@iu-bremen.de`

Department of Mathematics, Royal Institute of Technology(KTH)
Lindstedtsvägen 25, Stockholm 10044 Sweden
Email: `lzhao@math.kth.se`