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A discrete de Rham complex with enhanced smoothness

Xue–Cheng Tai[1], **Ragnar Winther**[2]

[1] Department of Mathematics, University of Bergen, 5007 Bergen, Norway
e-mail: Xue-Cheng.Tai@mi.uib.no
[2] Centre of Mathematics for Applications and Department of Informatics, University of Oslo, 0316 Oslo, Norway
e-mail: ragnar.winther@cma.uio.no

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Abstract. Discrete de Rham complexes are fundamental tools in the construction of stable elements for some finite element methods. The purpose of this paper is to discuss a new discrete de Rham complex in three space dimensions, where the finite element spaces have extra smoothness compared to the standard requirements. The motivation for this construction is to produce discretizations which have uniform stability properties for certain families of singular perturbation problem. In particular, we show how the spaces constructed here lead to discretizations of Stokes type systems which have uniform convergence properties as the Stokes flow approaches a Darcy flow.

Keywords: Discrete exact sequences, nonconforming finite elements, Darcy–Stokes flow, uniform error estimates.

Mathematics Subject Classification (1991): Primary 65N12, 65N15, 65N30

1 Introduction

In [10] a robust finite element discretization of Darcy–Stokes flow in two space dimensions was proposed. More precisely, for a domain $\Omega \subset \mathbb{R}^2$ the following singular perturbation problem was studied:

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$$
\begin{aligned}
(\boldsymbol{I} - \varepsilon^2 \boldsymbol{\Delta})\boldsymbol{u} - \mathbf{grad}\, p &= \boldsymbol{f} \quad && \text{in } \Omega, \\
\operatorname{div} \boldsymbol{u} &= g && \text{in } \Omega, \\
\boldsymbol{u} &= 0 && \text{on } \partial\Omega,
\end{aligned} \tag{1}
$$

where $\varepsilon \in (0, 1]$ is the perturbation parameter. The unknowns are the vector field $\boldsymbol{u}$ and the scalar field p, which in flow problems correspond to velocity and pressure, respectively. We note that when ε is not too small this problem is simply a standard linear Stokes problem, but with an additional non-harmful lower-order term. However, if ε approaches zero the model problem formally tends to a mixed formulation of the Poisson equation with homogeneous Neumann boundary conditions, i.e., a Darcy flow. Hence, the model covers the transition from fluid flow to porous medium flow. In this respect, the singular perturbation system (1) is a prototype for problems arising in multiscale modelling.

The main motivation for the finite element method constructed in [10] was to construct a discretization which has convergence properties that are uniform with respect to the perturbation parameter ε. Hence, for ε bounded away from zero the method should behave as a finite element method for the linear Stokes problem, while as ε tends to zero the method should approach a mixed method for the Poisson equation. The approach taken in [10] was to construct a pair of finite element spaces $(\boldsymbol{V}_h, Q_h)$, for approximating the solution $(\boldsymbol{u}, p)$, such that the Brezzi stability conditions, cf. [4], are satisfied with stability constants independent of ε. The purpose of the present paper is to design a corresponding finite element method in three space dimensions.

The construction and analysis presented in [10] are closely related to discrete de Rham complexes. In two space dimensions the de Rham complex, with minimal smoothness measured in L^2, can be stated as

$$
\mathbb{R} \xrightarrow{\subset} H^1 \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\operatorname{div}) \xrightarrow{\operatorname{div}} L^2 \longrightarrow 0, \tag{2}
$$

where **curl** in the two-dimensional case denotes the operator which maps a scalar field ϕ to the vector field $(-\partial_{x_2}\phi, \partial_{x_1}\phi)$. The precise definitions of the spaces involved is given in the next section. Note that the function spaces (2) have exactly the property that they consist of all L^2 fields for which the image of the differential operator mapping to the right also is in L^2. The statement that this is a complex just means that the composition of two consecutive maps is zero. If the domain Ω is simply connected the sequence (2) is exact in the sense that the range of each map is exactly the null space of the succeeding map.

The Sobolev spaces H^1, $\boldsymbol{H}(\operatorname{div})$, and L^2 occurring in (2) are fundamental function spaces used for weak formulations of a large collection of differential systems. Furthermore, corresponding finite element spaces, and, in particular, various discrete de Rham complexes, are important tools in designing stable finite element discretizations of these systems.

A discrete de Rham complex in two dimensions can be written in the form

$$\mathbb{R} \xrightarrow{\subset} W_h \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h \xrightarrow{\text{div}} Q_h \longrightarrow 0, \tag{3}$$

where $W_h \subset H^1$, $\boldsymbol{V}_h \subset \boldsymbol{H}(\text{div})$, and $Q_h \subset L^2$ are finite element spaces with respect to a given triangulation $\mathcal{T}_h$ of Ω. The best known examples involve the Raviart–Thomas spaces [15] or the Brezzi–Douglas–Marini spaces [5] as the choice of $\boldsymbol{V}_h$, while W_h and Q_h consist of standard piecewise polynomial scalar fields which are globally continuous or discontinuous, respectively.

In [10] we constructed a discrete sequence of the form (3), but with the additional property that the finite element spaces are nonconforming approximations of spaces with extra smoothness. More precisely, $\boldsymbol{V}_h \subset \boldsymbol{H}(\text{div})$, i.e., the elements of $\boldsymbol{V}_h$ have continuous normal components over all edges of the mesh. In addition, at each edge the tangential components of the vector fields in $\boldsymbol{V}_h$ have continuous mean value. Correspondingly, $W_h \subset H^1$ is a nonconforming approximation of H^2. Hence, the spaces constructed in [10] are a discrete analogs of those of the complex

$$\mathbb{R} \xrightarrow{\subset} H^2 \xrightarrow{\mathbf{curl}} \boldsymbol{H}^1 \xrightarrow{\text{div}} L^2 \longrightarrow 0, \tag{4}$$

which is an exact sequence if the domain Ω is simply connected.

In three space dimensions the Sobolev space version of the de Rham complex can be written in the form

$$\mathbb{R} \xrightarrow{\subset} H^1 \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl}) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0, \tag{5}$$

which is an exact sequence if the domain Ω is contractable. Here $\boldsymbol{H}(\mathbf{curl})$ consists of all vector fields $\boldsymbol{u} \in \boldsymbol{L}^2$ with $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{L}^2$. A corresponding discrete de Rham complex of the form

$$\mathbb{R} \xrightarrow{\subset} S_h \xrightarrow{\mathbf{grad}} \boldsymbol{W}_h \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h \xrightarrow{\text{div}} Q_h \longrightarrow 0 \tag{6}$$

where $S_h \subset H^1$, $\boldsymbol{W}_h \subset \boldsymbol{H}(\mathbf{curl})$, $\boldsymbol{V}_h \subset \boldsymbol{H}(\text{div})$, and $Q_h \subset L^2$ is referred to as a conforming approximation of the complex (5). Well-known examples of such finite element spaces are the Nédélec families described in [12] and [13]; cf. also [1,2].

A three-dimensional example of a complex with extra smoothness, corresponding to (4), is given by

$$\mathbb{R} \xrightarrow{\subset} H^2 \xrightarrow{\mathbf{grad}} \boldsymbol{H}^1(\mathbf{curl}) \xrightarrow{\mathbf{curl}} \boldsymbol{H}^1 \xrightarrow{\text{div}} L^2 \longrightarrow 0. \tag{7}$$

Here $\boldsymbol{H}^1(\mathbf{curl})$ consists of all vector fields $\boldsymbol{u} \in \boldsymbol{H}^1$ with $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{H}^1$. The sequence (7) is obviously a complex. Furthermore, if Ω is a convex polyhedron then the sequence is exact; cf. [9, Chapter I.3.5].

The main purpose of the present paper is to construct an analog to the one given in [10] for three space dimensions. Given a tetrahedral mesh $\mathcal{T}_h$

we construct a conforming approximation of the complex (5) of the form (6), which, at the same time, is a nonconforming approximation of (7) in the sense that the discrete spaces of (6) are nonconforming approximations of H^2, $\boldsymbol{H}^1(\mathbf{curl})$, and $\boldsymbol{H}^1$, respectively. We show that the constructed spaces $\boldsymbol{V}_h$ and Q_h lead to a robust discretization of the Darcy–Stokes system (1) in the sense that the method is uniformly stable both with respect to the perturbation parameter ε and the discretization parameter h. We should mention here that an alternative approach to obtain discretizations of (1) which are stable uniformly in ε is to use an augmented Lagrangian approach as in [7]. In this formulation, any stable Stokes element indeed leads to uniformly stable discretizations for (1).

In a similar manner the finite element spaces S_h and $\boldsymbol{W}_h$, constructed below, can potentially be used to design uniform discretizations of other singular perturbation problems. For example, the space S_h is a three-dimensional analog of the finite element space used in [14] to discretize fourth-order problems which are perturbations of a second-order problem. However, we do not discuss this here.

In §2 we introduce the notation used in the paper, and we define the finite element spaces S_h, $\boldsymbol{W}_h$, $\boldsymbol{V}_h$, and Q_h. The properties of these discrete spaces are discussed in §3, and then in §4 we proceed to show that the pair of spaces $(\boldsymbol{V}_h, Q_h)$ leads to a uniformly stable discretization of the Darcy–Stokes system (1). It can also be proven that the finite element space $\boldsymbol{W}_h$ is a uniform stable element for a singular perturbed fourth-order elliptic equation studied in [14]. As the proof is essentially the same as that in [14], we omit the proof here.

2 Notation and preliminaries

We use $H^m = H^m(\Omega)$ to denote the L^2–based Sobolev spaces of order m on the polygonal domain $\Omega \subset \mathbb{R}^3$, and the corresponding norm by $\|\cdot\|_m$. The subspace H_0^m is the closure in H^m of $C_0^\infty(\Omega)$, while L_0^2 consists of all elements of L^2 with mean value zero. The notation $(\cdot,\cdot)$ is used to denote the standard L^2 inner product over the domain Ω. In general we use boldface symbols for vector fields and function spaces of vector fields. In particular $\boldsymbol{H}(\mathbf{curl}) = \boldsymbol{H}(\mathbf{curl};\Omega)$ is the space of all $\boldsymbol{L}^2$ vector fields $\boldsymbol{v}$ with $\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2$; $\boldsymbol{H}(\mathrm{div}) = \boldsymbol{H}(\mathrm{div};\Omega)$ is defined in a similar manner. The gradient of a vector field $\boldsymbol{v}$ is denoted by $\boldsymbol{D}\boldsymbol{v}$, i.e., $\boldsymbol{D}\boldsymbol{v}$ is the 3×3 matrix with elements

$$(\boldsymbol{D}\boldsymbol{v})_{i,j} = \partial \boldsymbol{v}_i/\partial x_j\ , \quad 1 \le i, j \le 3.$$

For a subset $T \subset \mathbb{R}^n$, the notation $\mathbb{P}_k = \mathbb{P}_k(T)$ is used for the space of polynomials of degree k defined on T, and $\mathbb{P}_k^n$ denotes the corresponding space of polynomial vector fields. If $T \subset \mathbb{R}^3$ is a tetrahedron then $\Delta_2(T)$

denotes the set of the four 2-dimensional faces, $\Delta_1(T)$ is the set of the six 1-dimensional edges, and $\Delta_0(T)$ the set of the four vertices.

In order to define the finite element spaces S_h, $\boldsymbol{W}_h$, $\boldsymbol{V}_h$, and Q_h we first define the restriction of these spaces to one tetrahedron. Througout this paper $\{\mathcal{T}_h\}$ is a family of shape-regular tetrahedral meshes, where h is the maximal diameter. For $T \in \mathcal{T}_h$, let $b = b_T \in \mathbb{P}_4$ be the quartic bubble function with respect to T, i.e., $b = \lambda_1\lambda_2\lambda_3\lambda_4$, where λ_i are the barycentric coordinates with respect to the vertices of T. The restriction of the space S_h to T is denoted by $S(T)$ and is given by

$$S(T) = \{s = s_2 + b\, s_1 : s_i \in \mathbb{P}_i, \quad i = 1, 2\}.$$

Hence, the space $S(T)$ is a linear space of dimension 14. The corresponding space $\boldsymbol{W}(T)$ is a space of dimension 36 given by

$$\boldsymbol{W}(T) = \boldsymbol{N}_1 + \mathbf{grad}(b\mathbb{P}_1) + b\mathbb{P}_1^3.$$

Here $\boldsymbol{N}_1 = \boldsymbol{N}_1(T)$ is the polynomial space which corresponds to the restriction of the second lowest order $\boldsymbol{H}(\mathbf{curl})$ space of Nédélec's first family to one tetrahedron; cf. [12]. Hence,

$$\boldsymbol{N}_1 = \{\boldsymbol{w} \in \mathbb{P}_2^3 : \boldsymbol{w} \cdot \boldsymbol{x} \in \mathbb{P}_2\}.$$

This space has dimension 20, and an element $\boldsymbol{w} \in \boldsymbol{N}_1$ is uniquely determined by the two lowest order moments of the tangential components on each edge, and the lowest order moment of the two tangential components on each face. We refer to [12] for more details. The restriction of the space $\boldsymbol{V}_h$ to T, $\boldsymbol{V}(T)$, is given as

$$\boldsymbol{V}(T) = \mathbb{P}_1^3 + \mathbf{curl}(b\mathbb{P}_1^3), \tag{8}$$

which is a space of dimension 24. Finally, $Q(T)$ is simply taken to be $\mathbb{P}_0$. It is straightforward to check that $\mathbf{grad}\, S(T) \subset \boldsymbol{W}(T)$, $\mathbf{curl}\, \boldsymbol{W}(T) \subset \boldsymbol{V}(T)$, and $\operatorname{div} \boldsymbol{V}(T) \subset Q(T)$. Hence, the polynomial sequence

$$\mathbb{R} \xrightarrow{\subset} S(T) \xrightarrow{\mathbf{grad}} \boldsymbol{W}(T) \xrightarrow{\mathbf{curl}} \boldsymbol{V}(T) \xrightarrow{\operatorname{div}} Q(T) \longrightarrow 0 \tag{9}$$

is a complex. In fact, it can be easily checked that (9) is exact.

The finite element spaces S_h, $\boldsymbol{W}_h$, $\boldsymbol{V}_h$, and Q_h are defined from the corresponding spaces of restrictions to a given tetrahedron, introduced above, by specifying degrees of freedom for these local spaces. As for the degree of freedom for the one-dimensional space $Q(T) = \mathbb{P}_0$ we use the mean value of the function over T. Hence, the corresponding global space Q_h is a subspace of L^2.

Any function $s \in S(T)$ is determined by the values of s at each vertex and

$$\int_e s\, dx_e\,, \quad e \in \Delta_1(T), \quad \int_f \frac{\partial s}{\partial \boldsymbol{n}}\, d\boldsymbol{x}_f\,, \quad f \in \Delta_2(T). \tag{10}$$

Here and below dx_e and $d\boldsymbol{x}_f$ denote the integration with respect to arc length or surface area, and $\boldsymbol{n}$ is a unit normal vector on f.

It is straightforward to check that these degrees of freedom uniquely determine an element $s \in S(T)$. If the degrees of freedom associated with $\Delta_0(T)$ and $\Delta_1(T)$ are all zero then $s = bs_1$, where $s_1 \in \mathbb{P}_1$. Furthermore, on a face $f \in \Delta_2(T)$

$$\frac{\partial s}{\partial \boldsymbol{n}} = c_f b_f s_1,$$

where $c_f \neq 0$ is a constant, and b_f is the cubic bubble function associated with the face f. However, b_f is nonzero in the interior of f. Hence, if the zero-order moment of $\partial s/\partial \boldsymbol{n}$ is zero on each face $f \in \Delta_2(T)$ there must exist an interior root of s_1 on each face f, and therefore $s = s_1 = 0$.

The local space $S(T)$ and the degrees of freedom determined by (10) define the corresponding global space S_h. It is clear that the elements of S_h are continuous, i.e., $S_h \subset H^1$. Furthermore, the normal derivatives are weakly continuous over inter–element faces f, in the sense that

$$\int_f \left[\frac{\partial s}{\partial \boldsymbol{n}}\right] d\boldsymbol{x}_f = 0,$$

where $[\cdot]$ denotes the jump across the face f. Hence, the space S_h is a nonconforming approximation of H^2.

Finally, we have to design proper degrees of freedom for the spaces of vector fields, $\boldsymbol{W}(T)$ and $\boldsymbol{V}(T)$. Recall that rigid motions in two- and three-dimensional spaces are vector fields $\boldsymbol{r} \in \mathbb{R}^n$, $n = 2, 3$, of the form

$$\boldsymbol{r}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b}\boldsymbol{x}, \tag{11}$$

where $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{b}$ is a skew symmetric $n \times n$ real matrix ($n = 2, 3$). The space of rigid motions is denoted by $\boldsymbol{RM}$. Furthermore, if $f \subset \mathbb{R}^3$ is a two-dimensional affine space, i.e., a plane in $\mathbb{R}^3$, then $\boldsymbol{RM}(f)$ denotes the rigid motions on f, i.e., $\boldsymbol{RM}(f)$ is the space which only contains tangentials to f and all vector fields from $\boldsymbol{RM}(f)$ are of the form (11). Hence, $\boldsymbol{RM}(f)$ is a linear space of dimension 3. In fact, all the vectors $\boldsymbol{r}(\boldsymbol{x})$ from $\boldsymbol{RM}(f)$ for $\boldsymbol{x} \in \mathbb{R}^3$ are of the form

$$\boldsymbol{r}(\boldsymbol{x}) = \boldsymbol{r}_0 + \beta(\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{n} \quad \forall \beta \in \mathbb{R},$$

where $\boldsymbol{x}_0 \in f$ is a fixed point and $\boldsymbol{r}_0$ is a fixed tangent vector.

We show below that a vector field $\boldsymbol{v} \in \boldsymbol{V}(T)$ is uniquely determined by 24 degrees of freedom. For all $f \in \Delta_2(T)$ we specify the moments

$$\int_f (\boldsymbol{v} \cdot \boldsymbol{n}) p \, d\boldsymbol{x}_f \,, \quad p \in \mathbb{P}_1(f), \quad \int_f \boldsymbol{v}_t \cdot \boldsymbol{r} \, d\boldsymbol{x}_f \,, \quad \boldsymbol{r} \in \boldsymbol{RM}(f). \tag{12}$$

In the above, $\mathbb{P}_1(f)$ is the space of linear functions on f. Here and also later, we use $\boldsymbol{v}_t$ to denote the tangential component of $\boldsymbol{v}$ on f, i.e.,

$$\boldsymbol{v}_t = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}. \tag{13}$$

Note that, if we introduce bases for the spaces $\mathbb{P}_1(f)$ and $\boldsymbol{RM}(f)$, then these moment conditions lead to 24 degrees of freedom for the elements $\boldsymbol{v} \in \boldsymbol{V}(T)$.

The proof that these degrees of freedom are unisolvent for $\boldsymbol{V}(T)$ is given in the next section. Once $V(T)$ is defined, the finite element space V_h is defined by these degree of freedoms over the finite elements of $\mathcal{T}_h$. It can easily be seen that the elements of the corresponding global space $\boldsymbol{V}_h$ have continuous normal derivatives. Therefore, $\boldsymbol{V}_h \subset \boldsymbol{H}(\mathrm{div})$. Furthermore, the tangential components are weakly continuous, so $\boldsymbol{V}_h$ is a nonconforming approximation of $\boldsymbol{H}^1$.

Remark 1 Note that if $\boldsymbol{v} \in \boldsymbol{V}_h$ then the jumps of $\boldsymbol{v}$ on the inter-element faces are orthogonal to rigid motions. From the observation of [11], based on the general nonconforming theory of [3], it follows that the elements of the nonconforming $\boldsymbol{H}^1$ space $\boldsymbol{V}_h$ indeed satisfy Korn's inequality.

The degrees of freedom for $\boldsymbol{W}(T)$ are determined by the moments

$$\int_e (\boldsymbol{w} \cdot \boldsymbol{t}) p \, dx_e \ , \quad e \in \Delta_1(T), \quad p \in \mathbb{P}_1(e), \tag{14}$$

where $\boldsymbol{t}$ is a tangent vector on e, and for all $f \in \Delta_2(T)$

$$\int_f \boldsymbol{w} \, d\boldsymbol{x}_f, \quad \text{and} \quad \int_f (\mathbf{curl}\, \boldsymbol{w})_t \cdot \boldsymbol{r} \, d\boldsymbol{x}_f \ , \quad \boldsymbol{r} \in \boldsymbol{RM}(f). \tag{15}$$

It is a consequence of the discussion in §3 below that these degrees of freedom determine an element of $\boldsymbol{W}(T)$ uniquely. It can also be seen that the elements of the corresponding global space, $\boldsymbol{W}_h$, have continuous tangential components, and therefore $\boldsymbol{W}_h \subset \boldsymbol{H}(\mathbf{curl})$. Furthermore, the normal components of $\boldsymbol{w}$, and the components of $\mathbf{curl}\, \boldsymbol{w}$ are weakly continuous, and hence the space $\boldsymbol{W}_h$ is a nonconforming approximation of $\boldsymbol{H}^1(\mathbf{curl})$.

3 The discrete de Rham complex

The purpose of this section is to complete the discussion of the finite element spaces S_h, $\boldsymbol{W}_h$, $\boldsymbol{V}_h$, and Q_h. In particular, we show that the corresponding complex (6) is exact. In order to show that elements $\boldsymbol{v} \in \boldsymbol{V}(T)$ and $\boldsymbol{w} \in \boldsymbol{W}(T)$ are uniquely determined by the degrees of freedom specified by (12), or by (14) and (15), respectively, we need preliminary results in two dimensions.

3.1 Preliminary results in two dimensions

Throughout this subsection $f \subset \mathbb{R}^2$ is a general triangle, and $\hat{f}$ the reference triangle given by

$$\hat{f} = \{\boldsymbol{x} \in \mathbb{R}^2 \,:\, x_1, x_2 \geq 0,\ x_1 + x_2 \leq 1\}.$$

We let $\lambda_i, i = 1, 2, 3$, be the barycentric cooordinates on f, and $b = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$ the cubic bubble function on f. The integral of b over f is denoted $|b|_f$. For example, if $f = \hat{f}$, then $b(\boldsymbol{x}) = x_1 x_2 (1 - x_1 - x_2)$ and $|b|_f = 1/120$.

Furthermore, for each triangle f there is a 1–1 linear transformation Φ of the form

$$\Phi(\hat{\boldsymbol{x}}) = B\hat{\boldsymbol{x}} + \boldsymbol{x}_0$$

mapping $\hat{f}$ onto f. If $\lambda_i(\boldsymbol{x})$ are the barycentric cooordinates on f and $\hat{\lambda}_i(\hat{\boldsymbol{x}})$ the corresponding functions on $\hat{f}$, then

$$\lambda_i(\boldsymbol{x}) = \hat{\lambda}_i(\Phi^{-1}(\boldsymbol{x})).$$

The corresponding Piola transform, $\mathcal{P}$, maps 2-vectorfields defined on $\hat{f}$ to corresponding vectorfields on f. If $\hat{z}$ is a vectorfield on $\hat{f}$ then

$$z(\boldsymbol{x}) = \mathcal{P}\hat{z}(x) = J^{-1} B \hat{z}(\Phi^{-1}(\boldsymbol{x})).$$

Here J is the determinant of B. The Piola transform maps constant vectors to constant vectors. In addition, $\mathcal{P}\hat{z}(\boldsymbol{x}) = J^{-1}(\boldsymbol{x} - \boldsymbol{x}_0)$ if $\hat{z}(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{x}}$.

The following identities, which can be established by straightforward calculations, are useful below.

Lemma 1 *If $f = \hat{f}$, then*

$$\int_f x_1 b \, d\boldsymbol{x} = \int_f x_2 b \, d\boldsymbol{x}_f = 1/360 = |b|_f/3,$$
$$\int_f x_1^2 b \, d\boldsymbol{x} = \int_f x_2^2 b \, d\boldsymbol{x}_f = |b|_f/7,$$
$$\int_f x_1 x_2 b \, d\boldsymbol{x} = 2|b|_f/21.$$

For a general triangle f, we define the barycenter $\boldsymbol{x}^b \in f$ by

$$\lambda_i(\boldsymbol{x}^b) = 1/3 \quad i = 1, 2, 3.$$

It is a direct consequence of the lemma above that the integration rule

$$\int_f bp \, d\boldsymbol{x} = |b|_f p(\boldsymbol{x}^b) \tag{16}$$

is exact for $p \in \mathbb{P}_1$ and $f = \hat{f}$. By a change of variables this formula then holds for any triangle f.

Assume that $\boldsymbol{v} \in \mathbb{P}_1^2(f)$ is of the form

$$\boldsymbol{v} = \sum_{i=1}^{3} c_i \left(\lambda_i - \frac{1}{3}\right) \mathbf{grad}\, \lambda_i. \tag{17}$$

By (16), it follows that

$$\int_f b(\boldsymbol{v} \cdot \boldsymbol{z})\, d\boldsymbol{x} = 0$$

for all constant vector fields $\boldsymbol{z}$. In addition we have the following characterization.

Lemma 2 *If* $\boldsymbol{v} \in \mathbb{P}_1^2(f)$ *is of the form* (17) *and satisfies*

$$\int_f b(\boldsymbol{v} \cdot \boldsymbol{x})\, d\boldsymbol{x} = 0\ ,$$

then $c_1 + c_2 + c_3 = 0$.

Proof Let $\hat{\boldsymbol{z}}$ and $\boldsymbol{\psi}$ be smooth vector fields on $\hat{f}$ and f, respectively. Then

$$\int_f b(\boldsymbol{\psi} \cdot \mathcal{P}\hat{\boldsymbol{z}})\, d\boldsymbol{x} = \int_{\hat{f}} \hat{b}(B^T \hat{\boldsymbol{\psi}} \cdot \hat{\boldsymbol{z}})\, d\hat{\boldsymbol{x}}\ , \tag{18}$$

where $\hat{\boldsymbol{\psi}} = \boldsymbol{\psi} \circ \Phi$, $\hat{b} = b \circ \Phi$ and B^T is the transpose of the matrix B. Note that, if $\boldsymbol{\psi} = \mathbf{grad}\, q$, then $\mathbf{grad}_{\hat{\boldsymbol{x}}}\, \hat{q} = B^T \hat{\boldsymbol{\psi}}$. Therefore, if $\boldsymbol{v} \in \mathbb{P}_1^2(f)$ is of the form (17), then

$$B^T \hat{\boldsymbol{v}} = \sum_{i=1}^{3} c_i \left(\hat{\lambda}_i - \frac{1}{3}\right) \mathbf{grad}_{\hat{\boldsymbol{x}}}\, \hat{\lambda}_i.$$

Hence, by the assumption and (18), the coefficients c_i satisfy

$$\begin{aligned}
\left(\sum_{i=1}^{3} c_i \int_{\hat{f}} \hat{b} \backslash \left(\hat{\lambda}_i - \frac{1}{3}\right) \mathbf{grad}_{\hat{\boldsymbol{x}}}\, \hat{\lambda}_i\right) \cdot \hat{\boldsymbol{x}}\, d\hat{\boldsymbol{x}} \\
&= \int_{\hat{f}} \hat{b}(B^T \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{x}})\, d\hat{\boldsymbol{x}} \\
&= J^{-1} \int_f b\boldsymbol{v} \cdot (\boldsymbol{x} - \boldsymbol{x}_0)\, d\boldsymbol{x} = 0,
\end{aligned}$$

where we have used the fact that $(\mathcal{P}\hat{\boldsymbol{x}})(\boldsymbol{x}) = J^{-1}(\boldsymbol{x} - \boldsymbol{x}_0)$. However, from the identities of Lemma 1 we easily compute

$$\left(\sum_{i=1}^{3} c_i \int_{\hat{f}} \hat{b}(\hat{\lambda}_i - \frac{1}{3}) \,\mathbf{grad}_{\hat{x}}\, \hat{\lambda}_i\right) \cdot \hat{\boldsymbol{x}}\, d\hat{\boldsymbol{x}} = \int_{\hat{f}} \left[c_1 x_1 \left(x_1 - \frac{1}{3}\right)\right.$$
$$\left. + c_2 x_2 \left(x_2 - \frac{1}{3}\right) + c_3 \left(x_1 + x_2 - \frac{2}{3}\right)(x_1 + x_2)\right] dx_1 dx_2$$
$$= \frac{2}{63} |b|_f (c_1 + c_2 + c_3),$$

and therefore $c_1 + c_2 + c_3 = 0$. □

3.2 Unisolvent degrees of freedom

We now return to the discussion of polynomial spaces defined on a tetrahedron $T \subset \mathbb{R}^3$. We recall that $b = b_T = \prod_{j=1}^{4} \lambda_j$ is the quartic bubble function on T. Furthermore, on the face $f = f_i = \{\boldsymbol{x} : \lambda_i(\boldsymbol{x}) = 0\} \in \Delta_2(T)$ we associate the cubic bubble function $b_f = \prod_{j \neq i} \lambda_j$. We need to show that the functions in the spaces $\boldsymbol{V}(T)$ and $\boldsymbol{W}(T)$ are uniquely determined by the moment conditions given by (12) and (14)–(15), respectively. We first establish the following lemma.

Lemma 3 *Assume that $\boldsymbol{v} \in \mathbb{P}_1^3(T)$ satisfies*

$$\int_f b_f (\boldsymbol{v} \times \boldsymbol{n}) \cdot \boldsymbol{r} d\boldsymbol{x}_f = 0\,, \quad \boldsymbol{r} \in \boldsymbol{RM}(f),\ f \in \Delta_2(T). \tag{19}$$

Then $\boldsymbol{v} = 0$.

Proof If $\boldsymbol{v} \in \mathbb{P}_1^3(T)$ satisfies (19), then

$$\int_f b_f \boldsymbol{v}_t \cdot \boldsymbol{z} d\boldsymbol{x}_f = 0\,, \quad \boldsymbol{z} \in \mathbb{P}_0^2(f),\ f \in \Delta_2(f).$$

This follows since $\boldsymbol{v} \times \boldsymbol{n} = R\boldsymbol{v}_t$, where the matrix R represents a rotation by 90 degrees in the tangent space of f. Since $\boldsymbol{RM}(f)$ contains all constant tangential vector fields, using (16), we conclude that

$$\boldsymbol{v}_t(x_f^b) = 0, \quad f \in \Delta_2(f), \tag{20}$$

where x_f^b is the barycenter of a face f. The space of functions in $\mathbb{P}_1^3(T)$ satisfying (20) is four-dimensional and is given as the span of the functions $(\lambda_i - \frac{1}{3})\,\mathbf{grad}\,\lambda_i$, $i = 1, 2, 3, 4$. Hence, $\boldsymbol{v}$ is of the form

$$\boldsymbol{v} = \sum_{i=1}^{4} c_i \left(\lambda_i - \frac{1}{3}\right) \mathbf{grad}\,\lambda_i \tag{21}$$

for some constants c_1, c_2, c_3, c_4. Restricting $\boldsymbol{v}$ to the face f_1, given by $\lambda_1 = 0$, the tangential component $\boldsymbol{v}_t$ has the form

$$\boldsymbol{v}_t = \sum_{i=2}^{4} c_i \left(\lambda_i - \frac{1}{3}\right) \mathbf{grad}_t\, \lambda_i,$$

where $\mathbf{grad}_t\, \lambda = (\mathbf{grad}\, \lambda)_t$ is the tangential component of $\mathbf{grad}\, \lambda$. Note that (19) implies that, for any fixed $\boldsymbol{x}_0 \in f_1$,

$$\int_{f_1} b_1 \boldsymbol{v}_t \cdot (\boldsymbol{x} - \boldsymbol{x}_0)\, d\boldsymbol{x}_f = \int_{f_1} b_1 (\boldsymbol{v} \times \boldsymbol{n}) \cdot \big((\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{n}\big)\, d\boldsymbol{x}_f = 0.$$

As a consequence of Lemma 2 we conclude that $c_2 + c_3 + c_4 = 0$. By considering all the four faces we conclude that

$$\sum_{i \neq j} c_i = 0\,, \quad j = 1, 2, 3, 4,$$

and this implies that $c_1 = c_2 = c_3 = c_4 = 0$. $\square$

Next we show that the elements of $\boldsymbol{V}(T)$ are uniquely determined by the 24 degrees of freedom given by (12).

Lemma 4 *Assume that* $\boldsymbol{v} \in \boldsymbol{V}(T) = \mathbb{P}_1^3 + \mathbf{curl}(b\mathbb{P}_1^3)$ *and that all the degrees of freedom represented by* (12) *are zero. Then* $v = 0$.

Proof Let $\boldsymbol{v} = \boldsymbol{p} + \mathbf{curl}\, b\boldsymbol{q}$, where $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_1^3$. On each face $f \in \Delta_2(T)$ the normal component of $\mathbf{curl}\, b\boldsymbol{q}$ is zero since it only depends on tangential derivatives of $b\boldsymbol{q}$. Therefore, $\boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{p} \cdot \boldsymbol{n}$. Hence, if the 12 constraints on the normal component of $\boldsymbol{v}$ are all zero, we can conclude that $\boldsymbol{p} \cdot \boldsymbol{n} = 0$ on each face. Since three faces meet at a vertex, we conclude that $\boldsymbol{p} = 0$ on each vertex. However, this means that $\boldsymbol{p} = 0$ or $\boldsymbol{v} = \mathbf{curl}(b\boldsymbol{q})$. As a consequence, on each face

$$\boldsymbol{v}_t = (\mathbf{curl}\, b\boldsymbol{q})_t = -\frac{\partial b}{\partial \boldsymbol{n}} (\boldsymbol{q} \times \boldsymbol{n}).$$

However, $\partial b / \partial \boldsymbol{n}$ is proportional to b_f. Therefore, if the conditions on $\boldsymbol{v}_t$ in (12) all vanish, then

$$\int_f b_f (\boldsymbol{q} \times \boldsymbol{n}) \cdot \boldsymbol{r}\, d\boldsymbol{x}_f = 0\,, \quad \boldsymbol{r} \in \boldsymbol{RM}(f)$$

for all $f \in \Delta_2(T)$, and by Lemma 3 this implies that $\boldsymbol{q} = 0$. $\square$

A similar argument can be given to show that the elements of $\boldsymbol{W}(T)$ are uniquely determined by the degrees of freedom given by (14)–(15). Recall that a vector field $\boldsymbol{w}$ is in $\boldsymbol{W}(T)$ if it is of the form

$$\boldsymbol{w} = \boldsymbol{w}^0 + \mathbf{grad}(bp) + b\boldsymbol{q} \ ,$$

where $p \in \mathbb{P}_1$, $\boldsymbol{q} \in \mathbb{P}_1^3$ and $\boldsymbol{w}^0 \in \boldsymbol{N}_1$. From the definition of $\boldsymbol{N}_1$, we see that $\boldsymbol{w}^0 \in \mathbb{P}_2^3$ and that $\boldsymbol{w}^0 \cdot \boldsymbol{x} = 0$. If all the degrees of freedom given by (14)–(15) are zero then we quickly conclude that $\boldsymbol{w}^0 = 0$ from the standard 20 degrees of freedom of $\boldsymbol{N}_1$ (two lowest order moments of the tangential component on each edge and the loweat order moment of the tangential components on each face). Furthermore, since $\boldsymbol{w} = \mathbf{grad}(bp) + b\boldsymbol{q}$ we see that $\boldsymbol{w} \cdot \boldsymbol{n}$ is proportional to $b_f p$ on each face. Hence, we conclude that

$$\int_f b_f p \, d\boldsymbol{x}_f = 0 \quad f \in \Delta_2(T)$$

and therefore, by (16), $p = 0$ at the barycenter of each face. Thus, it is true that $p = 0$. Finally, if $\boldsymbol{w} = b\boldsymbol{q}$ then the tangential component $(\mathbf{curl}\, \boldsymbol{w})_t$ is proportional to $b_f(\boldsymbol{q} \times \boldsymbol{n})$ on each face, and therefore Lemma 3 again implies that $\boldsymbol{q} = 0$.

3.3 The discrete complex

We have seen that the polynomial spaces $S(T)$, $\boldsymbol{W}(T)$, $\boldsymbol{V}(T)$, and $Q(T)$, defined on a single tetrahedron T, all have a set of unisovent degrees of freedom as specified in §2. For a tetrahedral mesh $\mathcal{T}_h$ the spaces S_h, $\boldsymbol{W}_h$, $\boldsymbol{V}_h$, and Q_h are all defined as the functions which belong to the corresponding polynomial spaces on each tetrahedron T, and the continuity conditions for the global space are implicitly defined by the degrees of freedom on vertices, edges, and faces.

It is straightforward to check that in the sequence

$$\mathbb{R} \xrightarrow{\subset} S_h \xrightarrow{\mathbf{grad}} \boldsymbol{W}_h \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h \xrightarrow{\mathrm{div}} Q_h \longrightarrow 0 \qquad (22)$$

each space is mapped into the succeeding space by the given operator, and that the sequence is a complex. Furthermore, if Ω is contractible the sequence is exact. In fact, this is an easy consequence of the analogous property for more standard discrete spaces. To see this, let S_h^0 be the standard continuous piecewise linear space with respect to the triangulation $\mathcal{T}_h$, $\boldsymbol{W}_h^0$ the second lowest order Nédélec space corresponding to piecewise polynomials in $\boldsymbol{N}_1$, and $\boldsymbol{V}_h^0$ the space of piecewise linear vector fields with $\boldsymbol{H}(\mathrm{div})$ continuity, i.e., $\boldsymbol{V}_h^0$ is the lowest order space in Nédélec's second family. For these spaces it is well-known that the sequence

$$\mathbb{R} \xrightarrow{\subset} S_h^0 \xrightarrow{\mathbf{grad}} \boldsymbol{W}_h^0 \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h^0 \xrightarrow{\mathrm{div}} Q_h \longrightarrow 0 \tag{23}$$

is exact; cf. [1,13]. By definition the restriction $\boldsymbol{v}_T$ of an element $\boldsymbol{v} \in \boldsymbol{V}_h$ to a tetrahedron $T \in \mathcal{T}_h$ is of the form

$$\boldsymbol{v}_T = \boldsymbol{v}_T^0 + \mathbf{curl}(b_T \boldsymbol{q}_T) \quad \text{with } \boldsymbol{v}_T^0, \boldsymbol{q}_T \in \mathbb{P}_1^3. \tag{24}$$

However, on each face $f \in \Delta_2(T)$ the normal component of $\mathbf{curl}(b_T \boldsymbol{q}_T)$ is zero. Therefore, the continuity requirements on $\boldsymbol{v}$ imply that the piecewise polynomial $\boldsymbol{v}^0$ has continuous normal components, and hence $\boldsymbol{v}^0$ is an element of the space $\boldsymbol{V}_h^0$. Furthermore, the weak continuity of the tangential components of $\boldsymbol{v}$ implies that, for each face f of $\mathcal{T}_h$,

$$\int_f [(\boldsymbol{v}^0 + \mathbf{curl}\, b\boldsymbol{q})_t] \cdot \boldsymbol{r}\, d\boldsymbol{x}_f = 0\,, \quad \boldsymbol{r} \in \boldsymbol{RM}(f), \tag{25}$$

where $[\cdot]$ denote the jump across f.

Assume now that $\mathrm{div}\, \boldsymbol{v} = 0$. In order to show that the sequence (22) is exact we need to show that there is $\boldsymbol{w} \in \boldsymbol{W}_h$ such that $\mathbf{curl}\, \boldsymbol{w} = \boldsymbol{v}$. However, if $\mathrm{div}\, \boldsymbol{v} = 0$ and $\boldsymbol{v}$ is of the form (24) then $\mathrm{div}\, \boldsymbol{v}^0 = 0$, and by the exactness of the sequence (23) we conclude that there is $\boldsymbol{w}^0 \in \boldsymbol{W}_h^0$ such that

$$\boldsymbol{v}_T = \mathbf{curl}(\boldsymbol{w}_T^0 + b_T \boldsymbol{q}_T)$$

on each triangle T. Furthermore, from (25) we obtain

$$\int_f \big[(\mathbf{curl}\, \boldsymbol{w}^0 + b\boldsymbol{q})_t\big] \cdot \boldsymbol{r}\, d\boldsymbol{x}_f = 0\,, \quad \boldsymbol{r} \in \boldsymbol{RM}(f).$$

Hence, if $\boldsymbol{w} = \boldsymbol{w}^0 + b\boldsymbol{q}$ we get from (14)–(15) that $\boldsymbol{w} \in \boldsymbol{W}_h$.

We can use a similar argument to show that all curl–free elements of $\boldsymbol{W}_h$ are gradients of functions in S_h. First note that any $\boldsymbol{w} \in \boldsymbol{W}_h$ is of the form

$$\boldsymbol{w}_T = \boldsymbol{w}_T^0 + \mathbf{grad}(b_T p_T) + b_T \boldsymbol{q}_T \tag{26}$$

on each tetrahedron T, where $\boldsymbol{w}_T^0$ is in the class $\boldsymbol{N}_1$ and p and $\boldsymbol{q}$ are linear. Furthermore, $\boldsymbol{w}^0 \in \boldsymbol{W}_h^0$ since the other two terms on the right-hand side of (26) vanish for the standard degrees of freedom of $\boldsymbol{W}_h^0$. If $\mathbf{curl}\, \boldsymbol{w} = 0$ then clearly

$$\mathbf{curl}\, \boldsymbol{w}^0 = 0 \quad \text{and} \quad \mathbf{curl}\, b\boldsymbol{q} = 0.$$

However, if $\mathbf{curl}\, b\boldsymbol{q} = 0$ then, in particular, the tangential component $(\mathbf{curl}\, b\boldsymbol{q})_t = 0$ on all faces, and hence, by (14)–(15), the element $b\boldsymbol{q}$ of $\boldsymbol{W}_h$ is zero. Furthermore, since $\mathbf{curl}\, \boldsymbol{w}^0 = 0$ we can use (23) to obtain $\boldsymbol{w}^0 = \mathbf{grad}\, s^0$ for a suitable $s^0 \in S_{h,0}$. So $\boldsymbol{w} = \mathbf{grad}\, s$, where $s = s^0 + bp \in S_h$.

3.4 Commuting diagrams

The finite element spaces S_h, $\boldsymbol{W}_h$, $\boldsymbol{V}_h$, and Q_h introduced above are subspaces of H^1, $\boldsymbol{H}(\mathbf{curl})$, $\boldsymbol{H}(\mathrm{div})$, and L^2, respectively. In addition, due to additional weak continuity over inter-element faces, the spaces S_h, $\boldsymbol{W}_h$, and $\boldsymbol{V}_h$ are nonconforming approximations of H^2, $\boldsymbol{H}^1(\mathbf{curl})$, and $\boldsymbol{H}^1$.

The degrees of freedom, or more precisely the moment conditions, specified above define canonical interpolation operators

$$\Pi_h^S : H^2 \to S_h, \quad \boldsymbol{\Pi}_h^W : \boldsymbol{H}^1(\mathbf{curl}) \to \boldsymbol{W}_h, \quad \boldsymbol{\Pi}_h^V : \boldsymbol{H}^1 \to \boldsymbol{V}_h,$$

and $\Pi_h^Q : L^2 \to Q_h$. Furthermore, the following diagram commutes:

$$\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & H^2 & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}^1(\mathbf{curl}) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}^1 & \xrightarrow{\mathrm{div}} & L^2 & \longrightarrow & 0 \\
 & & \Big\downarrow \Pi_h^S & & \Big\downarrow \boldsymbol{\Pi}_h^W & & \Big\downarrow \boldsymbol{\Pi}_h^V & & \Big\downarrow \Pi_h^Q & & \\
\mathbb{R} & \longrightarrow & S_h & \xrightarrow{\mathbf{grad}} & \boldsymbol{W}_h & \xrightarrow{\mathbf{curl}} & \boldsymbol{V}_h & \xrightarrow{\mathrm{div}} & Q_h & \longrightarrow & 0
\end{array}$$

In other words the identities

$$\mathbf{grad}\,\Pi_h^S = \boldsymbol{\Pi}_h^W\,\mathbf{grad}, \quad \mathbf{curl}\,\boldsymbol{\Pi}_h^W = \boldsymbol{\Pi}_h^V\,\mathbf{curl}, \quad \mathrm{div}\,\boldsymbol{\Pi}_h^V = \Pi_h^Q\,\mathrm{div}$$

all hold. It is a straightforward and standard argument to verify these identities, and we therefore omit the details here.

In the analysis for the finite element solutions, we need the corresponding spaces to have homogeneous boundary conditions. Hence, the complex (7) should be replaced by

$$0 \xrightarrow{\subset} H_0^2 \xrightarrow{\mathbf{grad}} \boldsymbol{H}_0^1(\mathbf{curl}) \xrightarrow{\mathbf{curl}} \boldsymbol{H}_0^1 \xrightarrow{\mathrm{div}} L_0^2 \longrightarrow 0, \tag{27}$$

where

$$\boldsymbol{H}_0^1(\mathbf{curl}) = \{\boldsymbol{w} \in \boldsymbol{H}_0^1 : (\mathbf{curl}\,\boldsymbol{w})_t = 0 \quad \text{on } \partial\Omega\}.$$

A corresponding discrete, nonconforming, approximation is obtained by restricting the spaces S_h, $\boldsymbol{W}_h$, and $\boldsymbol{V}_h$ to the subspaces with vanishing degrees of freedom on the boundary $\partial\Omega$. For example, the space $\boldsymbol{V}_h$ is replaced by $\boldsymbol{V}_{h,0}$ given as all $\boldsymbol{v} \in \boldsymbol{V}_h$ such that

$$\int_f (\boldsymbol{v}\cdot\boldsymbol{n})p\,d\boldsymbol{x}_f = 0\,, \quad p \in \mathbb{P}_1(f), \qquad \int_f \boldsymbol{v}_t \cdot \boldsymbol{r}\,d\boldsymbol{x}_f = 0\,, \quad \boldsymbol{r} \in \boldsymbol{RM}(f),$$

for all faces f in $\partial\Omega$. Hence, $\boldsymbol{v}\cdot\boldsymbol{n}$ vanishes on the boundary for any $\boldsymbol{v} \in \boldsymbol{V}_{h,0}$, while the tangential component is zero in a weak sense. Hence, $\boldsymbol{V}_{h,0}$ is contained in $\boldsymbol{H}_0(\mathrm{div})$, where

$$\boldsymbol{H}_0(\mathrm{div}) = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}) : \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega\}.$$

Furthermore, $\boldsymbol{V}_{h,0}$ is a nonconforming approximation of $\boldsymbol{H}_0^1$. We also let $Q_{h,0} = Q_h \cap L_0^2$.

As above we obtain the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & H_0^2 \arrow[r, "\mathbf{grad}"] \arrow[d, "\Pi_h^S"] & \boldsymbol{H}_0^1(\mathbf{curl}) \arrow[r, "\mathbf{curl}"] \arrow[d, "\boldsymbol{\Pi}_h^W"] & \boldsymbol{H}_0^1 \arrow[r, "\mathrm{div}"] \arrow[d, "\boldsymbol{\Pi}_h^V"] & L_0^2 \arrow[r] \arrow[d, "\Pi_h^Q"] & 0 \\
0 \arrow[r] & S_{h,0} \arrow[r, "\mathbf{grad}"] & \boldsymbol{W}_{h,0} \arrow[r, "\mathbf{curl}"] & \boldsymbol{V}_{h,0} \arrow[r, "\mathrm{div}"] & Q_{h,0} \arrow[r] & 0
\end{tikzcd}$$

where the upper and lower rows are complexes.

4 Uniform error estimates for the Darcy–Stokes system

In this section we discuss how the finite element space $\boldsymbol{V}_{h,0} \times Q_{h,0}$ contained in $\boldsymbol{H}_0(\mathrm{div}) \times L_0^2$ can be used to construct a discretization of the singular perturbation problem (1) in $\mathbb{R}^3$ with uniform convergence properties with respect to the parameter ε. The results are similar to the corresponding results obtained in [10] for the two-dimensional case. Therefore, the discussion here is brief, and we only focus attention on the parts where the analysis from [10] needs to be modified essentially.

In order to avoid technical difficulties, we restrict the discussion to the case when the source term $g = 0$. The standard weak formulation of the system (1) is to find $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1 \times L_0^2$ such that

$$\begin{aligned} a_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) + (p, \mathrm{div}\, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v})\ , \qquad && \boldsymbol{v} \in \boldsymbol{H}_0^1, \\ (\mathrm{div}\, \boldsymbol{u}, q) &= 0\ , && q \in L_0^2. \end{aligned} \tag{28}$$

Here we assume that data $\boldsymbol{f} \in \boldsymbol{H}^{-1} \equiv (\boldsymbol{H}_0^1)^*$ and a_ε is the bilinear form

$$a_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v}) + \varepsilon^2(\boldsymbol{D}\boldsymbol{u}, \boldsymbol{D}\boldsymbol{v}),$$

defined on $\boldsymbol{H}_0^1 \times \boldsymbol{H}_0^1$. The corresponding finite element solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_{h,0} \times Q_{h,0}$ is given by the equation

$$\begin{aligned} a_\varepsilon(\boldsymbol{u}_h, \boldsymbol{v}) + (p_h, \mathrm{div}\, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v})\ , \qquad && \boldsymbol{v} \in \boldsymbol{V}_{h,0}, \\ (\mathrm{div}\, \boldsymbol{u}_h, q) &= 0\ , && q \in Q_{h,0}. \end{aligned} \tag{29}$$

As $\boldsymbol{V}_{h,0}$ is nonconforming, the bilinear form $a_\varepsilon(\cdot, \cdot)$ is understood to be the sum of the corresponding integrals over each tetrahedron of $\mathcal{T}_h$. Recall that for a smooth vector field $\boldsymbol{v}$

$$\boldsymbol{\Delta}\boldsymbol{v} = \mathbf{grad}\, \mathrm{div}\, \boldsymbol{v} - \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{v},$$

and, as a consequence,

$$(\boldsymbol{D}\boldsymbol{u}, \boldsymbol{D}\boldsymbol{v}) = (\mathrm{div}\, \boldsymbol{u}, \mathrm{div}\, \boldsymbol{v}) + (\mathbf{curl}\, \boldsymbol{u}, \mathbf{curl}\, \boldsymbol{v}), \boldsymbol{u} \in \boldsymbol{H}_0^1, \boldsymbol{v} \in \boldsymbol{H}^1.$$

If $\boldsymbol{u}$ is the solution of (28), we define the consistency error by

$$E_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) = a_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) + (p, \operatorname{div} \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v}), \quad \boldsymbol{v} \in \boldsymbol{V}_{h,0}. \tag{30}$$

From Green's formula we obtain

$$E_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) = \varepsilon^2 \sum_{f \in \Delta_2^h} \int_f (\mathbf{curl}\, \boldsymbol{u}) \;\; [\boldsymbol{v} \times \boldsymbol{n}]\, dx_f. \tag{31}$$

Here, Δ_2^h denotes all the faces for the tetrahedral mesh $\mathcal{T}_h$.

The uniform error analysis of the discretization of the system (1) is based on the ε-dependent function space $(\boldsymbol{H}_0(\operatorname{div}) \cap \varepsilon \cdot \boldsymbol{H}_0^1) \times L_0^2$. The corresponding norm is given by

$$|||\boldsymbol{v}|||_\varepsilon^2 = \|\boldsymbol{v}\|_0^2 + \|\operatorname{div} \boldsymbol{v}\|_0^2 + \varepsilon^2 \|\boldsymbol{D}\boldsymbol{v}\|_0^2.$$

For convenience we also introduce $\|\cdot\|_a$ as the norm associated with the bilinear form a_ε. For elements of $\boldsymbol{V}_{h,0}$ these norms should be interpreted as the corresponding broken norms.

Using the commuting diagram property $\operatorname{div} \boldsymbol{\Pi}_h^V = \Pi_h^Q \operatorname{div}$ and the $\boldsymbol{H}^1$ boundedness of $\boldsymbol{\Pi}_h^V$ we obtain that there exists a constant $\alpha_1 > 0$, independent of h and ε, such that

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}_{h,0}} \frac{(q, \operatorname{div} \boldsymbol{v})}{|||\boldsymbol{v}|||_\varepsilon} \geq \sup_{\boldsymbol{v} \in \boldsymbol{V}_{h,0}} \frac{(q, \operatorname{div} \boldsymbol{v})}{|||\boldsymbol{v}|||_1} \geq \alpha_1 \|q\|_0 \quad \text{for all } q \in Q_{h,0}. \tag{32}$$

Hence, the proper uniform inf–sup condition is satisfied.

Remark 2 Recall from [10] that most standard Stokes elements do not lead to a uniformly stable discretization in the present case. This is due to the fact that the bilinear form a_ε is not uniformly coercive with respect to the energy norm $|||\cdot|||_\varepsilon$ on the space of weakly divergence-free vector fields, i.e., the second Brezzi condition is violated. However, in the present case, where the divergence operator maps $\boldsymbol{V}_{h,0}$ onto $Q_{h,0}$, this condition is obvious. In particular, $\operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{u}_h = 0$.

Taking $\boldsymbol{v} = \boldsymbol{\Pi}_h^V \boldsymbol{u} - \boldsymbol{u}_h$ in the first equation of (29) and (30), we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_a \leq 2\left(\|\boldsymbol{u} - \boldsymbol{\Pi}_h^V \boldsymbol{u}\|_a + \sup_{\boldsymbol{v} \in \boldsymbol{V}_{h,0}} \frac{|E_{\varepsilon,h}(\boldsymbol{u}, \boldsymbol{v})|}{\|\boldsymbol{v}\|_a} \right). \tag{33}$$

Since the polynomial space $\boldsymbol{V}(T)$ contains all linears, and the family $\{\mathcal{T}_h\}$ is shape-regular, we see from a scaling argument that

$$\|\boldsymbol{\Pi}_h^V \boldsymbol{v} - \boldsymbol{v}\|_a \leq c(h^2 + \varepsilon h)\|\boldsymbol{v}\|_2, \quad \boldsymbol{v} \in \boldsymbol{H}^2 \cap \boldsymbol{H}_0^1, \tag{34}$$

where the constant c is independent of $\boldsymbol{v}$, ε and h. Under the assumption that the solution $\boldsymbol{u}$ of (28) is in $\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1$ we can use a trace theorem and a scaling argument (cf. [10, Lemma 5.1]), to conclude that

$$\sup_{\boldsymbol{v}\in \boldsymbol{V}_{h,0}} \frac{|E_\varepsilon(\boldsymbol{u},\boldsymbol{v})|}{\|\boldsymbol{v}\|_a} \le c\,\varepsilon \begin{cases} h\|\,\mathbf{curl}\,\boldsymbol{u}\|_1 \\ h^{1/2}\|\,\mathbf{curl}\,\boldsymbol{u}\|_1^{1/2}\|\,\mathbf{curl}\,\boldsymbol{u}\|_0^{1/2}. \end{cases} \tag{35}$$

By combining (32)–(35) we obtain the following error estimates (cf. [10, Theorem 5.1]):

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 + \varepsilon\|\,\mathbf{curl}(\boldsymbol{u}-\boldsymbol{u}_h)\|_0 \le c(h^2+\varepsilon h)\|\boldsymbol{u}\|_2, \tag{36}$$

$$\|p-p_h\|_0 \le c\,h(\|p\|_1 + (\varepsilon+h)\|\boldsymbol{u}\|_2). \tag{37}$$

These estimates are uniform in the sense that the constant c is independent of $\boldsymbol{u}$, ε and h. However, in general the term $\|\boldsymbol{u}\|_2$ is not bounded uniformly in ε. A real uniform estimate, corresponding to a result obtained in [10] in the two-dimensional case, is of the form

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 + \varepsilon\|\,\boldsymbol{u}-\boldsymbol{u}_h\|_1 + \|p-p_h\|_0 \le c\,h^{1/2}\|\boldsymbol{f}\|_1. \tag{38}$$

As illustrated by the development in [10] the key ingredient in deriving such an estimate is proper uniform bounds on the solution $\boldsymbol{u}$; cf. Lemma 5 below. However, the argument given in [10] for this result cannot easily be extended to three dimensions. For completeness, we therefore give an alternative proof here, valid in both two and three dimensions.

4.1 A uniform estimate

Throughout this section we assume that the domain Ω is a convex polyhedron. Then we have the following uniform error estimate.

Theorem 1 *If Ω is convex and $\boldsymbol{f}\in \boldsymbol{H}^1(\Omega)$, then the uniform error estimate (38) holds with a constant c independent of h and ε.*

The key ingredient to prove this result is a uniform regularity estimate. By an energy argument it is straightforward to show that the weak solution $(\boldsymbol{u},p)$ of (1) satifies the uniform bound

$$|\!|\!|\boldsymbol{u}|\!|\!|_\varepsilon + \|p\|_0 \le c\,\|\boldsymbol{f}\|_0. \tag{39}$$

Hence, for a fixed $\boldsymbol{f}\in \boldsymbol{L}^2$, the quantity $\|\boldsymbol{D}\boldsymbol{u}\|_0$ is at most proportional to ε^{-1} as ε tends to zero. However, if $\boldsymbol{f}$ is more regular an improved estimate can be obtained. To see this we let $(\boldsymbol{u}^0, p^0) \in \boldsymbol{H}_0(\mathrm{div}) \times L_0^2$ be the weak solution of the corresponding reduced problem:

$$\begin{aligned} \boldsymbol{u}^0 - \mathbf{grad}\, p^0 &= \boldsymbol{f} && \text{in } \Omega, \\ \mathrm{div}\, \boldsymbol{u}^0 &= 0 && \text{in } \Omega, \\ \boldsymbol{u}^0 \cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{40}$$

Below we consider this problem with $\boldsymbol{f} \in \boldsymbol{H}^1$. Hence, it is an immediate consequence of elliptic regularity that $(\boldsymbol{u}^0, p^0)$ is a classical solution in $(\boldsymbol{H}^1 \cap \boldsymbol{H}_0(\mathrm{div})) \times H^2$, with corresponding norms depending continuously on $\|\boldsymbol{f}\|_1$.

Lemma 5 *Assume that Ω is convex and that $\boldsymbol{f} \in \boldsymbol{H}^1$. There exists a constant $c > 0$, independent of ε and $\boldsymbol{f}$, such that*

$$\varepsilon^{1/2}\|\boldsymbol{u}\|_1 + \varepsilon^{3/2}\|\boldsymbol{u}\|_2 \le c\,\|\boldsymbol{f}\|_1, \tag{41}$$

$$\|\boldsymbol{u} - \boldsymbol{u}^0\|_0 + \|p - p^0\|_1 \le c\,\varepsilon^{1/2}\|\boldsymbol{f}\|_1. \tag{42}$$

Proof When Ω is convex the solution of the standard Stokes problem

$$\begin{aligned} -\Delta\bar{\boldsymbol{u}} - \mathbf{grad}\, \bar{p} &= \boldsymbol{f} && \text{in } \Omega, \\ \mathrm{div}\, \bar{\boldsymbol{u}} &= 0 && \text{in } \Omega, \\ \bar{\boldsymbol{u}} &= 0 && \text{on } \partial\Omega \end{aligned} \tag{43}$$

satisfies the regularity estimate (cf. [8])

$$\|\bar{\boldsymbol{u}}\|_2 + \|\bar{p}\|_1 \le c\,\|\boldsymbol{f}\|_0. \tag{44}$$

By considering the pair $(\boldsymbol{u}, \varepsilon^{-2}(p - p^0))$ as a weak solution of the system

$$\begin{aligned} -\Delta\boldsymbol{u} - \mathbf{grad}(\varepsilon^{-2}(p - p^0)) &= -\varepsilon^{-2}(\boldsymbol{u} - \boldsymbol{u}^0)\ , \\ \mathrm{div}\, \boldsymbol{u} &= 0, \end{aligned} \tag{45}$$

we see from (44) that $\boldsymbol{u} \in \boldsymbol{H}_0^1 \cap \boldsymbol{H}^2$, $p \in L_0^2 \cap H^1$ and

$$\varepsilon^2\|\boldsymbol{u}\|_2 + \|p - p^0\|_1 \le c\,\|\boldsymbol{u} - \boldsymbol{u}^0\|_0 \tag{46}$$

with constant c independent of ε. Due to the enhanced regularity of the solution $\boldsymbol{u}$ we obtain from (45) that

$$a_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) - (\boldsymbol{u}^0, \boldsymbol{v}) + (p - p^0, \mathrm{div}\, \boldsymbol{v}) = \left\langle \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}, \boldsymbol{v} \right\rangle, \quad \boldsymbol{v} \in \boldsymbol{H}^1 \cap \boldsymbol{H}_0(\mathrm{div}),$$

where, $\langle \cdot, \cdot \rangle$ is the L^2 inner product on $\partial\Omega$. For $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}^0$ this gives

$$\|\boldsymbol{u} - \boldsymbol{u}^0\|_0^2 + \varepsilon^2\|\boldsymbol{D}\boldsymbol{u}\|_0^2 = \varepsilon^2(\boldsymbol{D}\boldsymbol{u}, \boldsymbol{D}\boldsymbol{u}^0) + \varepsilon^2\left\langle \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}, \boldsymbol{u}^0 \right\rangle. \tag{47}$$

By using a standard trace inequality we further obtain

$$
\begin{aligned}
\varepsilon^2 \langle \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}, \boldsymbol{u}^0 \rangle &\le c\, \varepsilon^2 \|\boldsymbol{u}\|_1^{1/2} \|\boldsymbol{u}\|_2^{1/2} \|\boldsymbol{u}^0\|_1 \\
&\le c_\delta \varepsilon \|\boldsymbol{u}^0\|_1^2 + \delta \varepsilon^3 \|\boldsymbol{u}\|_1 \|\boldsymbol{u}\|_2 \\
&\le c_\delta \varepsilon \|\boldsymbol{u}^0\|_1^2 + \frac{\delta \varepsilon^2}{2} \|\boldsymbol{u}\|_1^2 + \frac{\delta \varepsilon^4}{2} \|\boldsymbol{u}\|_2^2 \\
&\le c_\delta \varepsilon \|\boldsymbol{f}\|_1^2 + C\, \delta \left(\varepsilon^2 \|\boldsymbol{D}\boldsymbol{u}\|_0^2 + \|\boldsymbol{u} - \boldsymbol{u}^0\|_0^2 \right),
\end{aligned} \tag{48}
$$

where we have used (46) and the H^1-regularity of $\boldsymbol{u}^0$ in the last step. Here both the constants C and c_δ are independent of ε, but c_δ depends continuously on δ. For the first term on the right-hand side of (47) we have

$$
\varepsilon^2 (\boldsymbol{D}\boldsymbol{u}, \boldsymbol{D}\boldsymbol{u}^0) \le \frac{\varepsilon^2}{4} \|\boldsymbol{D}\boldsymbol{u}\|_0^2 + \varepsilon^2 \|\boldsymbol{D}\boldsymbol{u}^0\|_0^2 \le \frac{\varepsilon^2}{4} \|\boldsymbol{D}\boldsymbol{u}\|_0^2 + c\, \varepsilon^2 \|\boldsymbol{f}\|_1^2,
$$

where the constant c is independent of ε. However, together with (47) and (48), and by choosing δ sufficiently small, this implies that

$$
\|\boldsymbol{u} - \boldsymbol{u}^0\|_0^2 + \varepsilon^2 \|\boldsymbol{D}\boldsymbol{u}\|_0^2 \le c\, \varepsilon \|\boldsymbol{f}\|_1^2 \tag{49}
$$

with c independent of ε. Together with (46) this implies the desired estimates (41) and (42). □

Proof of Theorem 1: From Lemma 5 and (35), we see that

$$
\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_{h,0}} \frac{|E_\varepsilon(\boldsymbol{u}, \boldsymbol{v}_h)|}{\|\boldsymbol{v}_h\|_a} \le c\, h^{\frac12} \varepsilon \|\,\mathbf{curl}\, \boldsymbol{u}\|_1^{\frac12} \|\,\mathbf{curl}\, \boldsymbol{u}\|_0^{\frac12} \le c\, h^{\frac12} \|\boldsymbol{f}\|_1.
$$

Furthermore, since the interpolation operator $\boldsymbol{\Pi}_h^V$ is defined from traces on the two-dimensional faces in Δ_2^h, the interpolation estimate

$$
\|\boldsymbol{\Pi}_h^V \boldsymbol{v} - \boldsymbol{v}\|_0 \le c\, h^{1/2} \|\boldsymbol{v}\|_0^{1/2} \|\boldsymbol{v}\|_1^{1/2},
$$

follows from a standard trace inequality and scaling. From this estimate, and by arguing exactly as in the proof of Theorem 6.1 of [10], we derive

$$
\|\boldsymbol{u} - \boldsymbol{\Pi}_h^V \boldsymbol{u}\|_0 + \varepsilon \|\boldsymbol{u} - \boldsymbol{\Pi}_h^V \boldsymbol{u}\|_1 \le c\, h^{\frac12} \|\boldsymbol{f}\|_1.
$$

Combining the two estimates above with the inf-sup condition (32) and error bound (33), we obtain the desired uniform estimate (38). □

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