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CONVERGENCE RATE ANALYSIS OF A MULTIPLICATIVE SCHWARZ METHOD FOR VARIATIONAL INEQUALITIES*

LORI BADEA[†], XUE-CHENG TAI[‡], AND JUNPING WANG[§]

Abstract. This paper derives a linear convergence for the Schwarz overlapping domain decomposition method when applied to constrained minimization problems. The convergence analysis is based on a minimization approach to the corresponding functional over a convex set. A general framework of convergence is established for some multiplicative Schwarz algorithm. The abstract theory is particularly applied to some obstacle problems, which yields a linear convergence for the corresponding Schwarz overlapping domain decomposition method of one and two levels. Numerical experiments are presented to confirm the convergence estimate derived in this paper.

Key words. domain decomposition, variational inequalities, finite element methods, obstacle problems

AMS subject classifications. 65N55, 65N30, 65J15

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1. Introduction. The study of domain decomposition methods was motivated by the increasing need of fast numerical solutions for problems in science and engineering. Such practical problems are often of very large scale and are extremely difficult to solve by using classical approaches. The domain decomposition method has the capability of providing new numerical algorithms which are efficient and parallelizable. The Schwarz overlapping domain decomposition method represents a typical thinking of parallelization and shall be the main focus of this paper.

The Schwarz method consists of two categories which have been traditionally classified as multiplicative and additive methods. The multiplicative Schwarz replicates the well-known Gauss–Seidel iteration for linear systems in a block fashion, while the additive Schwarz method resembles the Jacobi iteration in numerical linear algebra. Both methods have been well studied for second order elliptic problems for the last two decades. Details can be found from [2, 4, 6, 13, 14, 15, 16, 18, 20, 32] and the references cited therein. However, to the authors’ knowledge, there are very few existing results which are satisfactorily developed for the Schwarz method when applied to constrained minimization problems.

The main objective of this paper is to establish a convergence rate estimate for the overlapping domain decomposition method for variational inequalities. The result is inspired by the classical analysis of [4] for linear second order elliptic problems and extends some of the new techniques for nonlinear problems of [26, 27, 29, 30]. The essential idea is to decompose the global approximating space into subspaces, which

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[†]Institute of Mathematics, Romanian Academy of Sciences, P.O. Box 1-764, RO-70700 Bucharest, Romania (lbadea@imar.ro). The research of this author was supported by IMAR under contract ICA1-CT-2000-70022 with the European Commission and by the Norwegian Research Council under contract 128224/431.

[‡]Department of Mathematics, University of Bergen, Johannes Brunsgate 12, 5007, Bergen, Norway (tai@mi.uib.no). The research of this author was supported by IMAR under contract ICA1-CT-2000-70022 with the European Commission and by the Norwegian Research Council under contract 128224/431.

[§]Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, CO 80401 (jwang@mines.edu).

is the key idea behind the latest convergence analysis for domain decomposition and multigrid methods. We shall first establish an abstract framework for the convergence of general minimization problems and then apply it to some obstacle problems by verifying the assumptions of the abstract theory.

A brief review of the existing work on the domain decomposition methods for variational inequalities is as follows. In [1], Badea proved a convergence of a domain decomposition algorithm which is based on minimizing quadratic functionals in a Hilbert space. A convergence rate was established there by using the maximum principle for the problem. A similar method was later proposed and analyzed in [3] as a new member of the additive Schwarz methods. Various one-level overlapping domain decomposition methods have been studied in [10, 12, 17, 19, 21, 24, 25]. A linear convergence for the one-level overlapping domain decomposition method was derived recently in [29, 22, 33] under the condition that the iterative solution increases or decreases monotonically to the true solution. It is known that we can linearize the obstacle problem first and then apply domain decomposition methods for the linearized problems; see, for example, [11]. Our approach is applied directly to the obstacle problem, and no linearization is necessary in the domain decomposition scheme.

Both the one-level and two-level domain decomposition methods are considered in this paper. As it is well known, the two-level method makes use of a coarse level, and its convergence is quite challenging in theory. In fact, the convergence for two-level algorithms has not been fully understood so far in the literature. The only ones we know are from [26, 27, 28]; see also [31] for a two-level algebraic method for the Signorini problem. The method proposed in [26, 27] relies on a decomposition of the convex set, which is different from the algorithm to be studied in the present paper. For the approach to be taken here, the subproblems can be solved in parallel or sequentially. Numerical tests and convergence rate analysis for the parallel version have been done in [28] for domain decomposition and multigrid methods. In this paper, we shall give a convergence rate estimate for the sequential method and concentrate only on the one-level and two-level domain decomposition methods. To the authors' knowledge, our result is the first that gives an explicit convergence rate estimate for this two-level Schwarz method for variational inequalities. For the one-level method, our estimate does not require any monotone property of the iterative solution. Moreover, we give an explicit relation between the convergence rate and the overlapping size. The convergence rate analysis for the multigrid method with the sequential approach is much more difficult and remains open.

This paper is organized as follows. In section 2, we present some abstract domain decomposition algorithms for general convex minimization problems over convex constraint sets. In section 3, we state an abstract result of convergence based on some assumptions for the spatial decomposition. In section 4, we apply the abstract convergence result to a specific obstacle problem by verifying all the conditions required for the abstract theory. To validate our convergence theory, we present some numerical results in section 5 for a two-sided obstacle problem. Finally, in section 6, we provide a complete proof for the main convergence estimate for constrained minimization problems.

2. Algorithm description. Given a reflexive Banach space V and a convex functional $F : V \mapsto \mathbb{R}$, we consider the following optimization problem:

$$(2.1) \quad \min_{v \in K} F(v), \quad K \subset V,$$

where K is a closed convex subset of V . We are interested in the case where the space V can be decomposed into a sum of subspaces V_i , i.e.,

$$(2.2) \quad V = V_1 + V_2 + \cdots + V_m = \sum_{i=1}^m V_i .$$

This means that for any $v \in V$, there exists $v_i \in V_i$ such that $v = \sum_{i=1}^m v_i$.

With the decomposition (2.2), there are two different ways to solve the nonlinear problem (2.1). The first approach is to decompose K into a sum of $K_i \subset V_i$, $i = 1, 2, \dots, m$, i.e.,

$$K = K_1 + K_2 + \cdots + K_m = \sum_{i=1}^m K_i,$$

and then to solve a minimization problem over each subset K_i in parallel or sequentially. The convergence rate analysis and numerical experiments for this approach have been conducted in [26, 27]. The approach of [26] could handle one- and two-level domain decomposition methods as well as the multigrid method. The second approach does not involve any decomposition of the convex set K and is illustrated in Algorithms 1 and 2.

ALGORITHM 1. For a given $u^n \in K$ and $\rho \in (0, 1/m)$, compute $e_i^{n+1} \in V_i$ in parallel for $i = 1, 2, \dots, m$ such that

$$(2.3) \quad e_i^{n+1} = \arg \min_{v_i + u^n \in K, v_i \in V_i} G(v_i) \quad \text{with} \quad G(v_i) = F(u^n + v_i)$$

and then update

$$u^{n+1} := u^n + \rho \sum_{i=1}^m e_i^{n+1}.$$

ALGORITHM 2. For a given $u^n \in K$, compute $e_i^{n+1} \in V_i$ sequentially for $i = 1, 2, \dots, m$ such that

$$(2.4) \quad e_i^{n+1} = \arg \min_{v_i + u^{n + \frac{i-1}{m}} \in K, v_i \in V_i} G(v_i) \quad \text{with} \quad G(v_i) = F(u^{n + \frac{i-1}{m}} + v_i)$$

and update

$$u^{n + \frac{i}{m}} := u^{n + \frac{i-1}{m}} + e_i^{n+1}.$$

The algorithms introduced in [1] and [3] are in the same spirit as Algorithms 1 and 2. A convergence rate analysis for Algorithm 1 has been established in [28] for domain decomposition and multigrid methods and in [3] for domain decomposition methods. The objective of this paper is to study Algorithm 2 and derive a linear convergence. The conditions for the convergence of Algorithm 2 differ from those for Algorithm 1. In addition, the analysis turns out to be more complicated than Algorithm 1. The techniques used in the analysis are extensions of those presented in [26, 29, 30].

3. An abstract theory of convergence. Assume that the minimization functional F is Gâteaux differentiable (see [8]) and that there exists a constant $\kappa > 0$ such that

$$(3.1) \quad \langle F'(w) - F'(v), w - v \rangle \geq \kappa \|w - v\|_V^2 \quad \forall w, v \in V.$$

Here $\langle \cdot, \cdot \rangle$ is the duality pairing between V and its dual space V' , i.e., the value of a linear function at an element of V . Under the condition (3.1), problem (2.1) has a unique solution; see [8, p. 35]. For some nonlinear problems, the constant κ may depend on v and w and the analysis given here is still applicable; see [30, Rem. 2.1] for more information. Our abstract convergence theory is based on the following two assumptions inspired from [1].

Assumption 1. There exists a constant $C_1 > 0$ such that for any $w, v \in K$ and $s_i \in V_i$ with $w + \sum_{j=1}^i s_j \in K, i = 1, \dots, m$, there exist $z_i \in V_i$ satisfying

$$(3.2) \quad \begin{cases} \text{(a) } v - w = \sum_{i=1}^m z_i, & \text{(b) } w + \sum_{j=1}^{i-1} s_j + z_i \in K \text{ for } i = 1, \dots, m, \\ \text{(c) } \left(\sum_{i=1}^m \|z_i\|_V^2 \right)^{\frac{1}{2}} \leq C_1 \left(\|v - w\|_V^2 + \sum_{j=1}^m \|s_j\|_V^2 \right)^{\frac{1}{2}}. \end{cases}$$

Assumption 2. There exists a constant $C_2 > 0$ which is the least constant satisfying the following inequality for any $w_{ij} \in V, u_i \in V_i$, and $v_j \in V_j$:

$$(3.3) \quad \sum_{i,j=1}^m |\langle F'(w_{ij} + u_i) - F'(w_{ij}), v_j \rangle| \leq C_2 \left(\sum_{i=1}^m \|u_i\|_V^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^m \|v_j\|_V^2 \right)^{\frac{1}{2}}.$$

Let u be the unique solution of (2.1). Our main result of the convergence estimate can be stated as follows.

THEOREM 3.1. *Assume that the space decomposition satisfies (3.2), (3.3), and assume that the functional F satisfies (3.1). Then for the iterative approximation $\{u^n\}_{n=1}^\infty$ given by Algorithm 2, we have*

$$(3.4) \quad \frac{|F(u^{n+1}) - F(u)|}{|F(u^n) - F(u)|} \leq 1 - \frac{1}{(\sqrt{1 + C^*} + \sqrt{C^*})^2}$$

and

$$(3.5) \quad \|u^n - u\|_V^2 \leq \frac{2}{\kappa} \left[1 - \frac{1}{(\sqrt{1 + C^*} + \sqrt{C^*})^2} \right]^n |F(u^0) - F(u)|,$$

where

$$(3.6) \quad C^* = \left((1 + C_1)C_2 + \frac{(C_1C_2)^2}{2\kappa} \right) \frac{2}{\kappa}.$$

In order to prove the theorem, we need to combine the special assumption (3.2) with the techniques used in [26]. The proof is tedious and rather complex, and it is postponed to section 6.

4. Application to obstacle problems. The objective of this section is to apply the abstract convergence theory to obstacle problems and to derive a linear convergence for the corresponding domain decomposition algorithm. To this end, let $\Omega \subset \mathbb{R}^d$ be an open bounded and connected domain with a polyhedral boundary. Consider the problem that seeks an unknown function $u = u(x)$ on Ω satisfying

$$(4.1) \quad a(u, v - u) \geq f(v - u) \quad \forall v \in K,$$

where

$$(4.2) \quad \begin{aligned} a(v, w) &= \int_{\Omega} \nabla v \cdot \nabla w \, dx, \\ K &= \{v \in H_0^1(\Omega) \mid \alpha(x) \leq v(x) \leq \beta(x) \text{ a.e. in } \Omega\}, \end{aligned}$$

$\alpha(x)$ and $\beta(x)$ are two obstacle functions in $L^\infty(\Omega)$, and $f(\cdot)$ is a bounded linear functional on the Sobolev space $H_0^1(\Omega)$. It is well known that the above problem is equivalent to the following minimization problem (see [9], for instance):

$$(4.3) \quad \min_{v \in K} F(v), \quad F(v) = \frac{1}{2}a(v, v) - f(v).$$

For the obstacle problem (4.1), the reflexive Banach space is given by $V = H_0^1(\Omega)$. Correspondingly, we have $\kappa = 1$ in assumption (3.1). We point out that our algorithms and the convergence estimate presented in the previous section are valid for a general class of optimization problems in which the optimization functional F is a strongly convex functional satisfying (3.1).

We use the standard notation for Sobolev spaces $H_0^k(\Omega)$ and $W_0^{k,p}(\Omega)$ and their norms and seminorms. In particular, for a given subdomain $D \subset \Omega$ and $v \in H_0^1(D)$, we shall always extend v with zero in $\Omega \setminus D$, i.e.,

$$H_0^1(D) = \{v \mid v \in H^1(\Omega), \quad v = 0 \text{ in } \Omega \setminus D\}.$$

Throughout the paper, C will be used to denote a generic constant that does not depend on mesh parameters of the finite element partitions introduced later.

4.1. Numerical approximation and technical tools. The domain Ω is first partitioned into a coarse mesh denoted \mathcal{T}_H with a mesh size H . Next, we refine the partition \mathcal{T}_H and obtain a fine mesh partition \mathcal{T}_h with a mesh size $h < H$. We assume that both the coarse and fine meshes are shape-regular (see [7]).

Let $S_H \subset W_0^{1,\infty}(\Omega)$ and $S_h \subset W_0^{1,\infty}(\Omega)$ be the continuous, piecewise linear finite element spaces associated with \mathcal{T}_H and \mathcal{T}_h , respectively. More precisely, we have

$$S_H = \left\{ v \in W_0^{1,\infty}(\Omega) \mid v|_{\tau} \in P_1(\tau) \forall \tau \in \mathcal{T}_H \right\}$$

and

$$S_h = \left\{ v \in W_0^{1,\infty}(\Omega) \mid v|_{\tau} \in P_1(\tau) \forall \tau \in \mathcal{T}_h \right\}.$$

The obstacle problem (4.1) is approximated by a finite element function $u_h(x) \in K \cap S_h$ satisfying

$$(4.4) \quad a(u_h, v - u_h) \geq f(v - u_h) \quad \forall v \in K \cap S_h.$$

Let $\{\Omega_i\}_{i=1}^M$ be a nonoverlapping domain decomposition for Ω , and each Ω_i is the union of some coarse mesh elements. For each Ω_i , we consider an enlarged subdomain Ω_i^δ consisting of elements $\tau \in \mathcal{T}_h$ with $\text{dist}(\tau, \Omega_i) \leq \delta \leq H$. The union of Ω_i^δ covers $\bar{\Omega}$ with overlaps of size δ . Let us denote the piecewise linear finite element space with vanishing values on the boundary $\partial\Omega_i^\delta$ as $S_h(\Omega_i^\delta)$. It is not hard to show that

$$(4.5) \quad S_h = \sum_{i=1}^M S_h(\Omega_i^\delta) \quad \text{and} \quad S_h = S_H + \sum_{i=1}^M S_h(\Omega_i^\delta).$$

For the overlapping subdomains, assume that there exist m colors such that each subdomain Ω_i^δ can be marked with one color, and the subdomains with the same color will not intersect with each other. For suitable decompositions, one can choose $m = 2$ if $d = 1$, $m \leq 4$ if $d = 2$, and $m \leq 8$ if $d = 3$. Let Ω_i^c be the union of the subdomains with the i th color, and

$$V_i = \{v \in S_h \mid v(x) = 0, \quad x \notin \Omega_i^c\}$$

for $i = 1, 2, \dots, m$. By denoting $V_0 = S_H$ and $V = S_h$, we see from (4.5) that

$$(4.6) \quad \text{(a) } V = \sum_{i=1}^m V_i \quad \text{and} \quad \text{(b) } V = V_0 + \sum_{i=1}^m V_i.$$

Associated with the subdomains, we consider some functions $\theta_j^i \in C^1(\bar{\Omega})$, $i = 1, 2, \dots, m$, $j = i, \dots, m$, such that for any $i = 1, 2, \dots, m$ we have

$$(4.7) \quad \text{supp}(\theta_j^i) \subset \bar{\Omega}_j^c, \quad 0 \leq \theta_j^i \leq 1 \quad \forall j = i, \dots, m, \quad \text{and} \quad \sum_{j=i}^m \theta_j^i = 1 \quad \text{in} \quad \bigcup_{j=i}^m \Omega_j^c.$$

More precisely, θ_j^1 is a partition of unity with respect to the subdomains Ω_j^c , $j = 1, 2, \dots, m$; θ_j^2 is a partition of unity with the subdomains Ω_j^c , $j = 2, \dots, m$; i.e., the subdomains with the first color are dropped. Accordingly, θ_j^i is a partition of unity with respect to the subdomains Ω_j^c , $j = i, \dots, m$, where the subdomains Ω_j^c , $j = 1, 2, \dots, i - 1$, are dropped. Due to the overlapping property, the preceding functions can be constructed to satisfy

$$(4.8) \quad |\nabla \theta_j^i| \leq C/\delta.$$

In the following, I_h denotes the Lagrangian interpolation operator which uses the function values at the nodes of a given mesh \mathcal{T}_h with a mesh size h . The following estimate is correct due to the special structure of the functions θ_j^i :

$$(4.9) \quad \|I_h(\theta_j^i v)\|_0 \leq C\|v\|_0, \quad |I_h(\theta_j^i v)|_1 \leq C\|v\|_1 + \frac{1}{\delta}\|v\|_0 \quad \forall i, j, \quad \forall v \in S_h.$$

We also need a nonlinear interpolation operator $I_H^\ominus : S_h \mapsto S_H$ introduced in [26, 27]. Denote $\mathcal{N}_H = \{x_0^i\}_{i=1}^{n_0}$ all the interior nodes for \mathcal{T}_H . For a given x_0^i , let ω_i be the union of the mesh elements of \mathcal{T}_H having x_0^i as one of its vertices, i.e.,

$$\omega_i := \bigcup \{\tau \in \mathcal{T}_H, x_0^i \in \bar{\tau}\}.$$

Let $\{\phi_0^i\}_{i=1}^{n_0}$ be the associated nodal basis functions. It is clear that ω_i is the support of ϕ_0^i . Given a nodal point $x_0^i \in \mathcal{N}_H$ and a $v \in S_h$, let $I_i v = \min_{\omega_i} v(x)$. The interpolation function is then defined as

$$(4.10) \quad I_H^\ominus v := \sum_{x_0^i \in \mathcal{N}_H} (I_i v) \phi_0^i(x).$$

From the definition, it is easy to see that

$$(4.11) \quad I_H^\ominus v \leq v \quad \forall v \in S_h,$$

$$(4.12) \quad I_H^\ominus v \geq 0 \quad \forall v \geq 0, v \in S_h.$$

Moreover, the interpolation for a given $v \in S_h$ on a finer mesh is always no smaller than the corresponding interpolation on a coarser mesh due to the fact that each coarser mesh element contains several finer mesh elements, i.e.,

$$(4.13) \quad I_{H_1}^\ominus v \leq I_{H_2}^\ominus v \quad \forall H_1 \geq H_2 \geq h, \quad \forall v \in S_h.$$

Define

$$c_d = \begin{cases} C & \text{if } d = 1, \\ C(1 + |\log(H/h)|^{\frac{1}{2}}) & \text{if } d = 2, \\ C(H/h)^{\frac{1}{2}} & \text{if } d = 3. \end{cases}$$

Using Lemma 2.3 in [5], it was proven in [26, 27] that the following approximation properties are correct for the nonlinear interpolation operator I_H^\ominus .

THEOREM 4.1. *For any $v, w \in S_h$, it is true that*

$$(4.14) \quad \|I_H^\ominus v - I_H^\ominus w - (v - w)\|_0 \leq c_d H |v - w|_1,$$

$$(4.15) \quad \|I_H^\ominus v - v\|_0 \leq c_d H |v|_1,$$

$$(4.16) \quad |I_H^\ominus v - I_H^\ominus w|_1 \leq c_d |v - w|_1.$$

4.2. Two-level domain decomposition methods. In the two-level domain decomposition method, the coarse level space S_H is used in the iterative scheme for correction. As a result, the analysis will be based on the space decomposition as given in (4.6.b). Our goal is to verify Assumptions 1 and 2. Notice that the verification for Assumption 2 is straightforward and is essentially the same as for linear problems. We are left with the verification of Assumption 1 by finding the smallest constant C_1 which satisfies (3.2). We use V_0 to denote the coarse mesh and, correspondingly, all the summation index in Assumptions 1 and 2 will start from 0 to m .

The following lemma is stated for a general convex constraint set K defined by constraints on the function values at the fine mesh nodes, and it originates from a similar one given in [1] for the Sobolev spaces. Assume that $v, w, w + \sum_{j=0}^i s_j \in K, s_i \in V_i, i = 0, 1, \dots, m$, holds true for a general convex subset. Choose a $v_0 \in V_0$ such that

$$(4.17) \quad v - v_0 \in K, \quad v_0 + w + s_0 \in K.$$

We then define $z_i, i = 0, 1, 2, \dots, m$, recursively by

$$(4.18) \quad z_0 = s_0 + v_0, \quad z_i = I_h \left(\theta_i^i \left(v - w - \sum_{j=0}^{i-1} z_j \right) + (1 - \theta_i^i) s_i \right), \quad i = 1, \dots, m.$$

LEMMA 4.2. For a general convex subset $K \subset H_0^1(\Omega)$, assume that $v, w, w + \sum_{j=0}^i s_j \in K, s_i \in V_i, i = 1, \dots, m$, and assume that v_0 satisfies (4.17). Then the functions $z_i, i = 1, \dots, m$, defined in (4.18) satisfy

$$(4.19) \quad z_i \in V_i, \quad z_i + w + \sum_{j=0}^{i-1} s_j \in K,$$

$$(4.20) \quad v - w - \sum_{j=0}^i z_j \in H_0^1 \left(\bigcup_{j=i+1}^m \Omega_j^c \right),$$

$$(4.21) \quad v - \sum_{j=0}^i z_j + \sum_{j=0}^i s_j \in K.$$

Proof. The conclusion shall be proved by induction. For $i = 1$, we get from (4.18) that

$$(4.22) \quad z_1 = I_h(\theta_1^1(v - w - z_0) + (1 - \theta_1^1)s_1).$$

Due to the fact that $\theta_1^1 = 0, s_1 = 0$ in $\Omega \setminus \Omega_1^c$, it is true that $z_i = 0$ in $\Omega \setminus \Omega_1^c$ and thus $z_1 \in V_1$. Using (4.17), the assumption that $w + s_0 + s_1 \in K$, and the fact that $0 \leq \theta_1^1 \leq 1$, it is not hard to see that

$$z_1 + w + s_0 = I_h(\theta_1^1(v - v_0) + (1 - \theta_1^1)(w + s_0 + s_1)) \in K.$$

As $I_h(v - w - z_0) = v - w - z_0$, one gets from (4.22) that

$$(4.23) \quad v - w - z_0 - z_1 = I_h((1 - \theta_1^1)(v - w - z_0 - s_1)).$$

From (4.7), one obtains that $\theta_1^1 = 1$ in $\Omega_1^c \setminus \cup_{j=2}^m \Omega_j$. Combining it with the above equality we get

$$(4.24) \quad v - w - z_0 - z_1 \in H_0^1 \left(\bigcup_{j=2}^m \Omega_j^c \right).$$

Furthermore, one gets from (4.17), the assumption that $w + s_0 + s_1 \in K$, the fact that $0 \leq \theta_1^1 \leq 1$, and (4.23) that

$$\begin{aligned} & v - z_0 - z_1 + s_0 + s_1 \\ &= I_h((1 - \theta_1^1)(v - z_0 + s_0) + \theta_1^1(w + s_0 + s_1)) \\ &= I_h((1 - \theta_1^1)(v - v_0) + \theta_1^1(w + s_0 + s_1)) \in K. \end{aligned}$$

In what follows, we shall assume that a z_i defined by (4.18) satisfies (4.19)–(4.21); then we shall prove that z_{i+1} also satisfies (4.19)–(4.21). From (4.18), we see that

$$(4.25) \quad z_{i+1} = I_h \left(\theta_{i+1}^{i+1} \left(v - w - \sum_{j=0}^i z_j \right) + (1 - \theta_{i+1}^{i+1})s_{i+1} \right).$$

Using the fact that

$$\theta_{i+1}^{i+1} \in H_0^1(\Omega_{i+1}^c), \quad s_{i+1} \in H_0^1(\Omega_{i+1}^c),$$

and from (4.20), we see that $z_{i+1} \in H_0^1(\Omega_{i+1}^c)$ and thus $z_{i+1} \in V_{i+1}$. In addition, one gets by using (4.20), (4.25), the assumption $w + \sum_{j=0}^i s_j \in K$, and the fact that $0 \leq \theta_{i+1}^{i+1} \leq 1$ that

$$(4.26) \quad \begin{aligned} & z_{i+1} + w + \sum_{j=0}^i s_j \\ &= I_h \left(\theta_{i+1}^{i+1} \left(v + \sum_{j=0}^i s_j - \sum_{j=0}^i z_j \right) + (1 - \theta_{i+1}^{i+1}) \left(w + \sum_{j=0}^{i+1} s_j \right) \right) \in K. \end{aligned}$$

From (4.25), it is easy to calculate that

$$(4.27) \quad \begin{aligned} & v - w - \sum_{j=0}^{i+1} z_j = v - w - \sum_{j=0}^i z_j - z_{i+1} \\ &= I_h \left((1 - \theta_{i+1}^{i+1}) \left(v - w - \sum_{j=0}^i z_j - s_{i+1} \right) \right). \end{aligned}$$

Using the fact that $s_{i+1} \in H_0^1(\Omega_{i+1}^c)$, $\theta_{i+1}^{i+1} = 1$ in $\Omega_{i+1}^c \setminus \cup_{k=i+2}^m \Omega_k^c$, and from (4.20), one obtains

$$v - w - \sum_{j=0}^{i+1} z_j \in H_0^1 \left(\bigcup_{j=i+2}^m \Omega_j^c \right).$$

To verify (4.21) for $i+1$, one gets from (4.27), (4.20), the assumption $w + \sum_{j=0}^{i+1} s_j \in K$, and the fact $0 \leq \theta_{i+1}^{i+1} \leq 1$ that

$$\begin{aligned} & v - \sum_{j=0}^{i+1} z_j + \sum_{j=0}^{i+1} s_j \\ &= I_h \left(\theta_{i+1}^{i+1} \left(v - \sum_{j=0}^i z_j + \sum_{j=0}^i s_j \right) + (1 - \theta_{i+1}^{i+1}) \left(w + \sum_{j=0}^{i+1} s_j \right) \right) \in K. \end{aligned}$$

Thus, we have proved by induction that (4.19)–(4.21) are correct for all z_i defined as in (4.18). \square

Assume from now on that the convex set K is given as in (4.2). For any $v, w + s_0 \in K$, let

$$\sigma^\oplus = I_h \max(0, v - w - s_0), \quad \sigma^\ominus = I_h \max(0, w + s_0 - v),$$

and define

$$(4.28) \quad v_0 = I_H^\ominus \sigma^\oplus - I_H^\ominus \sigma^\ominus.$$

Due to the special structure of σ^\ominus and σ^\oplus , it is not hard to show that

$$(4.29) \quad |\sigma^\oplus|_1 \leq C|v - w - s_0|_1, \quad |\sigma^\ominus|_1 \leq C|v - w - s_0|_1.$$

Thus, from (4.14)–(4.16) and the fact that $v - w - s_0 = \sigma^\oplus - \sigma^\ominus$ one obtains

$$\begin{aligned}
 (4.30) \quad & \|v_0 - (v - w - s_0)\|_l \\
 & \leq \|I_H^\ominus \sigma^\oplus - I_H^\ominus \sigma^\ominus - (\sigma^\oplus - \sigma^\ominus)\|_l \\
 & \leq c_d H^{1-l} |\sigma^\oplus|_1 + c_d H^{1-l} |\sigma^\ominus|_1 \\
 & \leq c_d H^{1-l} |v - w - s_0|_1, \quad l = 0, 1.
 \end{aligned}$$

As $\alpha(x) \leq v, w + s_0 \leq \beta(x)$, there follows that

$$v - w - s_0 \leq \min(\beta - w - s_0, v - \alpha), \quad w + s_0 - v \leq \min(\beta - v, w + s_0 - \alpha).$$

Note that $\min(\beta - w - s_0, v - \alpha) \geq 0$ and $\min(\beta - v, w + s_0 - \alpha) \geq 0$. It follows from properties (4.11) and (4.12) that

$$\begin{aligned}
 0 & \leq I_H^\ominus \sigma^\oplus \leq \min(\beta - w - s_0, v - \alpha), \\
 0 & \leq I_H^\ominus \sigma^\ominus \leq \min(\beta - v, w + s_0 - \alpha),
 \end{aligned}$$

which implies that $v_0 = I_H^\ominus \sigma^\oplus - I_h^\ominus \sigma^\ominus$ satisfies

$$\max(v - \beta, \alpha - w - s_0) \leq v_0 \leq \min(\beta - w - s_0, v - \alpha).$$

The above inequality shows that

$$(4.31) \quad \alpha(x) \leq v_0 + w + s_0 \leq \beta(x), \quad \alpha(x) \leq v - v_0 \leq \beta(x),$$

which means that v_0 , defined in (4.28), satisfies (4.17) when K is given as in (4.2).

LEMMA 4.3. *Let v_0 be given as in (4.28). Then the functions $z_i, i = 0, 1, 2, \dots, m$, defined in (4.18) satisfy*

$$(4.32) \quad \|v - w - z_0\|_0 \leq c_d H (|v - w|_1 + |s_0|_1),$$

$$(4.33) \quad |v - w - z_0|_1 \leq c_d (|v - w|_1 + |s_0|_1),$$

$$(4.34) \quad \left\| v - w - \sum_{j=0}^i z_j \right\|_0 \leq c_d H \left(|v - w|_1 + \sum_{j=0}^i |s_j|_1 \right), \quad i = 1, \dots, m,$$

$$(4.35) \quad \left| v - w - \sum_{j=0}^i z_j \right|_1 \leq c_d \left(1 + \frac{H}{\delta} \right) \left(|v - w|_1 + \sum_{j=0}^i |s_j|_1 \right), \quad i = 1, \dots, m.$$

Proof. The estimates (4.32) and (4.33) follow from (4.30). We shall establish (4.34) and (4.35) by induction. Since $s_i \in H_0^1(\Omega_i^c)$ and $\Omega_i^c, i = 1, \dots, m$, contains many disjoint subdomains with size proportional to H , then the Friedrich–Poincaré inequality can be employed to yield

$$(4.36) \quad \|s_i\|_0 \leq CH |s_i|_1, \quad i = 1, 2, \dots, m.$$

Now applying (4.9), (4.30), and (4.36) to (4.23) gives

$$\begin{aligned}
 (4.37) \quad & \|v - w - z_0 - z_1\|_0 \quad (\text{using (4.9) and (4.23)}) \\
 & \leq C \|v - w - z_0 - s_1\|_0 \quad (\text{using } z_0 = v_0 + s_0 \text{ and (4.30)}) \\
 & \leq c_d H |v - w - s_0|_1 + \|s_1\|_0 \quad (\text{using (4.36)}) \\
 & \leq c_d H (|v - w|_1 + |s_0|_1 + |s_1|_1).
 \end{aligned}$$

Similarly, one arrives at

$$\begin{aligned}
 (4.38) \quad & |v - w - z_0 - z_1|_1 \\
 & \leq C\|v - w - v_0 - s_0 - s_1\|_1 + \frac{C}{\delta}\|v - w - v_0 - s_0 - s_1\|_0 \\
 & \leq C\|v - w - s_0 - v_0\|_1 + C\|s_1\|_1 \\
 & \quad + \frac{C}{\delta}\|v - w - s_0 - v_0\|_0 + \frac{C}{\delta}\|s_1\|_0 \\
 & \leq c_d\left(1 + \frac{H}{\delta}\right)|v - w - s_0|_1 + C\left(1 + \frac{H}{\delta}\right)|s_1|_1 \\
 & \leq c_d\left(1 + \frac{H}{\delta}\right)(|v - w|_1 + |s_0|_1 + |s_1|_1).
 \end{aligned}$$

Now, let us assume that (4.35) and (4.34) are correct for i , and we shall show that they are also correct for $i + 1$. To this end, it follows from (4.9) and (4.27) that

$$\left\|v - w - \sum_{j=0}^{i+1} z_j\right\|_0 \leq C\left(\left\|v - w - \sum_{j=0}^i z_j\right\|_0 + \|s_{i+1}\|_0\right),$$

which, with the help of (4.36), shows that (4.34) is correct for $i + 1$ if it is correct for i . Finally, using again (4.9) and (4.27), we have

$$\begin{aligned}
 \left|v - w - \sum_{j=0}^{i+1} z_j\right|_1 & \leq +C\left(\left|v - w - \sum_{j=0}^i z_j\right|_1 + |s_{i+1}|_1\right) \\
 & + \left(C + \frac{1}{\delta}\right)\left(\left\|v - w - \sum_{j=0}^i z_j\right\|_0 + \|s_{i+1}\|_0\right).
 \end{aligned}$$

Thus, it follows from (4.34) and (4.36) that (4.35) is correct for $i + 1$ if it is correct for the index i . This completes the proof of the lemma. \square

THEOREM 4.4. *The estimate (3.2) in Assumption 1 holds true for the decomposition (4.6.b) with*

$$(4.39) \quad C_1 = c_d\left(1 + \frac{H}{\delta}\right).$$

Proof. Since $\theta_m^m \equiv 1$, then from (4.27) we conclude that

$$v - w - \sum_{j=0}^m z_j = 0 \text{ in } \Omega,$$

which shows that (3.2.a) is valid. Condition (3.2.b) has been shown to be valid for z_0 and z_i in (4.31) and (4.19). It follows from

$$\begin{aligned}
 \|z_0\|_1 & \leq \|v - w - z_0\|_1 + \|v - w\|_1, \\
 \|z_i\|_1 & \leq \left\|v - w - \sum_{j=0}^i z_j\right\|_1 + \left\|v - w - \sum_{j=0}^{i-1} z_j\right\|_1, \quad i = 1, \dots, m,
 \end{aligned}$$

and (4.32)–(4.35) that (3.2.c) holds true with C_1 being given in (4.39). We point out that the generic constant depends on m , which is the number of colors for the subdomains. \square

The estimate (3.3) in Assumption 2 has been shown to be correct for the decomposition (4.6.b) with $C_2 = \sqrt{m + 1}$ and m being the number of colors; see [30] and [20, 32] for details. Thus, all the conditions of the abstract convergence Theorem 3.1 are verified for the proposed domain decomposition method for the obstacle problem. As a consequence of Theorem 3.1, we see that the convergence rate of Algorithm 2 for the obstacle problem is given by

$$\frac{F(u^{n+1}) - F(u)}{F(u^n) - F(u)} \leq 1 - \frac{1}{1 + (c_d H/\delta)^2}$$

or

$$\|u^n - u\|_1^2 \leq \frac{2}{\kappa} \left[1 - \frac{1}{1 + (c_d H/\delta)^2} \right]^n [F(u^0) - F(u)].$$

4.3. Domain decomposition methods without coarse levels. When no coarse levels are used in the domain decomposition method, the finite element space $V = S_h$ can be decomposed into subspaces as given in (4.6.a). In this case, Algorithm 2 turns out to be the classical Schwarz alternating method for the corresponding minimization problem. We want to show that the abstract convergence Theorem 3.1 can be applied to yield a linear convergence for the Schwarz method, in which the rate of convergence depends only on the overlapping size. Furthermore, our result is more useful than those presented in [22, 29, 33] since no monotonicity is assumed on the iterative approximations.

Let $v, w \in K$ and $s_i \in V_i$ satisfy $w + \sum_{j=1}^i s_j \in K$. We define z_i recursively by

$$(4.40) \quad z_i = I_h \left(\theta_i^i \left(v - w - \sum_{j=1}^{i-1} v_j \right) + (1 - \theta_i^i) s_i \right), \quad i = 1, \dots, m.$$

By repeating the proof as for Lemma 4.2, we obtain the following result.

LEMMA 4.5. *For a general convex subset $K \subset H_0^1(\Omega)$, assume that $v, w, w + \sum_{j=1}^i s_j \in K, s_i \in V_i$ for $i = 1, \dots, m$. Let $z_i, i = 1, \dots, m$, be defined as in (4.40). Then we have*

$$(4.41) \quad z_i \in V_i, \quad z_i + w + \sum_{j=1}^{i-1} s_j \in K,$$

$$(4.42) \quad v - w - \sum_{j=1}^i z_j = 0 \text{ in } H_0^1 \left(\bigcup_{j=i+1}^m \Omega_j^c \right),$$

$$(4.43) \quad v - \sum_{j=1}^i z_j + \sum_{j=1}^i s_j \in K.$$

In fact, the above lemma is a consequence of Lemma 4.2 by taking $v_0 = 0$ and $s_0 = 0$. Now, using (4.9), from (4.27) in which the summation index i starts from 1, we obtain the following estimate.

LEMMA 4.6. *With $z_i, i = 1, 2, \dots, m$, being defined in (4.40) we have*

$$(4.44) \quad \left\| v - w - \sum_{j=1}^i z_j \right\|_0 \leq C \left(|v - w|_1 + \sum_{j=1}^i |s_j|_1 \right),$$

$$(4.45) \quad \left| v - w - \sum_{j=0}^i z_j \right|_1 \leq C(1 + \delta^{-1}) \left(|v - w|_1 + \sum_{j=1}^i |s_j|_1 \right).$$

Consequently, the following result has been proved.

THEOREM 4.7. *The estimate (3.2) in Assumption 1 is valid for the decomposition (4.6.a) with*

$$(4.46) \quad C_1 = C(1 + \delta^{-1}).$$

An application of Theorem 3.1 indicates that the one-level Schwarz method has the following convergence rate estimate for the obstacle problem:

$$\frac{F(u^{n+1}) - F(u)}{F(u^n) - F(u)} \leq 1 - \frac{1}{1 + C(1 + \delta^{-2})}$$

or

$$\|u^n - u\|_1^2 \leq \frac{2}{\kappa} \left[1 - \frac{1}{1 + C(1 + \delta^{-2})} \right]^n [F(u^0) - F(u)].$$

5. Numerical example. To support the convergence theory developed in the previous sections, we present some numerical results here for the obstacle problem approximated by piecewise linear finite elements. To this end, consider the homogeneous problem (4.1) and (4.2) which seeks $u \in H_0^1(\Omega)$ such that

$$(5.1) \quad \alpha \leq u \leq \beta : \int_{\Omega} \nabla u \nabla (v - u) \geq 0 \quad \forall v \in H_0^1(\Omega), \alpha \leq v \leq \beta,$$

where $\alpha(x)$ and $\beta(x)$ are two obstacle functions and $\Omega = (0, 4) \times (0, 3)$.

The two finite element partitions \mathcal{T}_H and \mathcal{T}_h contain right triangles, which are obtained through a uniform refinement of Ω as illustrated in Figure 5.1. In Figure 5.1, the coarse partition \mathcal{T}_H comprises 6×6 rectangles (i.e., 72 triangles) and the fine-level partition \mathcal{T}_h contains 30×30 rectangles (i.e., 1800 triangles). As for the nonoverlapping structure $\{\Omega_i\}_{i=1}^M$ for Ω , we take $M = 9$, and $\{\Omega_i\}_{i=1}^9$ is obtained as a uniform partition of Ω into rectangles. The overlapping decomposition $\{\Omega_i^\delta\}_{i=1}^M$ is constructed by extending each Ω_i with a width of two triangles in \mathcal{T}_h . Roughly speaking, the width δ is given by $2h$.

The obstacles $\alpha(x)$ and $\beta(x)$ are shown in Figure 5.2. More precisely, we have

$$(5.2) \quad \begin{aligned} \alpha(x, y) &= 3 + \sqrt{\left(\frac{1}{6}\right)^2 - (x - 2)^2 - (y - 1.5)^2} \\ &\text{if } (x - 2)^2 + (y - 1.5)^2 \leq \left(\frac{1}{6}\right)^2, \text{ or else } \alpha(x, y) = 0; \\ \beta(x, y) &= 1/6 - \sqrt{\left(\frac{1}{6}\right)^2 - \left(x - \frac{4}{3}\right)^2 - \left(y - \frac{3}{4}\right)^2} \\ &\text{if } \left(x - \frac{4}{3}\right)^2 + \left(y - \frac{3}{4}\right)^2 \leq \left(\frac{1}{6}\right)^2, \text{ or else } \beta(x, y) = \frac{19}{6}. \end{aligned}$$

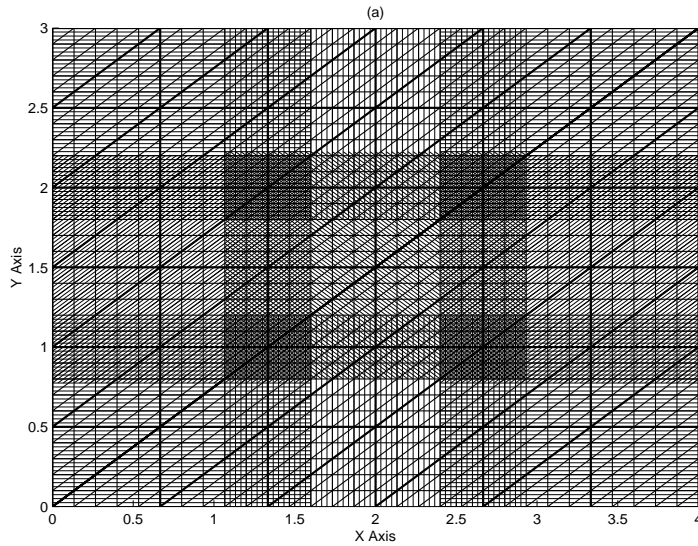


FIG. 5.1. Meshes T_h and T_H and domain decomposition.

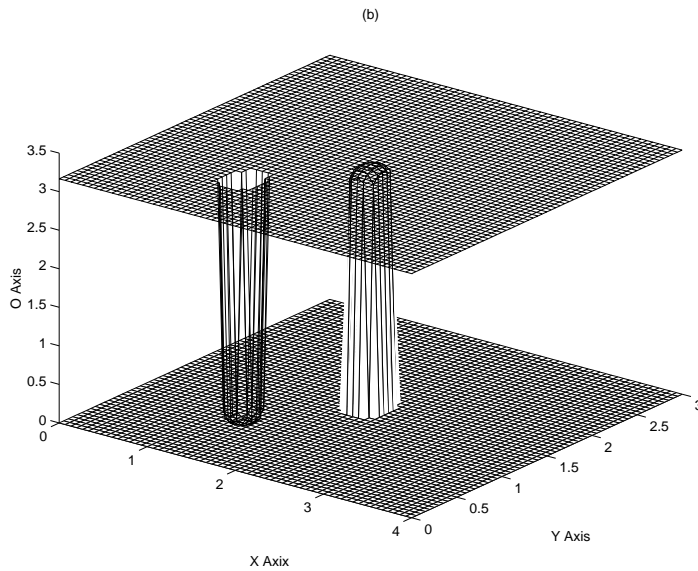
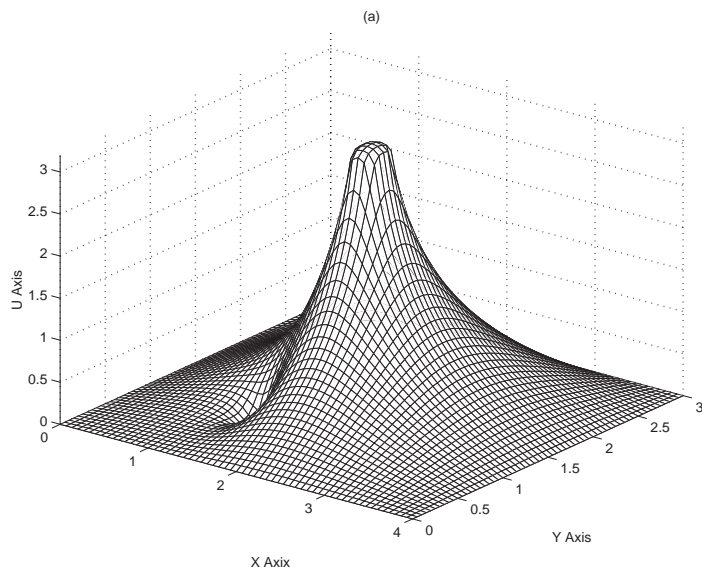


FIG. 5.2. Obstacle functions $\alpha(x)$ (lower one) and $\beta(x)$ (upper one).

In the numerical simulations, the obstacles are replaced by their finite element approximations. Corresponding to this obstacle, the finite element solution for (5.1) is as shown in Figure 5.3.

We have seen that the constant C_1 in the convergence estimate of Theorem 4.7 depends on δ^{-1} as given by (4.46) when one-level domain decomposition methods are considered. For two-level domain decomposition methods, the constant C_1 depends on H/h and H/δ as given in (4.39). One of the goals of this section is to numerically verify this dependence by taking various values of H , h , and δ . In all of our numerical tests

FIG. 5.3. *Solution.*

the iteration is stopped when the maximum error between two consecutive computed solutions is smaller than the tolerance $\epsilon = 0.001$. The solution for each subdomain problem is calculated by using the Gauss–Seidel iteration, which itself is a particular case of the Schwarz domain decomposition method in which each subdomain is merely the support of a nodal basis function of the finite element space. When solving subdomain problems, the calculation is terminated at a relative maximum error of $\epsilon = 10^{-5}$ at the nodes of \mathcal{T}_h between two consecutive computed solutions.

For the results shown in Figure 5.4, the coarse mesh size H varies, while the ratios $H/h = 6$ and $H/\delta = 2$ stay unchanged. The plot shows the total number of iterations in the Schwarz method when the partition \mathcal{T}_H has 20, 18, 16, \dots , 2 elements in the x - and y -directions. Starting from six elements, the number of iterations is almost constant for the two-level method, which is in concordance with the convergence theory. It can also be seen that the number of iterations is a decreasing function of H for the one-level method. Since H/δ is constant, it follows that the number of iterations is an increasing function of $1/\delta$, and this is in concordance with the estimate for C_1 in (4.46).

For the results in Figure 5.5, we have taken $H = \frac{5}{12}$, $h = \frac{5}{120}$, and $\delta = h, 2h, \dots, 10h$. For both one- and two-level methods, the number of iterations is a decreasing function of δ . This observation is in concordance with the estimate on the constant C_1 .

For the results shown in Figure 5.6, the values for H, δ are chosen as $H = \frac{5}{6}$ and $\delta = \frac{5}{12}$. The value of h assumes the mesh size of the partition \mathcal{T}_h with $2 \times 6, 4 \times 6, 6 \times 6, \dots, 20 \times 6$ elements in the x - and y -directions. For the one-level Schwarz method, the number of iterations is constant for $h \leq \frac{5}{12}$, and this confirms the observation that the constant C_1 does not depend on h for the one-level method. For the two-level method, the number of iterations is an decreasing function of h , which is in concordance with the $\log(H/h)$ -dependence estimate of C_1 in (4.39).

Finally, we see from the above numerical tests that the number of iterations for the two-level method is significantly less than for the one-level method. We remark

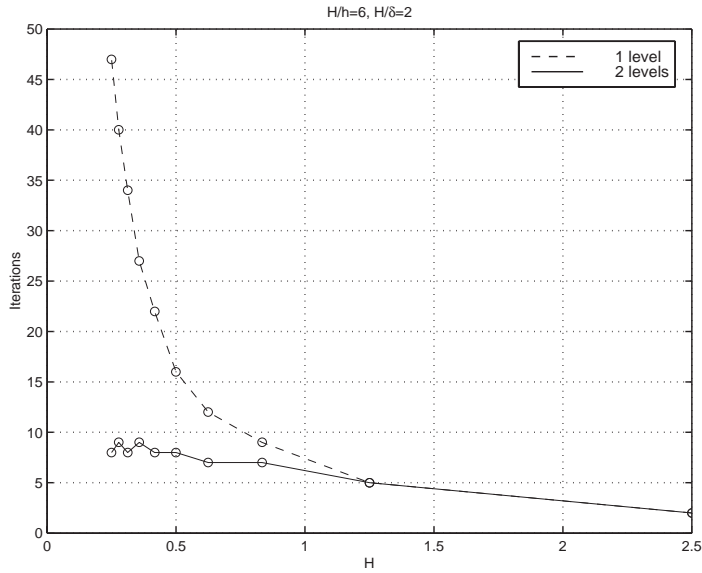


FIG. 5.4. Number of iterations as a function of H for the Schwarz method when H/h and H/δ are fixed.

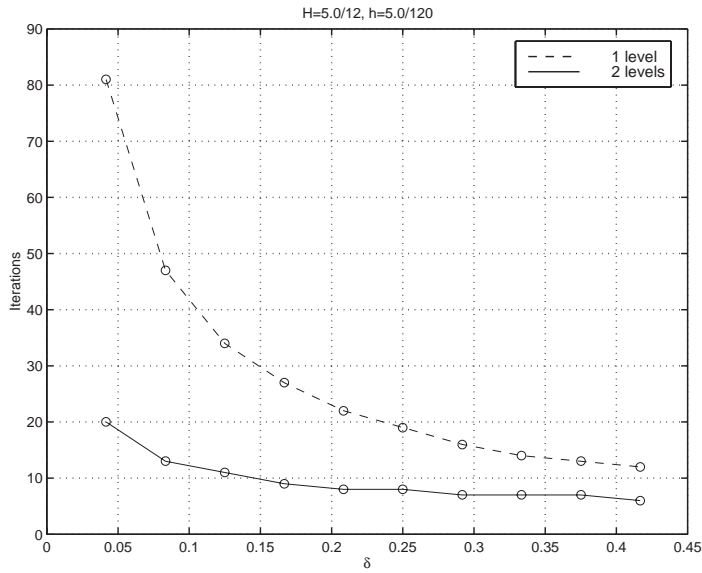


FIG. 5.5. Number of iterations as a function of δ for the Schwarz method when H and h are fixed.

that for one-sided obstacle problems, numerical tests using the two-level domain decomposition method have been shown in [28].

In the rest of this section, we make comments on the relaxation method that was used to solve the minimization problem on each subdomain. Notice that in the relaxation method, we have a one-dimensional minimization problem to solve on each support of the nodal basis functions. The solution of these one-dimensional problems

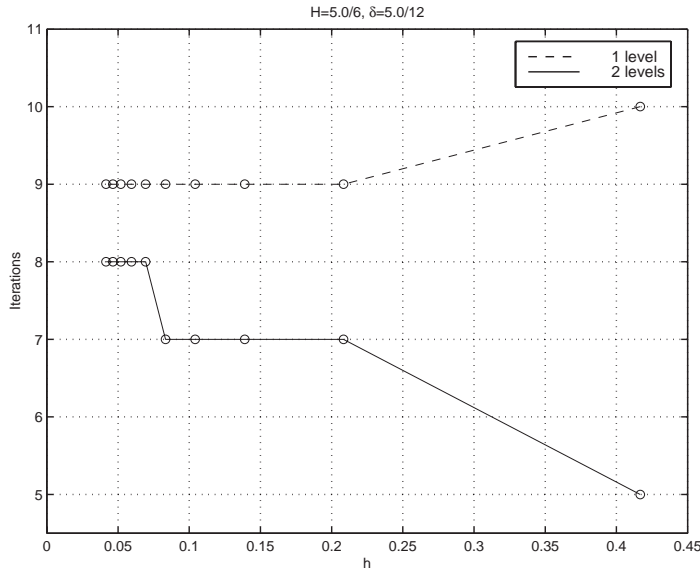


FIG. 5.6. Number of iterations of the Schwarz method as a function of h when H and δ are fixed.

was obtained by first solving the one-dimensional problem without constraint and then projecting it to the interval that presents the constraint for this one-dimensional problem. To be more precise, we use two vectors $u(k)$ and $e(k)$, where k runs from 1 to the number of interior nodes in \mathcal{T}_h for the values of $u^{n+\frac{i-1}{m}}$ and e_i^{n+1} obtained from Algorithm 2. Naturally, we have two vectors $\alpha(k)$ and $\beta(k)$ containing the values of the two obstacles at the interior nodal points in \mathcal{T}_h . Assume now that we are computing the solution on the subdomain Ω_i and we are seeking the value $e(k)$ of the correction at the node k of \mathcal{T}_h . Let $\tilde{e}(k)$ be the value obtained from the one-dimensional problem without constraint. The projection is simply given by

$$(5.3) \quad e(k) = \min(\beta(k) - u(k), \max(\alpha(k) - u(k), \tilde{e}(k))).$$

For the problem associated with the coarse mesh, the minimization function (i.e., the correction value e) comes from the coarse mesh finite element space, with constraints imposed on the fine mesh. A relaxation method is employed to solve this problems in which one-dimensional problems associated with interior coarse mesh nodal basis functions $\phi_j^0(x)$, $j = 1, 2, \dots, n_0$, are solved. As this is a one-dimensional minimization problem with a constraint, we can first compute the minimizer without constraint and then project this number into the interval which represents the constraint. The computation of the one-dimensional problem without constraint can be done in the same way as for the standard Schwarz method [20, 28]. The computation of the constraint interval can be done similarly as explained in [28, p. 136] for one-sided obstacle problems. To explain the idea more clearly, let us use $u^{old}(x)$ to denote the computed solution, and we need to solve the following problem to get an updated value for a coarse mesh nodal basis function $\phi_j^0(x)$:

$$(5.4) \quad e(j) = \arg \min_{\{\lambda \in R \mid \alpha(x) \leq u^{old} + \lambda \phi_j^0(x) \leq \beta(x) \forall x \in \text{supp}(\phi_j^0)\}} F(u^{old} + \lambda \phi_j^0),$$

where $supp(\phi_j^0)$ is the support set of the function ϕ_j^0 . Let $\tilde{e}(j)$ be the minimizer of the one-dimensional unconstrained problem, i.e.,

$$(5.5) \quad \tilde{e}(j) = \arg \min_{\lambda \in R} F(u^{old} + \lambda \phi_j^0).$$

The solution $\tilde{e}(j)$ is found by solving the one-dimensional algebraic equation associated with this minimization problem. Since F is convex, $e(j)$ is the projection of $\tilde{e}(j)$ over the interval

$$[\alpha_j, \beta_j] = \{ \lambda \in R \mid \alpha(x) \leq u^{old} + \lambda \phi_j^0(x) \leq \beta(x) \forall x \in supp(\phi_j^0) \},$$

where

$$(5.6) \quad \alpha_j = \sup_{x \in supp(\phi_j^0)} \frac{\alpha(x) - u^{old}(x)}{\phi_j^0(x)}, \quad \beta_j = \inf_{x \in supp(\phi_j^0)} \frac{\beta(x) - u^{old}(x)}{\phi_j^0(x)}.$$

Evidently, we have

$$(5.7) \quad e(j) = \min(\beta_j, \max(\alpha_j, \tilde{e}(j))).$$

We notice that, since $\alpha(x) \leq u^{old}(x) \leq \beta(x)$, we have $0 \in [\alpha_j, \beta_j]$, and, consequently, this interval is not empty. Naturally, the above $\inf_{x \in supp(\phi_j^0)}$ and $\sup_{x \in supp(\phi_j^0)}$ are calculated only for the mesh nodes of \mathcal{T}_h .

Similar relaxation methods have been employed in the domain decomposition for unconstrained minimization problems such as the Dirichlet problem for second order elliptic problems. For the constrained problem, the relaxation method involves an additional step which computes the lower and upper bounds α_j and β_j as given in (5.6) and the projections (5.3) and (5.7).

The projection for the two-level method is more complicated than for the one-level method. However, since the convergence of the two-level method is much faster than the one-level method, the two-level method is more preferable for practical use. For instance, for $H = 5.0/10$, $h = 5.0/60$, and $\delta = 5.0/20$, the number of iterations is 16 for the one-level method, and it is 8 for the two-level method. The computing CPU time on a PC with one processor (Intel Pentium III, 600MHz) is 5.2 minutes for the one-level method, and it is 3.7 minutes for the two-level method. The finite element discretization problem in these numerical tests involves 3481 unknowns.

We shall mention that the subproblems associated with the subdomains and the coarse mesh problem can also be solved by methods other than the relaxation method. In the numerical tests of [26] and [28], the subproblems are solved by the augmented Lagrangian method, which is also rather efficient for handling the constraints both for the subdomain and coarse mesh problems.

6. Proof of Theorem 3.1. Since e_i^{n+1} minimizes (2.4), it satisfies (see [8])

$$(6.1) \quad \langle F'(u^{n+\frac{i-1}{m}} + e_i^{n+1}), v_i - e_i^{n+1} \rangle \geq 0 \quad \forall v_i \in V_i \text{ satisfying } v_i + u^{n+\frac{i-1}{m}} \in K.$$

Using assumption (3.1), we can prove that (see [23, Lem. 3.2])

$$(6.2) \quad F(w) - F(v) \geq \langle F'(v), w - v \rangle + \frac{\kappa}{2} \|w - v\|_V^2 \quad \forall v, w \in V.$$

Taking $v_i = 0$ in (6.1), we get from the above two inequalities that

$$(6.3) \quad F(u^{n+\frac{i-1}{m}}) - F(u^{n+\frac{i}{m}}) \geq \frac{\kappa}{2} \|e_i^{n+1}\|_V^2.$$

It follows from (6.3) that

$$(6.4) \quad F(u^n) - F(u^{n+1}) = \sum_{i=1}^m (F(u^{n+\frac{i-1}{m}}) - F(u^{n+\frac{i}{m}})) \geq \frac{\kappa}{2} \sum_{i=1}^m \|e_i^{n+1}\|_V^2.$$

Thus, we have

$$F(u^n) \geq F(u^{n+1}).$$

Denote, for a given n ,

$$\nu_j^i = \begin{cases} u^n + \sum_{k=1}^i e_k^{n+1}, & j \leq i; \\ u^n + \sum_{k=1}^j e_k^{n+1}, & j > i. \end{cases}$$

It can be seen that ν_j^i satisfies

$$(6.5) \quad \begin{aligned} \nu_j^i - \nu_{j-1}^i &= 0, & j \leq i; \\ \nu_j^i - \nu_{j-1}^i &= e_j^{n+1}, & j > i; \\ F'(u^{n+1}) - F'(u^{n+\frac{i}{m}}) &= \sum_{j=i+1}^m (F'(\nu_j^i) - F'(\nu_{j-1}^i)). \end{aligned}$$

As $u, u^n, u^n + \sum_{j=1}^i e_j^{n+1} \in K, i = 1, 2, \dots, m$, we get from assumption (3.2) that there exist $z_i^n \in V_i$ such that

$$(6.6) \quad \begin{cases} \text{(a) } u - u^n = \sum_{i=1}^m z_i^n, & \text{(b) } u^n + \sum_{j=1}^{i-1} e_j^{n+1} + z_i^n \in K, \quad i = 1, \dots, m, \\ \text{(c) } \left(\sum_{i=1}^m \|z_i^n\|_V^2 \right)^{\frac{1}{2}} \leq C_1 \left(\|u^n - u\|_V^2 + \sum_{j=1}^m \|e_j^{n+1}\|_V^2 \right)^{\frac{1}{2}}. \end{cases}$$

We use (3.3), (6.6), and (6.1) to get

$$(6.7) \quad \begin{aligned} \langle F'(u^{n+1}), u^{n+1} - u \rangle &= \left\langle F'(u^{n+1}), \sum_{i=1}^m e_i^{n+1} + u^n - u \right\rangle \\ &= \sum_{i=1}^m \langle F'(u^{n+1}), e_i^{n+1} - z_i^n \rangle \quad (\text{using (6.6.a)}) \\ &\leq \sum_{i=1}^m \langle F'(u^{n+1}) - F'(u^{n+i/m}), e_i^{n+1} - z_i^n \rangle \\ &\quad (\text{using (6.6.b) and (6.1)}) \\ &= \sum_{i=1}^m \sum_{j=i+1}^m \langle F'(\nu_j^i) - F'(\nu_{j-1}^i), e_i^{n+1} - z_i^n \rangle \quad (\text{using (6.5)}) \end{aligned}$$

$$\begin{aligned} &\leq C_2 \left(\sum_{j=1}^m \|e_j^{n+1}\|_V^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^m \|e_i^{n+1} - z_i^n\|_V^2 \right)^{\frac{1}{2}} \quad (\text{using (3.3)}) \\ &\leq C_2 \left(\sum_{j=1}^m \|e_j^{n+1}\|_V^2 \right)^{\frac{1}{2}} \left((1 + C_1) \left(\sum_{i=1}^m \|e_i^{n+1}\|_V^2 \right)^{\frac{1}{2}} + C_1 \|u^n - u\|_V \right) \\ &\quad (\text{using (6.6.c) and the triangular inequality}). \end{aligned}$$

The rest of the proof is the same as in [26, 27]. As u is the unique minimizer for (2.1), we use (6.2) and the optimality condition to obtain

$$(6.8) \quad F(u^n) - F(u) \geq \langle F'(u), u^n - u \rangle + \frac{\kappa}{2} \|u - u^n\|_V^2 \geq \frac{\kappa}{2} \|u - u^n\|_V^2.$$

The following estimate needs to use (6.2), (6.4), (6.7), and (6.8):

$$\begin{aligned} &F(u^{n+1}) - F(u) \\ &\leq \langle F'(u^{n+1}), u^{n+1} - u \rangle \quad (\text{using (6.2)}) \\ &\leq (1 + C_1) C_2 \frac{2}{\kappa} (F(u^n) - F(u^{n+1})) \quad (\text{using (6.4) and (6.7)}) \\ &\quad + C_1 C_2 \frac{2}{\kappa} \sqrt{F(u^n) - F(u^{n+1})} \sqrt{F(u^n) - F(u)} \quad (\text{using (6.8) and (6.7)}). \end{aligned}$$

Denote $d_n = F(u^n) - F(u)$ for all $n \geq 0$. Let $\mu \in (0, 1)$ be a constant to be determined later. Apply the inequality $ab \leq \frac{1}{4\mu} a^2 + \mu b^2$ for all $a, b \in \mathbb{R}$ to the last term of the above estimate to get

$$\begin{aligned} d_{n+1} &\leq (1 + C_1) \frac{2C_2}{\kappa} (d_n - d_{n+1}) + C_1 C_2 \frac{2}{\kappa} \sqrt{d_n - d_{n+1}} \sqrt{d_n} \\ &\leq \left((1 + C_1) \frac{2C_2}{\kappa} + \frac{[C_1 C_2]^2}{\mu \kappa^2} \right) (d_n - d_{n+1}) + \mu d_n \\ &\leq C^* \mu^{-1} (d_n - d_{n+1}) + \mu d_n. \end{aligned}$$

As a consequence, we see that

$$\frac{d_{n+1}}{d_n} \leq \frac{C^* \mu^{-1} + \mu}{1 + C^* \mu^{-1}} = 1 - \frac{\mu(1 - \mu)}{\mu + C^*}.$$

For a given $C^* > 0$, the function $g(\mu) = \frac{\mu(1-\mu)}{\mu+C^*}$ has a unique maximizer in $[0, 1]$, and the maximizer is given by $\mu^* = \sqrt{(C^*)^2 + C^*} - C^* \in (0, 1)$. Moreover, the maximum value is given by $g(\mu^*) = \frac{1}{(\sqrt{C^*+1} + \sqrt{C^*})^2}$. Consequently, (3.4) holds. The error estimation (3.5) is obtained using (6.8) and (3.4). This completes the proof of the theorem. \square

REFERENCES

[1] L. BADEA, *On the Schwarz alternating method with more than two subdomains for nonlinear monotone problems*, SIAM J. Numer. Anal., 28 (1991), pp. 179–204.
 [2] L. BADEA, *A generalization of the Schwarz alternating method to an arbitrary number of subdomains*, Numer. Math., 55 (1989), pp. 61–81.
 [3] L. BADEA AND J. WANG, *An additive Schwarz method for variational inequalities*, Math. Comp., 69 (2000), pp. 1341–1354.

- [4] J.H. BRAMBLE, J.E. PASCIAK, J. WANG, AND J. XU, *Convergence estimates for multigrid algorithms without regularity assumptions*, Math. Comp., 57 (1991), pp. 23–45.
- [5] J.H. BRAMBLE AND J. XU, *Some estimate for a weighted L^2 projection*, Math. Comp., 56 (1991), pp. 463–476.
- [6] T.F. CHAN AND T.P. MATHEW, *Domain decomposition algorithms*, in Acta Numerica 1994, Cambridge University Press, Cambridge, UK, 1994, pp. 61–143.
- [7] P.G. CIARLET, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
- [8] I. EKELAND AND R. TEMAM, *Convex analysis and variational problems*, North-Holland, Amsterdam, 1976.
- [9] R.S. FALK, *Error estimate for the approximation of a class of variational inequalities*, Math. Comput., 28 (1974), pp. 963–971.
- [10] K. H. HOFFMANN AND J. ZOU, *Parallel algorithms of Schwarz variant for variational inequalities*, Numer. Funct. Anal. Optim., 13 (1992), pp. 449–462.
- [11] T. KARKKAINEN, K. KUNISCH, AND P. TARVAINEN, *Primal-Dual Active Set Methods for Obstacle Problems*, Reports of the Department of Mathematical Information Technology, University of Jyväskylä, Series B, No. 2, Jyväskylä, Finland, 2000.
- [12] Y. KUZNETSOV, P. NEITTAANMÄKI, AND P. TARVAINEN, *Block relaxation method for algebraic obstacle problems with M -matrices*, East-West J. Numer. Math., 2 (1994), pp. 75–89.
- [13] P. LE TALLEC, *Domain decomposition methods in computational mechanics*, Comput. Mech. Adv., 1 (1994), pp. 121–220.
- [14] P.L. LIONS, *On the Schwarz alternating method. I*, in Proceedings of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Periaux, eds., SIAM, Philadelphia, 1988, pp. 2–42.
- [15] P.L. LIONS, *On the Schwarz alternating method. II*, in Domain Decomposition Methods, T. F. Chan, R. Glowinski, J. Periaux, and O. B. Widlund, eds., SIAM, Philadelphia, 1989, pp. 47–70.
- [16] P.L. LIONS, *On the Schwarz alternating method. III*, in Proceedings of the Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, R. Glowinski, J. Periaux, and O. B. Widlund, eds., SIAM, Philadelphia, 1990, pp. 202–223.
- [17] T. LÜ, C. LIEM, AND T. SHIH, *Parallel algorithms for variational inequalities based on domain decomposition*, System Sci. Math. Sci., 4 (1991), pp. 341–348.
- [18] A. QUARTERONI AND A. VALLI, *Domain Decomposition Methods for Partial Differential Equations*, Oxford University Press, New York, 1999.
- [19] I.A. SHARPOV, *Multilevel Subspace Correction for Large Scale Optimization Problems*, Technical report CAM-97-31, UCLA, Department of Computational and Applied Mathematics, 1997.
- [20] B.F. SMITH, P.E. BJØRSTAD, AND W. GROPP, *Domain Decomposition: Parallel Multilevel Methods for Elliptic Differential Equations*, Cambridge University Press, Cambridge, UK, 1996.
- [21] X.-C. TAI, *Parallel function decomposition and space decomposition methods with applications to optimisation, splitting and domain decomposition*, preprint 231-1992, Institut für Mathematik, Technische Universität Graz, Austria, 1992. Available online at <http://www.mi.uib.no/%7Eetai/pub.html>.
- [22] X.-C. TAI, *Convergence rate analysis of domain decomposition methods for obstacle problems*, East-West J. Numer. Anal., 9 (2001), pp. 233–252.
- [23] X.-C. TAI AND M. ESPEDAL, *Rate of convergence of some space decomposition methods for linear and nonlinear problems*, SIAM J. Numer. Anal., vol. 35, no. 4 (1998), pp. 1558–1570.
- [24] X.-C. TAI, *Parallel function and space decomposition methods. Part I. Function decomposition*, Beijing Math., 1 (1991), pp. 104–134.
- [25] X.-C. TAI, *Parallel function and space decomposition methods. Part II. Space decomposition*, Beijing Math., 1 (1991), pp. 135–152.
- [26] X.-C. TAI, *Rate of convergence for some constraint decomposition methods for nonlinear variational inequalities*, Numer. Math., 93 (2003), pp. 755–786.
- [27] X.-C. TAI, *Some new domain decomposition and multigrid methods for variational inequalities*, in Proceedings of 14th International Conference on Domain Decomposition Methods, Cocoyoc Morelos, Mexico, 2002, pp. 323–330.
- [28] X.-C. TAI, B. OVE HEIMSUND, AND J. XU, *Rate of convergence for parallel subspace correction methods for nonlinear variational inequalities*, in Proceeding of the 13th International

- Conference on Domain Decomposition Methods, Lyon, France, 2001, pp. 127–138.
- [29] X.-C. TAI AND P. TSENG, *Convergence rate analysis of an asynchronous space decomposition method for convex minimization*, Math. Comp., 71 (2001), pp. 1105–1135.
 - [30] X.-C. TAI AND J. XU, *Global and uniform convergence of subspace correction methods for convex optimization problems*, Math. Comp., 71 (2002), pp. 105–124.
 - [31] P. TARVAINEN, *Two-level Schwarz method for unilateral variational inequalities*, IMA J. Numer. Anal., 14 (1998), pp. 1–18.
 - [32] J. XU, *Iterative methods by space decomposition and subspace correction*, SIAM Rev., 34 (1992), pp. 581–613.
 - [33] J. ZENG AND S. ZHOU, *On monotone and geometric convergence of Schwarz methods for two-sided obstacle problems*, SIAM J. Numer. Anal., 35 (1998), pp. 600–616.