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Extended F_4 -buildings and the Baby Monster

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Abstract

Let Θ be the Baby Monster graph which is the graph on the set of $\{3, 4\}$ -transpositions in the Baby Monster group B in which two such transpositions are adjacent if their product is a central involution in B . Then Θ is locally the commuting graph of central (root) involutions in ${}^2E_6(2)$. The graph Θ contains a family of cliques of size 120. With respect to the incidence relation defined via inclusion these cliques and the non-empty intersections of two or more of them form a geometry $\mathcal{E}(B)$ with diagram

$$c.F_4(t) : \begin{array}{c} \text{c} \\ \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \\ 1 \quad 2 \quad 2 \quad t \quad t \end{array}$$

for $t = 4$ and the action of B on $\mathcal{E}(B)$ is flag-transitive. We show that $\mathcal{E}(B)$ contains subgeometries $\mathcal{E}({}^2E_6(2))$ and $\mathcal{E}(Fi_{22})$ with diagrams $c.F_4(2)$ and $c.F_4(1)$. The stabilizers in B of these subgeometries induce on them flag-transitive actions of ${}^2E_6(2) : 2$ and $Fi_{22} : 2$, respectively. The geometries $\mathcal{E}(B)$, $\mathcal{E}({}^2E_6(2))$ and $\mathcal{E}(Fi_{22})$ possess the following properties: (a) any two elements of type 1 are incident to at most one common element of type 2 and (b) three elements of type 1 are pairwise incident to common elements of type 2 if and only if they are incident to a common element of type 5. The paper addresses the classification problem of $c.F_4(t)$ -geometries satisfying (a) and (b). We construct three further examples for $t = 2$ with flag-transitive automorphism groups isomorphic to $3 \cdot {}^2E_2(2) : 2$, $E_6(2) : 2$ and $2^{26}.F_4(2)$ and one for $t = 1$ with flag-transitive automorphism group $3 \cdot Fi_{22} : 2$. We also study the graph of an arbitrary (non-necessary flag-transitive) $c.F_4(t)$ -geometry satisfying (a) and (b) and obtain a complete list of possibilities for the isomorphism type of subgraph induced by the common neighbours of a pair of vertices at distance 2. Finally, we prove that $\mathcal{E}(B)$ is the only $c.F_4(4)$ -geometry, satisfying (a) and (b).

1 Introduction

The paper contributes to the geometric theory of sporadic simple groups. Our notation and terminology is mostly standard (see [Pasi94] and [I99]). On a diagram the types of elements increase rightwards from 1 to the rank of the geometry. If an element of type 2 in a geometry $\mathcal{G}$ is incident to exactly two elements of type 1 then in the *graph* $\Gamma(\mathcal{G})$ of $\mathcal{G}$ the vertices and edges are the elements of type 1 and 2 in $\mathcal{G}$ subject to the natural incidence relation (in general $\Gamma(\mathcal{G})$ might contain multiple edges). Recall that if Γ is a graph and $x \in \Gamma$ is a vertex, then $\Gamma_i(x)$ is the set of vertices at distance i from x in Γ . Sometimes we write $\Gamma(x)$ instead of $\Gamma_1(x)$. If Σ is a subset of the vertex-set of Γ , then Σ also denotes the subgraph in Γ induced by Σ . If for every $x \in \Gamma$ the subgraph in $\Gamma(x)$ is isomorphic

*A part of the research was conducted whilst this author was visiting Department of Mathematics and Statistics, Bowling Green State University.

to a fixed graph Δ then Γ is said to be *locally* Δ . A graph Γ whose vertices are involutions in a group G is said to be the *commuting graph* if $a, b \in \Gamma$ are adjacent if and only if the order $o(ab)$ of the product of a and b is 2 (equivalently if a and b commute). A graph Γ is said to be a *m-clique extension* of a graph Δ if there is a mapping ψ of the vertex-set of Γ onto the vertex-set of Δ such that $|\psi^{-1}(x)| = m$ for every $x \in \Delta$ and two distinct vertices u, v in Γ are adjacent if and only if the images $\psi(u)$ and $\psi(v)$ are either equal or adjacent in Δ . A graph Γ is said to be the *distance 1-or-2 graph* of Δ if there is a bijection χ of the vertex-set of Γ onto the vertex-set of Δ with the following property: two vertices x and y in Γ are adjacent if and only if the vertices $\chi(x)$ and $\chi(y)$ are adjacent or at distance 2 in Δ . By a 2-path in a graph Γ we mean an ordered sequence (x, y, z) of vertices, such that $x, z \in \Gamma(y)$ and $z \in \Gamma_2(x)$. If G is a group, acting by permutations on a set X , then for $x \in X$ and $Y \subseteq X$ by $G(x)$ and $G[Y]$ we denote the stabilizer of x and (the setwise) stabilizer of Y in G (the upper and lower indexes will have various different meanings).

Fig. 1

The sporadic simple group B known as the *Baby Monster* contains a conjugacy class Θ of involutions (with centralizers of the form $2 \cdot {}^2E_6(2) : 2$) such that for any two involutions $a, b \in \Theta$ the order $o(ab)$ of their product is at most 4. This is to say, Θ is a class of $\{3, 4\}$ -transpositions in B . Let Θ denote also the graph on this set, in which two transpositions are adjacent whenever their product belongs to the class of central involutions in B (with centralizers of the form $2_+^{1+22}.Co_2$). We call Θ the *Baby Monster graph*. The suborbit diagram of Θ with respect to the action of B is given on Fig 1.

Let $a \in \Theta$ and $B(a) = C_B(a)$ be the stabilizer of a in the action of B on Θ by conjugation. The

diameter of Θ is 3; $B(a)$ acts transitively on $\Theta_1(a)$; has two orbits $\Theta_2^3(a)$ and $\Theta_2^4(a)$ on $\Theta_2(a)$, where

$$\Theta_2^m(a) = \{b \mid b \in \Theta_2(a), o(ab) = m\};$$

a commutes with every involution from $\Theta_3(a)$ and $B(a)$ acts transitively on this set. This implies particularly that a (considered as an element of B) fixes $\{a\} \cup \Theta_1(a) \cup \Theta_3(a)$ and acts on $\Theta_2(a)$ with orbits of length 2.

Locally Θ is the commuting graph Δ of central (root) involutions in $B(a)/\langle a \rangle \cong {}^2E_6(2) : 2$. In terms of the F_4 -building $\mathcal{F}$ with the digram

the vertices of Δ are the elements of type 1 with two of them adjacent if incident to a common element of type 4.

The group B contains a maximal 2-local subgroup $S \sim 2^{9+16}.Sp_8(2)$. The centre of $O_2(S)$ contains exactly 120 $\{3, 4\}$ -transpositions which induce in Θ a clique (a maximal complete subgraph) Q . Let $\mathcal{E}(B)$ be an incidence system formed by the set $\mathcal{Q}$ of images of Q under B together with the non-empty intersections of two or more cliques from $\mathcal{Q}$; the incidence relation is via inclusion and the type of an element is determined by its size. Then $\mathcal{E}(B)$ is a geometry with the diagram

for $t = 4$ mentioned in [B85] and B induces on $\mathcal{E}(B)$ a flag-transitive action. The elements of type 1 are the vertices of Θ and the residue of such a vertex a is isomorphic to the building $\mathcal{F}$.

In [I92] within the characterization proof for the rank 5 Petersen geometry of B the following result has been established.

Proposition 1.1 *Let Γ be a graph which is locally the commuting graph of central involutions in ${}^2E_6(2)$. Suppose that Γ possesses an automorphism group G such that for every $x \in \Gamma$ the stabilizer $G(x)$ of x in G induces on $\Gamma(x)$ an action containing ${}^2E_6(2)$. Then Γ is isomorphic to the Baby Monster graph Θ . $\square$*

The main goal of this paper is to prove the assertion in Proposition 1.1 without assuming anything about the automorphism group of Γ (see Theorem 1 below). One of the intended applications of Theorem 1 is using it as an identification tool in the classification of (non-necessary flag-transitive) P -geometries of rank 5, as a continuation of the project started in [HS00].

Let $a \in \Theta$ and $b \in \Theta_2^3(a)$. Then the maximal intersection with $\Theta_3(a)$ (resp. with $\Theta_3(a) \cap \Theta_3(b)$) of a clique from $\mathcal{Q}$ is of size 64 (resp. 36). Let $\mathcal{E}({}^2E_6(2))$ and $\mathcal{E}(Fi_{22})$ be the incidence systems formed by the maximal intersections of the sets $\Theta_3(a)$ and $\Theta_3(a) \cap \Theta_3(b)$, respectively with the cliques from $\mathcal{Q}$ together with the elements of $\mathcal{E}(B)$ contained in these sets. The incidence relation and type function are as in $\mathcal{E}(B)$.

Proposition 1.2 (i) $\mathcal{E}({}^2E_6(2))$ is a $c.F_4(2)$ -geometry, on which $B(a)/\langle a \rangle \cong {}^2E_6(2) : 2$ induces a flag-transitive action;

(ii) $\mathcal{E}(Fi_{22})$ is a $c.F_4(1)$ -geometry, on which $B(a) \cap B(b) \cong Fi_{22} : 2$ induces a flag-transitive action.

By the construction $\mathcal{E}({}^2E_6(2))$ is a subgeometry in $\mathcal{E}(B)$ and $\mathcal{E}(Fi_{22})$ is a subgeometry in $\mathcal{E}({}^2E_6(2))$. By (1.1) the geometry $\mathcal{E}(B)$ is simply connected which is not the case for $\mathcal{E}({}^2E_6(2))$ and $\mathcal{E}(Fi_{22})$. Let $\mathcal{A}$ be the amalgam of maximal parabolics associated with the action of ${}^2E_6(2) : 2$ on $\mathcal{E}({}^2E_6(2))$. Then $\mathcal{A}$ is embedded into the unique non-split extension $3 \cdot {}^2E_6(2) : 2$ and hence there is a (3-fold) covering

$$\varphi : \mathcal{E}(3 \cdot {}^2E_6(2)) \rightarrow \mathcal{E}({}^2E_6(2))$$

of geometries. The preimage of $\mathcal{E}(Fi_{22})$ with respect to φ is connected and it is a flag-transitive triple cover $\mathcal{E}(3 \cdot Fi_{22})$ of $\mathcal{E}(Fi_{22})$.

The subdegrees of the action of $E_6(2)$ on the cosets of $F_4(2)$ calculated in [L93] show that one of the orbitals of this action is the graph of a flag-transitive $c.F_4(2)$ -geometry $\mathcal{E}(E_6(2))$. In a similar way the orbit lengths of $F_4(2)$ on the vectors of a 26-dimensional $GF(2)$ -module calculated in [CC88] prove existence of a $c.F_4(2)$ -geometry $\mathcal{E}(2^{26}.F_4(2))$.

We are focussed on the c -extensions $\mathcal{E}$ of buildings with diagrams $c.F_4(t)$, satisfying the following two conditions:

- (a) any two elements of type 1 are incident to at most one common element of type 2;
- (b) three elements of type 1 are pairwise incident to common elements of type 2 if and only if they are incident to a common element of type 5.

Let $\Gamma = \Gamma(\mathcal{E})$ be the graph of $\mathcal{E}$. Then the above conditions are equivalent, respectively, to the following ones:

- (a') Γ has no multiple edges;
- (b') for an element z of type 5 the vertices in Γ incident to z induce a complete graph $Q(z)$ and every triangle in Γ is contained in $Q(z)$ for some z .

The following theorems constitute the main result of the paper.

Theorem 1 *Let Γ be a graph which is locally the commuting graph of the central involutions in ${}^2E_6(2)$. Then Γ is isomorphic to the Baby Monster graph Θ .*

Theorem 2 *Let $\mathcal{E}$ be a $c.F_4(4)$ -geometry satisfying the conditions (a) and (b). Then $\mathcal{E}$ is the Baby Monster geometry $\mathcal{E}(B)$.*

We emphasise again that in Theorems 1 and 2 no automorphisms are assumed *a priori*. The condition (a) excludes some possible degenerate cases and it holds in every flag-transitive $c.F_4(t)$ -geometry. On the other hand the examples presented at the end of Section 4 show that the class of $c.F_4(t)$ -geometries which do not satisfy (b) is rather large and even in the flag-transitive case it is difficult to expect the complete classification.

The paper is organized as follows. In Section 2 we recall some basic properties of the F_4 -buildings with 3 points per a line. In Section 3 we discuss some general properties of $c.F_4(t)$ -geometries and their graphs. In Section 4 we present and discuss some examples of $c.F_4(t)$ -geometries. In Section 5 we construct an important class of subgraphs in a graph of $c.F_4(t)$ -geometry. In Sections 6 and 7 we characterize the μ -graphs which are subgraphs induced by the common neighbours of a pair of vertices at distance 2. Finally in Section 8 (which consists of a few subsections) we restrict ourselves to the case $t = 4$ and eventually prove the main theorems.

2 Some F_4 -buildings

Let $\mathcal{F} = {}^t\mathcal{F}$ be a building with diagram

By the classification [T74] of spherical buildings we obtain that $t = 4, 2$ or 1 and the (flag-transitive) automorphism group $F = {}^tF$ of $\mathcal{F}$ is isomorphic to

$${}^2E_6(2) : S_3, \quad F_4(2) \quad \text{or} \quad \Omega_8^+(2) : S_3,$$

respectively. The elements of type 1, 2, 3 and 4 in $\mathcal{F}$ will also be called *points*, *lines*, *planes* and *symplecta*, respectively. The points of $\mathcal{F}$ are the central (root) involutions in F for $t = 1$ and 4 and one of the two classes of such involutions for $t = 2$. Let ${}^t\Psi$ be the collinearity graph of $\mathcal{F}$, that is the graph on points in which two of them are adjacent if incident to a common line. The following result is well known ([Coh83] and [Coo83]).

Lemma 2.1 *The suborbit diagram of Ψ is the one given on Fig. 2, where $\{p\}$, $\Psi_1(p)$, $\Psi_2^2(p)$, $\Psi_2^4(p)$ and $\Psi_3(p)$ are the orbits of $F(p) := C_F(p)$ on Ψ (p is a point). Furthermore*

- (i) $F(p) \cong 2_+^{1+20} : U_6(2).S_3$, $2^{1+6+8} : Sp_6(2)$ and $2_+^{1+8} : (S_3 \wr S_3)$ for $t = 4, 2$ and 1 ;
- (ii) $q \in \Psi_1(p)$ if and only if $q \in O_2(F(p))$;
- (iii) $q \in \Psi_2^2(p)$ if and only if $q \in F(p) \setminus O_2(F(p))$, in which case there is a unique symplecton incident to p and q ;
- (iv) $q \in \Psi_2^4(p)$ if and only if $o(pq) = 4$, in which case $(pq)^2$ is the unique point collinear to both p and q ;
- (v) $q \in \Psi_3(p)$ if and only if $o(pq) = 3$, in which case $F(p) \cap F(q)$ is a complement to $O_2(F(p))$ in $F(p)$, if $\{p, r_1, r_2\}$ is a line containing p , then up to renumbering $r_1 \in \Psi_2^4(q)$, $r_2 \in \Psi_3(q)$;
- (vi) the orbits of $O_2(F(p))$ on $\Psi_1(p)$, $\Psi_2^2(p)$, $\Psi_2^4(p)$ and $\Psi_3(p)$ are of length 2, $2^4 \cdot t$, $2^5 \cdot t^3$ and $2^9 \cdot t^6 = |\Psi_3(p)|$, respectively;
- (vii) the orbits of $O_2(F(p))$ on $\Psi_1(p)$ and $\Psi_2^4(p)$ naturally correspond to the lines incident to p while the orbits of $O_2(F(p))$ on $\Psi_2^2(p)$ naturally correspond to the symplecta incident to p . $\square$

The points lines, planes and symplecta in $\mathcal{F}$ can be identified with the sets (of size 1, 3, 7 and $28t + 7$, respectively) of points incident to these elements, so that the incidence relation is via inclusion. The geometry $\mathcal{F}$ can be reconstructed from Ψ in the following way. If $q \in \Psi_2^2(p)$, then the unique symplecton containing p and q is the minimal geodetically closed subgraph in Ψ containing p and q . The remaining elements of $\mathcal{F}$ are the non-empty intersections of two or more symplecta.

There are the following embeddings among the three geometries:

$${}^1\mathcal{F} \subset {}^2\mathcal{F} \subset {}^4\mathcal{F}$$

which can be seen as follows. The group 4F contains an (outer) involution x such that $C_{{}^4F}(x) \cong \langle x \rangle \times {}^2F$. Furthermore there is a conjugate y of x in 4F such that $\langle x, y \rangle \cong S_3$ and $C_{{}^4F}(\langle x, y \rangle) \cong {}^1F$. The subgraph in ${}^4\Psi$ induced by the vertices fixed by x is isomorphic to ${}^2\Psi$ and the subgraph induced by the vertices fixed by $\langle x, y \rangle$ (equivalently by xy) is isomorphic to ${}^1\Psi$.

Fig. 2

Let $\widehat{\Psi}$ be a graph on the point-set of $\mathcal{F}$ in which two points p and q are adjacent if they are incident to a common symplecton but not to a common line, equivalently if $q \in \Psi_2^2(p)$. The suborbit diagram of $\widehat{\Psi}$ computed from that of Ψ is presented on Fig. 3.

Let $\Delta = {}^t\Delta$ be the graph on the point-set of $\mathcal{F}$ in which two points are adjacent whenever they are incident to a common symplecton. This means that two points p and q are adjacent in Δ if $q \in \Psi_1(p) \cup \Psi_2^2(p)$, so that the edge-set of Δ is the union of the edge-sets of Ψ and $\widehat{\Psi}$. By (2.1) Δ is the commuting graph on points of $\mathcal{F}$ (considered as involutions in F).

The next two lemmas follow directly from the diagrams of Ψ and $\widehat{\Psi}$.

Lemma 2.2 *If (p, r, q) is a 2-path in Δ such that $q \in \Psi_3(p)$, then $p, q \in \Psi_2^2(r)$.* $\square$

Lemma 2.3 *Let $\{p, q\}$ be an edge of Δ . Then exactly one of the following holds:*

- (i) $|\Delta(p) \cap \Delta(q)| = (24t + 4)(t^2 + t + 1) + 1$ (which is 2101, 365 and 85 for $t = 4, 2$ and 1) and $\{p, q\}$ is an edge of Ψ ;
- (ii) $|\Delta(p) \cap \Delta(q)| = 8t^4 + 12t^3 + 4t^2 + 28t + 5$ (which is 2997, 391 and 57 for $t = 4, 2$ and 1) and $\{p, q\}$ is an edge of $\widehat{\Psi}$. $\square$

Fig. 3

The subgraph in Ψ induced by (the points incident to) a symplecton has the following suborbit diagram.

Comparing this diagram with that of Ψ and $\widehat{\Psi}$ we obtain the following

Lemma 2.4 *Let $p \in \Delta$, $q \in \Psi_2^2(p)$ and let Υ be the subgraph in Ψ induced by (the points incident to) the unique symplecton containing p and q . Then*

$$\Upsilon_1(p) = \Delta(q) \cap \Psi_1(p) \quad \text{and}$$

$$\Upsilon_2(p) = \{r \mid r \in \Psi_2^2(p), \Delta(r) \cap \Psi_1(p) = \Upsilon_1(p)\}.$$

Lemma 2.5 *The automorphism group of $\Delta = {}^t\Delta$ is isomorphic to tF (which is ${}^2E_6(2) : S_3$, $F_4(2)$ or $\Omega_8^+(2) : S_3$ for $t = 4, 2$ and 1 , respectively).*

Proof. By (2.3) the automorphism group of Δ does not mix the edges of Ψ with the edges of $\widehat{\Psi}$ and by (2.4) the set of symplecta of $\mathcal{F}$ can be reconstructed from Δ . The lines and planes of $\mathcal{F}$ can

be reconstructed as the non-empty intersections of two or more symplecta; the incidence relation in $\mathcal{F}$ is via inclusion. Hence every automorphism of Δ induces a uniquely determined automorphism of $\mathcal{F}$ and the result follows. $\square$

3 c -extensions of $\mathcal{F}$

Throughout this section $\mathcal{E} = {}^t\mathcal{E}$ is a geometry with diagram $c.F_4(t)$, where $t = 4, 2$ or 1 , such that the residue of an element of type 1 is isomorphic to the building ${}^t\mathcal{F}$ and the conditions (a), (b) formulated in Section 1 are satisfied. Since these conditions are equivalent to conditions (a') and (b'), we have the following

Lemma 3.1 *The graph $\Gamma(\mathcal{E})$ is locally Δ .* $\square$

We will see that a converse of this lemma holds.

Let $\Delta = {}^t\Delta$ for $t = 4, 2, 1$ and let Γ be a graph which is locally Δ . For every $x \in \Gamma$ let us fix a bijection

$$i_x : \Gamma(x) \rightarrow \Delta$$

which induces an isomorphism of the subgraph in Γ induced by $\Gamma(x)$ onto Δ (so that if $y \in \Gamma(x)$ then $i_x(y)$ is an involution in tF). The following lemma is a direct consequence of (2.3).

Lemma 3.2 *Let $\{x_1, x_2, x_3\}$ be a triangle in Γ . Then one of the following holds:*

- (i) $i_{x_j}(x_k) \in \Psi_1(i_{x_j}(x_l))$ for all $\{j, k, l\} = \{1, 2, 3\}$;
- (ii) $i_{x_j}(x_k) \in \Psi_2^2(i_{x_j}(x_l))$ for all $\{j, k, l\} = \{1, 2, 3\}$. $\square$

The triangles in Γ as in (3.2 (i)) will be called *short* while those in (3.2 (ii)) will be called *long*.

Lemma 3.3 [7] *Let Γ be a graph which is locally Δ . Then $\Gamma = \Gamma(\mathcal{E})$, for a $c.F_4(t)$ -geometry $\mathcal{E}$ as above.*

Proof. Let x be a vertex of Γ and P be the point-set of a symplecton in $\mathcal{F}$. Then the set

$$Q_{x,P} = \{x\} \cup \{y \in \Gamma(x) \mid i_x(y) \in P\}$$

is a complete subgraph of size $28t + 8$ in Γ . It is easy to deduce using (2.4) and (3.2) that for every $y \in Q_{x,P}$ we have $Q_{x,P} = Q_{y,R}$ for a symplecton R in $\mathcal{F}$.

Define $\mathcal{E}$ to be a rank 5 geometry in which the elements of type 5 are the subgraphs $Q_{x,P}$ taken for all $x \in \Gamma$ and all symplecta P in $\mathcal{F}$; the remaining elements of $\mathcal{E}$ are the non-empty intersections of two or more such subgraphs; the elements of type 1, 2, 3 and 4 are the intersections of size 1, 2, 4 and 8, respectively. The incidence is via inclusion. Then it is immediately seen that $\mathcal{E}$ has the required properties and that Γ is the graph of $\mathcal{E}$. $\square$

In what follows the elements of type 1, 2, 3, 4 and 5 in $\mathcal{E}$ will be identified with the corresponding complete subgraphs in $\Gamma = \Gamma(\mathcal{E})$ as in (3.3) of size 1, 2, 4, 8 and $28t + 8$, respectively.

By (3.3) it is clear that $\mathcal{E}$ satisfies the intersection property (cf. [Pasi94]) and hence by [CP92] the residue $\mathcal{H}$ of an element of type 5 in $\mathcal{E}$ is a standard quotient of an affine polar space of type C_4 . On the other hand by condition (b) in the considered situation $\mathcal{H}$ is a 1-point extension of its C_3 -residue. This gives the following.

Lemma 3.4 *Let ${}^t\mathcal{H}$ be the residue of an element of type 5 in $\mathcal{E}$. Then ${}^t\mathcal{H}$ is flag-transitive with the automorphism group isomorphic to*

$$Sp_8(2), \quad 2^6 : Sp_6(2), \quad Sp_6(2)$$

for $t = 4, 2, 1$, respectively, and acting doubly transitively on the set of elements of type 1 in ${}^t\mathcal{H}$. Furthermore ${}^1\mathcal{H}$ is a subgeometry in ${}^2\mathcal{H}$ and the latter is a subgeometry in ${}^4\mathcal{H}$. $\square$

Lemma 3.5 *Let $L = \{x, z, x_1, z_1\}$ be an element of type 3 in $\mathcal{E}$ and y be an element of type 1, not in L , but incident with L to a common element Q of type 5 and $N = L \cup \{y\}$. Then exactly one of the following holds:*

- (i) *N is contained in an element of type 4 and every triangle in N is short;*
- (ii) *N is not contained in an element of type 4, there are exactly two short triangles in N , say T_1 and T_2 , which contain y and $N = T_1 \cup T_2$.*

Proof. Consider $\mathcal{F} = \text{res}_{\mathcal{E}}(x)$. Then $M := L \setminus \{x\}$ is a line and y is a point, both contained in the symplecton $Q \setminus \{x\}$. Then by the basic property of polar spaces, either y is collinear to every point in M , in which case $M \cup \{y\}$ is contained in a plane of $\mathcal{F}$ and we have (i); or y is collinear to a unique point, say z in M , in which case $\{y, x, z\}$ is short and the remaining two triangles in N containing $\{y, x\}$ are long. Since a similar assertion holds in $\text{res}_{\mathcal{E}}(x_1)$, the triangle $\{y, x_1, z_1\}$ must also be short. $\square$

From (3.5) we deduce that the short triangles contained in an element of type 5 define a 2-graph, which is equivalent to the following

Corollary 3.6 *If N is a 4-element subset of (the set of elements of type 1 in) an element of type 5 in $\mathcal{F}$, then the number of short triangles in N is even. $\square$*

The following lemma is nothing but a more detailed version of (3.6).

Lemma 3.7 *Let $\{x, z\}$ and $\{u, v\}$ be disjoint elements of type 2 contained in an element of type 5. Then exactly one of the following holds:*

- (i) *$\{x, z, u, v\}$ is contained in an element of type 4;*
- (ii) *both $\{x, z, u\}$ and $\{x, z, v\}$ are short and both $\{x, u, v\}$ and $\{z, u, v\}$ are long;*
- (iii) *exactly one of $\{x, z, u\}$ and $\{x, z, v\}$ is short and exactly one of $\{x, u, v\}$ and $\{z, u, v\}$ is short;*
- (iv) *both $\{x, z, u\}$ and $\{x, z, v\}$ are long and $\{x, u, v\}$ and $\{z, u, v\}$ are of the same type (both short or both long). $\square$*

The following result can alternatively be checked in the residue ${}^t\mathcal{H}$, known by (3.4).

Lemma 3.8 *Let $X = \{x, z, u_1, u_2\}$ and $Y = \{x, z, v_1, v_2\}$ be distinct elements of type 3 in $\mathcal{E}$ incident to the common element $\{x, z\}$ of type 2 and contained in a common element of type 5. Then the symmetric difference $Z = \{u_1, u_2, v_1, v_2\}$ of X and Y is also an element of type 3 in $\mathcal{E}$.*

Proof. If X and Y are contained in an element N of type 4, then the result is immediate from the structure of $\text{res}_{\mathcal{E}}(N)$, which is the 3-dimensional $GF(2)$ -space. Suppose that X and Y are not in an element of type 4. Then applying (3.5) to $\{x, u_1, u_2, v_1\}$, we observe that the triangle $\{u_1, u_2, v_1\}$ is short. By the obvious symmetry every triangle in Z is short and hence Z is an element of type 3. $\square$

4 The examples

By (3.1) and (3.3) there is a natural bijection between the $c.F_4(t)$ -geometries satisfying (a) and (b) and the graphs which are locally ${}^t\Delta$. Thus in order to construct a $c.F_4(t)$ -geometry it is sufficient to construct a graph which is locally ${}^t\Delta$.

It is known (cf. Lemma 5.10.6 in [I99]) that the Baby Monster graph is locally ${}^4\Delta$ which gives the geometry $\mathcal{E}(B)$ as in [B85].

For $t = 4, 2, 1$ put $n(t) = |{}^t\Delta|$. The following well-known result comes, for instance, from information on the maximal 2-locals in tF (compare [ATLAS]).

Lemma 4.1 *Up to conjugation in the automorphism group of tF every transitive action of tF of degree $n(t)$ is similar to the action on the vertex-set of ${}^t\Delta$. $\square$*

Consider the action of ${}^2E_6(2) : 2$ on the cosets of $F_4(2) \times 2$. The subdegrees of this action are calculated in [L93]. In fact this action is similar to the action of $O^2(B(a))$ on $\Theta_3(a)$ where B is the Baby Monster and Θ is the Baby Monster graph. This observation enables one to deduce the subdegrees from the structure constants of the centralizer algebra of B acting on Θ (cf. p. 128 in [PS97]).

Lemma 4.2 *The action of ${}^2E_6(2) : 2$ on the cosets of $F_4(2) \times 2$ has rank 4 with subdegrees 1, $n(2) = (2^4 + 1)(2^{12} - 1)$, $2^4(2^8 - 1)(2^{12} - 1)$ and $2^{12}(2^4 + 1)(2^8 + 2^4 + 1)$. $\square$*

Consider the orbital Γ of valency $n(2)$ of the action in (4.2). Comparing the subdegrees in (4.2) with the subdegrees of the action of $F_4(2)$ on ${}^2\Delta$ and since the action in (4.2) is by conjugation on a class of outer involutions in ${}^2E_6(2) : 2$ it is easy to see that Γ is locally ${}^2\Delta$ which gives the geometry $\mathcal{E}({}^2E_6(2))$.

Similarly [L93] gives the following.

Lemma 4.3 *The action of $E_6(2) : 2$ on $F_4(2) \times 2$ has rank 6 with subdegrees 1, $n(2) = (2^4 + 1)(2^{12} - 1)$, $2^4(2^8 - 1)(2^{12} - 1)$, $2^{12}(2^{12} - 1)$ and $2^{12}(2^4 - 1)(2^8 - 1)$ (twice). $\square$*

Again we observe that the orbital of valency $n(2)$ is the graph of a $c.F_4(2)$ -geometry $\mathcal{E}(E_6(2))$.

The following result follows from Table 2 in [CC88]

Lemma 4.4 *Let V be a 26-dimensional $GF(2)$ -module of $F \cong F_4(2)$. Then the orbit lengths of F on the set of vectors of V are the following: 1, $n(2) = (2^4 + 1)(2^{12} - 1)$, $2^8(2^8 + 2^4 + 1)$, $2^4(2^{12} - 1)(2^8 - 1)$, $2^8(2^{12} - 1)(2^4 + 1)$, $2^{12}(2^8 - 1)(2^4 - 1)$ and $2^{12}(2^{12} - 1)$. $\square$*

The orbit lengths in (4.4) are the subdegrees of the natural action on V of the semidirect product $V : F \cong 2^{26} : F_4(2)$ and similarly to the above case we observe that the orbital of valency $n(2)$ of this action is locally ${}^2\Delta$ and hence it is the graph of a $c.F_4(2)$ -geometry $\mathcal{E}(2^{26} : F_4(2))$.

From Lemma 2.17.1 in [ILLSS] or p. 112 in [PS97] we obtain the suborbits of the action of $Fi_{22} : 2$ on the cosets of $\Omega_8^+(2).S_3 \times 2$. Notice, that this action is similar to the action of $B(a) \cap B(b)$ on $\Theta_3(a) \cap \Theta_3(b)$ for $b \in \Theta_2^3(a)$ (compare the intersection number 61 776 in the top matrix on p. 128 in [PS97]).

Lemma 4.5 *The action of $Fi_{22} : 2$ on the cosets of $\Omega_8^+(2).S_3 \times 2$ has rank 4 with subdegrees 1, $n(1) = 1\,575$, 22 400, 37 800. $\square$*

It is immediate from the structure constants of the action from (4.5) given in [ILLSS] that the orbital of valency $n(1)$ is locally ${}^1\Delta$ and hence it is the graph of a $c.F_4(1)$ -geometry $\mathcal{E}(Fi_{22})$.

Let ${}^2H \cong {}^2E_6(2) : 2$, ${}^1H \cong Fi_{22} : 2$ and Ω be the set of cosets of ${}^2K \cong F_4(2) \times 2$ and ${}^1K \cong \Omega_8^+(2).S_3 \times 2$, respectively. Then the elements of Ω are identified with a conjugacy class of outer involutions in tH so that the action is by conjugation. Let $d \in \Omega$ and $\{d\}, \Omega_1(d), \Omega_2^3(d), \Omega_2^4(d)$ be the orbits of tK on Ω (compare (4.2) and (4.5)). Then under a suitable numbering of the orbits we have $o(de) = 2, 3, 4$ for $e \in \Omega_1(d), \Omega_2^3(d), \Omega_2^4(d)$, respectively (so that ${}^tK = C_{{}^tH}(d)$ and $|\Omega_1(d)| = n(t)$). For $t = 2, 1$ let ${}^t\tilde{H}$ be the unique non-split extension of tH by a normal subgroup Z of order 3 (recall that the 3-part of the Schur multiplier of $O^2({}^tH)$ is of order 3). Let $\tilde{d}$ be an involution in the preimage of $\langle d \rangle$ in ${}^t\tilde{H}$. Then $\tilde{d}$ inverts Z and hence $C_{{}^t\tilde{H}}(\tilde{d})$ maps isomorphically onto (and will be identified with) tK . Let $\tilde{\Omega}$ be the set of cosets of tK in ${}^t\tilde{H}$. Notice that $\tilde{\Omega}$ is a conjugacy class of involutions in ${}^t\tilde{H}$ containing $\tilde{d}$. It is easy to see that the preimage of $\Omega_1(d)$ in $\tilde{\Omega}$ splits into two tK -orbits with lengths $n(t)$ and $2 \cdot n(t)$. Let $\tilde{\Gamma}$ be the orbital of valency $n(t)$. We claim that $\tilde{\Gamma}$ is locally ${}^t\Delta$. Let $\tilde{e}, \tilde{f} \in \tilde{\Gamma}(\tilde{d})$, e, f be their respective images in Ω and suppose that $o(e\tilde{f}) = 2$. Since $\tilde{d}$ is an involution which commutes with $\tilde{e}$ and $\tilde{f}$, we have $o(\tilde{e}\tilde{f}) = o((\tilde{e}\tilde{d})(\tilde{f}\tilde{d}))$. Since Z commutes with the product of any two involutions from $\tilde{\Omega}$ the claim follows and we have the following.

Lemma 4.6 *The geometries $\mathcal{E}({}^2E_6(2))$ and $\mathcal{E}(Fi_{22})$ possess 3-fold covers $\mathcal{E}(3 \cdot {}^2E_6(2))$ and $\mathcal{E}(3 \cdot Fi_{22})$, respectively. $\square$*

The subdegrees of $3 \cdot {}^2E_6(2)$ on the cosets of $F_4(2)$ are given in Table 3 in [L93].

To the end of this section we discuss some $c.F_4(t)$ -geometries which do not satisfy condition (b). One class of such examples can be constructed as follows. Consider the action of the Baby Monster B on the cosets of a subgroup $2 \cdot {}^2E_6(2)$. Then one of the orbitals of valency $n(4)$ with respect to the action (the graph $\tilde{\Delta}_2$ on p. 98 in [I94]) is the graph of a $c.F_4(4)$ -geometry which is a double cover of $\mathcal{E}(B)$. The residue of an element of type 5 in this cover is a double cover of the complete graph on 120 vertices. The preimages of $\mathcal{E}({}^2E_6(2))$ - and $\mathcal{E}(Fi_{22})$ -subgeometries in $\mathcal{E}(B)$ are also proper double covers. One can construct analogous double covers of $\mathcal{E}(3 \cdot Fi_{22})$ and $\mathcal{E}(3 \cdot {}^2E_6(2))$.

Let $\mathcal{F} = {}^t\mathcal{F}$ be an F_4 -building with 3 points per a line and let (R, φ) be a representation of $\mathcal{F}$, which means that R is a group and φ is the a mapping $\varphi : p \mapsto z_p$ of the point-set of $\mathcal{F}$ into R such that (i) R is generated by the image of φ ; (ii) $z_p^2 = 1$ for every point p ; (iii) $z_p z_q z_r = 1$ whenever $\{p, q, r\}$ is a line. For an arbitrary element $u \in \mathcal{F}$ define $\varphi(u)$ to be the subgroup of R generated by the elements z_p taken for all points p incident to u . Suppose that (R, φ) is separable which means that the mapping $u \mapsto \varphi(u)$ of $\mathcal{F}$ into the set of subgroups of R is injective. Then there is a standard procedure which enables one to construct a c -extension $\mathcal{E}$ of $\mathcal{F}$: the elements of $\mathcal{E}$ are the elements of R and all cosets of the subgroups $\varphi(u)$ taken for all $u \in \mathcal{F}$; the incidence is via inclusion. Notice that $\mathcal{E}(2^{26} : F_4(2))$ can be constructed along these lines.

If F is the automorphism group of $\mathcal{F}$ then (F^∞, ψ) is a separable representation of $\mathcal{F}$ where ψ is the identity mapping. In addition for each $t = 4, 2$ and 1 the geometry $\mathcal{F}$ possesses an abelian representation. By this reason the classification of all (even flag-transitive) $c.F_4(t)$ -geometries seems to be far too difficult.

5 A subgraph Ξ

We follow the notation introduced in Section 3 so that $\mathcal{E}$ is a $c.F_4(t)$ -geometry for $t = 4, 2, 1$ whose elements of type 1, 2, 3, 4 and 5 are identified with the corresponding complete subgraphs of size 1,

2, 4, 8 and $28t + 8$ in the graph $\Gamma = \Gamma(\mathcal{E})$, which is locally ${}^t\Delta$.

Let $\tilde{\Xi} = {}^t\tilde{\Xi}$ be the graph on the set of elements of type 2 in $\mathcal{E}$ in which two such elements are adjacent if they are incident to a common element of type 3 but not to a common element of type 1 (equivalently if their union is an element of type 3). Let $e = \{x, y\}$ be an element of type 2 in $\mathcal{E}$ (which is an edge of Γ and a vertex of $\tilde{\Xi}$). Then the set $\tilde{\Xi}(e)$ of neighbours of e in $\tilde{\Xi}$ is in the natural bijection with the set of elements of type 3 incident to e via the correspondence

$$\psi : e_1 \rightarrow e \cup e_1$$

for $e_1 \in \tilde{\Xi}(e)$. Let Λ be the graph on the set of elements of type 3 incident to e in which two of them are adjacent if incident to a common element of type 4. Then Λ is the collinearity graph of the residual dual polar space $\text{res}_{\mathcal{E}}^+(e)$ with the diagram $\begin{smallmatrix} \circ & \text{---} & \circ & \text{---} & \circ \\ 2 & & t & & t \end{smallmatrix}$ and the suborbit diagram of Λ is

Let $e_1, e_2 \in \tilde{\Xi}(e)$, where $e_1 = \{x_1, y_1\}$ and $e_2 = \{x_2, y_2\}$. If $\psi(e_1)$ and $\psi(e_2)$ are at distance 1 or 2 in Λ , then they are incident to a common element of type 5 in $\mathcal{E}$ and by (3.8) $e_1 \cup e_2$ is an element of type 3 which means that e_1 and e_2 are adjacent in $\tilde{\Xi}$. On the other hand, if $\psi(e_1), \psi(e_2)$ are at distance 3 in Λ then $i_x(x_2) \in \Psi_2^4(i_x(x_1))$, in which case e_1 and e_2 are certainly not adjacent in $\tilde{\Xi}$. This implies the following.

Lemma 5.1 *Locally $\tilde{\Xi}$ is the distance 1-or-2 graph of Λ .* □

Now by Theorem 1.3 (vi), (vii), (ix) in [C99] we obtain the following.

Proposition 5.2 *Let e be an element of type 2 in $\mathcal{E}$ and $\tilde{\Xi}^e$ be the connected component of $\tilde{\Xi}$ containing e . Then*

- (i) ${}^1\tilde{\Xi}^e$ has 40 vertices, its automorphism group is isomorphic to $U_4(2) : 2$ and the suborbit diagram is

- (ii) ${}^2\tilde{\Xi}^e$ has 256 vertices, its automorphism group is isomorphic to $2^8 : Sp_6(2)$ and the suborbit diagram is

- (iii) ${}^4\tilde{\Xi}^e$ has 2300 vertices, its automorphism group is isomorphic to Co_2 and the suborbit diagram is

Furthermore, ${}^1\tilde{\Xi}^e$ is a subgraph in ${}^2\tilde{\Xi}^e$ and the latter is a subgraph in ${}^4\tilde{\Xi}^e$. $\square$

Let Ξ^e be the subgraph in Γ induced by the elements of type 1 in $\mathcal{E}$ incident to the elements of type 2 in $\tilde{\Xi}^e$. Since the diameter of $\tilde{\Xi}^e$ is 2, in view of the paragraph before (5.1), every vertex in Ξ^e is incident to exactly one element of type 2 in $\tilde{\Xi}^e$ which gives the following.

Lemma 5.3 *The subgraph Ξ^e is the 2-clique extension of $\tilde{\Xi}^e$.* $\square$

6 μ -graphs of D_8 -type

In this and next sections for a pair $\{x, y\}$ of vertices at distance 2 in Γ we analyze the subgraph $\Gamma(x, y)$ induced in Γ by the common neighbours of x and y (the subgraphs $\Gamma(x, y)$ are commonly known as μ -graphs).

First we introduce some terminology. Recall that $\pi = (x, z, y)$ is a 2-path in Γ if $x, y \in \Gamma(z)$ and $y \in \Gamma_2(x)$. Such a 2-path π is said to be of D_6 - or D_8 -type if

$$i_z(y) \in \Psi_3(i_z(x)) \quad \text{or} \quad i_z(y) \in \Psi_2^4(i_z(x)),$$

(equivalently if $\langle i_z(x), i_z(y) \rangle \cong D_6$ or D_8), respectively. Clearly the paths (x, z, y) and (y, z, x) have the same type.

For $j = 3$ or 4 and a vertex x of Γ let $\Gamma_2^j(x)$ denote the set of vertices y at distance 2 from x in Γ such that there is a path of D_{2j} -type joining x and y . Notice that $\Gamma_2(x) = \Gamma_2^3(x) \cup \Gamma_2^4(x)$ and *a priori* the sets $\Gamma_2^3(x)$ and $\Gamma_2^4(x)$ might intersect.

The next lemma follows from the fact that Γ is locally Δ and from the suborbit diagrams of Ψ and $\hat{\Psi}$.

Lemma 6.1 *Let $\pi = (x, z, y)$ be a 2-path in Γ . Then*

- (i) *if π is of D_8 -type, then the valency of z in $\Gamma(x, y)$ is $(2t+6)(t^2+t+1)+1$ (which is 295, 71 and 25 for $t = 4, 2$ and 1);*
- (ii) *if π is of D_6 -type, then the valency of z in $\Gamma(x, y)$ is $(2t^2+1)(t^2+t+1)$ (which is 693, 63 and 9 for $t = 4, 2$ and 1).* $\square$

Lemma 6.2 *Let $\pi = (x, z, y)$ be a 2-path of D_8 -type. Then*

- (i) *there is an element e of type 2 in $\mathcal{E}$ incident to x such that $y, z \in \Xi^e$;*
- (ii) *every 2-path contained in Ξ^e is of D_8 -type;*
- (iii) *the connected component of $\Gamma(x, y)$ containing z is contained in Ξ^e .*

Proof. Since π is of D_8 -type by (2.1 (iv)) there is a (unique) vertex u adjacent to z , such that $i_z(x), i_z(y) \in \Psi_1(i_z(u))$. Then clearly $\pi \in \Xi^e$ where $e = \{z, u\}$ and (i) follows. Now (ii) follows from the paragraph before (5.1) and the fact that the diameter of Ξ^e is 2. In order to prove (iii) it is sufficient to show that the valency k_1 of z in $\Gamma(x, y) \cap \Xi^e$ is equal to the valency of z in $\Gamma(x, y)$ given in (6.1 (i)). By (5.1) the graph $\tilde{\Xi}$ is locally the distance 1-or-2 graph of Λ . The latter is a near hexagon with classical quads. This shows that for $f \in \tilde{\Xi}$, $g \in \tilde{\Xi}_2(f)$ the subgraph in $\tilde{\Xi}$ induced by $\tilde{\Xi}(f) \cap \tilde{\Xi}(g)$ has valency $k_2 = (t+3)(t^2+t+1)$. Since Ξ^e is the 2-clique extension of $\tilde{\Xi}^e$ we have $k_1 = 2 \times k_2 + 1$, hence (iii) follows. $\square$

Lemma 6.3 *Let $\pi = (x, z, y)$ and $\sigma = (x, u, y)$ be 2-paths in Γ such that z and u are in the same connected component of $\Gamma(x, y)$ and $\delta = (z, v, w)$ be a 2-path contained in $\Gamma(x, y)$. Then π , σ and δ have the same type.*

Proof. If one of the paths π , σ , δ is of D_8 -type then by (6.2) all three paths are of D_8 -type and hence the result. $\square$

Let $y \in \Gamma_2^4(x)$, $e = \{x, z\}$ be an element of type 2 in $\mathcal{E}$ such that $y \in \Xi^e$ (compare (6.2 (i))), $p = i_x(z)$, $\Omega = \{i_x(v) \mid v \in \Gamma(x, y) \cap \Xi^e\}$ and I be the setwise stabilizer of Ω in $F = {}^t F$. It is clear that $I \leq F(p)$.

Lemma 6.4 (i) I contains $O_2(F(p))$;

(ii) for every element α in $O^2(F(p))$ there is an automorphism β of Ξ^e stabilizing e such that the action of α on the set of elements of type 3 in $\mathcal{E}$ incident to e is similar to the action of β on $\tilde{\Xi}(e)$;

(iii) $I \cong 2_+^{1+20} : U_4(3) : 2^2$, $2^{1+6+8} : U_3(3) : 2$ and $2_+^{1+8} . 3^{1+2} . 2^2$ for $t = 4, 2$ and 1 , respectively.

Proof. An element e_1 of type 2 in $\mathcal{E}$ is an orbit of $O_2(F(p))$ on $\Psi_1(p)$ if and only if e_1 is the image under i_x of an element from $\tilde{\Xi}(e)$, we obtain (i). Now (ii) and (iii) are immediate from (5.1). $\square$

With Ω and I as above put $\mu_1 = |\Omega|$. Our next goal is to describe the orbits of I on Δ .

Lemma 6.5 *Let L and S be the set of lines and the set of symplecta in $\mathcal{F}$ incident to p , so that $|L| = 3(2t^2 + 1)(2t + 1)$ and $|S| = (2t^2 + 1)(t^2 + t + 1)$. Then*

(i) I has two orbits, L_1 and L_2 on L with lengths $\mu_1/2$ and $|L| - \mu_1/2$, respectively;

(ii) if $t = 1$ or 2 then I acts transitively on S ;

(iii) if $t = 4$ then I has two orbits S_1 and S_2 on S with lengths 126 and 567, furthermore,

(a) the symplecta incident to a line from L_1 are contained in S_2 ;

(b) a line from L_2 is incident to 6 symplecta from S_1 and to 15 symplecta from S_2 .

Proof. The result follows from well-known properties of $F(p)/O_2(F(p))$. For the case $t = 4$ see Lemma 5.10.15 in [I99]. $\square$

The next result is a direct consequence of (2.1 (vii)), (6.4 (i)) and (6.5).

Lemma 6.6 *The following assertions hold:*

- (i) I has two orbits $\Omega_1 = \Omega$ and Ω_2 on $\Psi_1(p)$;
- (ii) if $t = 1, 2$ then I acts transitively on $\Psi_2^2(p)$ while if $t = 4$, I has two orbits Ω_3 and Ω_4 on $\Psi_2^2(p)$ with lengths $2^6 \cdot 126$ and $2^6 \cdot 567$, respectively;
- (iii) I has two orbits Ω_5 and Ω_6 on $\Psi_2^4(p)$ with lengths $16t^3 \cdot |\Omega_1|$ and $16t^3 \cdot |\Omega_2|$;
- (iv) I acts transitively on $\Omega_7 = \Psi_3(p)$. □

Notice that in terms of (6.5) the orbits Ω_1 and Ω_5 correspond to L_1 , Ω_2 and Ω_6 correspond to L_2 , Ω_3 corresponds to S_1 and Ω_4 corresponds to S_2 .

Lemma 6.7 *Let $u \in \Psi_1(p) \cup \Psi_2^2(p) \cup \Psi_2^4(p)$ for $t = 1, 2$ and $u \in \Psi_1(p) \cup \Omega_4 \cup \Psi_2^4(p)$ for $t = 4$. Then u is adjacent in Δ to a vertex from Ω .*

Proof. If $u \in \Psi_1(p)$, then the claim follows from the fact that the subgraph in Δ induced by $\Psi_1(p)$ is connected. A vertex from Ω is clearly adjacent in Δ to vertices in $\Psi_2^2(p)$. Thus the claim follows for $t = 1, 2$ and $u \in \Psi_2^2(p)$ as well as for $t = 4$ and $u \in \Omega_4$ (compare (6.5) (a)). Let $u \in \Psi_2^4(p)$ and v be the unique vertex in $\Psi_1(p) \cap \Psi_1(u)$ (compare the diagram of Ψ). Let A and B be the orbits of $O_2(F(p))$ containing u and v , respectively (so that by (2.1) (vi)) we have $|A| = 2^5 \cdot t^3$ and $|B| = 2$). Then by (2.1) (vii) the stabilizer of A in $F(p)$ coincides with the stabilizer of B in $F(p)$ and it is the stabilizer of the unique line l of $\mathcal{F}$ containing p and v . Let $\mathcal{C}$ be the set of orbits of $O_2(F(p))$ on $\Psi_1(p)$ containing vertices adjacent to u in Δ . Since $|\Delta(p) \cap \Delta(u)| = 1 + 2(t^2 + t + 1)$ while the suborbits of the action of $F(p)$ on the set of lines incident to p are $1, 2(t^2 + t + 1), 4t(t^2 + t + 1), 8t^3$ (compare the diagram of Λ before (5.1)), we conclude that $|\mathcal{C}| = 1 + 2(t^2 + t + 1)$ and $\mathcal{C} \setminus B$ correspond to the lines incident with l to a common plane. Since every vertex from $\Psi_1(p)$ is adjacent in Ψ to a vertex from Ω , this completes the proof. □

As a consequence of the above proof in view of (2.1) (v) and (6.5) (iii) (a)) we obtain the following.

Corollary 6.8 *If $u \in \Psi_3(p)$ for $t = 1, 2$ and $u \in \Omega_3 \cup \Psi_3(p)$ for $t = 4$, then no vertex in Ω is adjacent to u in Δ .* □

Lemma 6.9 *In terms introduced before (6.4) if $w \in \Gamma(x, y) \setminus \Xi^e$, then (x, w, y) is of D_6 -type and either $t = 4$ or $i_x(w) \in \Psi_3(p)$.*

Proof. By (6.2) (iii) $\Gamma(x, y) \cap \Xi^e$ is a connected component of $\Gamma(x, y)$ and hence by (6.7) $i_x(w) \in \Psi_3(p)$ for $t = 1, 2$ and $i_x(w) \in \Psi_3(p) \cup \Omega_3$ for $t = 4$. Suppose that (x, w, y) is of D_8 -type. By (6.2) (i), (iii) there is an element $f = \{x, v\}$ of type 2 in $\mathcal{E}$ such that $y, w \in \Xi^f$ and $\Gamma(x, y) \cap \Xi^f$ is another connected component of $\Gamma(x, y)$. Put $q = i_x(v)$ and $\Omega' = \{i_x(s) \mid s \in \Gamma(x, y) \cap \Xi^f\}$. Recall that Ω' is the union of some $\mu_1/2$ lines containing q with q removed. It is immediate from the diagram of Ψ that a line intersecting $\Psi_3(p)$ contains two points from $\Psi_3(p)$ and one point from $\Psi_2^4(p)$. Hence, assuming that $\Omega' \cap \Psi_3(p) \neq \emptyset$, we must have $q \in \Psi_2^4(p)$. In this case let r be the unique vertex in $\Psi_1(p) \cap \Psi_1(q)$. Then by (6.7) r is adjacent in Δ to a vertex from Ω' and by (2.2) r is not adjacent to any vertex from $\Psi_3(p)$. Hence $\Omega' \cap \Omega_3 \neq \emptyset$, in particular $t = 4$. Let $\{q, s_1, s_2\}$ be a line such that $s_1, s_2 \in \Omega'$ and $s_1 \in \Omega_3$. Then one can see from Fig. 2 that $s_2 \notin \Psi_3(p)$ and hence $s_2 \in \Omega_3$. Comparing the diagram of Ψ and the diagram of the subgraph in Ψ induced by a symplecton given before (2.4) we observe that s_1 and s_2 are contained in the same symplecton, containing p and $q \in \Psi_1(p)$, in particular $\Omega' \cap \Psi_3(p) = \emptyset$. By (6.5) (iii) (b)) q is contained in exactly 6 symplecta intersecting Ω_3 and each of these symplecta contains just 16 lines containing q and intersecting $\Psi_2^2(p)$. Since $6 \times 16 < 324$, we have a contradiction, hence (x, w, y) must be of D_6 -type. □

The results in (5.2), (6.2) and (6.9) can be summarized as follows.

Proposition 6.10 *Let $y \in \Gamma_2^4(x)$. Then*

- (i) *there is a unique element e of type 2 in $\mathcal{E}$ incident to x such that $y \in \Xi^e$;*
- (ii) *for any two elements e and f of type 2 in $\mathcal{E}$ the intersection $\Xi^e \cap \Xi^f$ induces in Γ a complete subgraph;*
- (iii) $|\Gamma_2^4(x)| = |\Delta| \cdot |\Psi_2^4(p)|/\mu_1$ *where $\mu_1 = 648, 144$ and 36 for $t = 4, 2$ and 1 .* □

7 μ -graphs of D_6 -type

In this section we analyze the common neighbours of x and $y \in \Gamma_2^3(x)$. Let (x, z, y) be a 2-path of D_6 -type, $p = i_z(x)$, $q = i_z(y)$ and ${}^t\Upsilon = \Upsilon = \{i_z(u) \mid u \in \Gamma(x, y) \cap \Gamma(z)\}$. Then $q \in \Psi_3(p)$ and by (2.1 (v)) $H = {}^tH := F(p) \cap F(q)$ is a complement to $O_2(F(p))$ in $F(p)$ isomorphic to $U_6(2).S_3$, $Sp_6(2)$ and $S_3 \wr S_3$ for $t = 4, 2$ and 1 .

Lemma 7.1 *The following assertions hold:*

- (i) $|\Upsilon| = (2t^2 + 1)(t^2 + t + 1)$;
- (ii) Υ *is the unique class of central involutions in H of size $|\Upsilon|$;*
- (iii) Υ *is a class of 3-transpositions in H , so that*
 - (a) ${}^1\Upsilon$ *is the complete 3-partite graph $K_{3 \times 3}$ with the suborbit diagram*

- (b) ${}^2\Upsilon$ *is the collinearity graph of the polar space $\mathcal{P}(Sp_6(2))$ of ${}^2H \cong Sp_6(2)$ with the diagram*
and the suborbit diagram of ${}^2\Upsilon$ is

- (c) ${}^4\Upsilon$ *is the collinearity graph of the polar space $\mathcal{P}(U_6(2))$ of ${}^4H \cong U_6(2).S_3$ with the diagram*
and the suborbit diagram of ${}^4\Upsilon$ is

Proof. The assertion (i) is just (6.1 (ii)). By the definition, Υ consists of all the involutions in Δ which commute with p and q . Hence Υ is contained in H and it is closed under conjugation by the elements of H . Hence Υ is a union of some conjugacy classes of involutions in H and since $|\Upsilon|$ is odd, at least one of these classes consists of central involutions. Now (ii) and (iii) follows from the well-known properties of H (cf. [ATLAS]). □

In terms introduced before (7.1) let Π be the connected component of $\Gamma(x, y)$ containing z and $\Sigma = \{i_x(u) \mid u \in \Pi\}$.

Lemma 7.2 *Let $u, v \in \Sigma$. Then*

- (i) $v \in \Psi_2^2(u)$ if u and v are adjacent in Σ and $v \in \Psi_3(u)$ if u and v are at distance 2 in Σ ;
- (ii) the graph Σ is locally Υ .

Proof. The assertion (ii) follows directly from the definition of Σ and Υ while (i) is immediate from (2.2) and (6.3). $\square$

Lemma 7.3 (i) ${}^1\Sigma$ is the complete 4-partite graph $K_{4 \times 3}$ with the automorphism group $S_3 \wr S_4$ and the suborbit diagram

(ii) ${}^2\Sigma$ is isomorphic to one of the following three graphs:

(a) the commuting graph ${}^2\Sigma^a$ of 3-transpositions in $\Omega_8^-(2) : 2$ with the suborbit diagram

(b) the commuting graph ${}^2\Sigma^b$ of 3-transpositions in $\Omega_8^+(2) : 2$ with the suborbit diagram

(c) the 2-fold antipodal cover ${}^2\Sigma^c$ of the complete graph with the automorphism group $2^7 : Sp_6(2)$ and the suborbit diagram

(iii) ${}^4\Sigma$ is the commuting graph of 3-transpositions in the Fischer group Fi_{22} with the suborbit diagram

Proof. By (7.2 (i)) Σ is locally Υ . A connected graph which is locally the complete multipartite graph $K_{t \times m}$ with $t \geq 2$ parts of size $m \geq 2$ each, is isomorphic to $K_{(t+1) \times m}$ (cf. Proposition 1.1.5 in [BCN89]) and hence (i) follows. For $t = 2$ and 4 let $\mathcal{P}$ be the polar space with diagram

as in (7.1 (iii) (b), (c)) so that Υ is the collinearity graph of $\mathcal{P}$. Let $\mathcal{R}$ be the geometry of rank 4, whose elements of type 4 are the maximal complete subgraphs (of size $t^2 + t + 2$) in Σ ; the elements of type 1 are the vertices, the elements of type 2 are the edges and the elements of type 3 are the complete subgraphs of size $t + 2$ contained in more than one maximal complete subgraph; the incidence relation is via inclusion. It is a standard fact that $\mathcal{R}$ is a c -extension of $\mathcal{P}$ with the diagram

If $t = 2$ then by [CP92] $\mathcal{R}$ is a standard quotient of an affinization of the polar space of $Sp_8(2)$. It is well known that $\mathcal{P}$ contains three classes of hyperplanes and we obtain the possibilities (a), (b) and (c) in (ii) (notice that the antipodal quotient of ${}^2\Sigma^c$ is the complete graph which is not locally ${}^2\Upsilon$). Finally the isomorphism type of ${}^4\Sigma$ follows from [Pase94] and [Pase95]. $\square$

Let us match the possibilities in (7.3) with the examples we know. Let $\Gamma = \Gamma(\mathcal{E})$ where $\mathcal{E}$ is one of the seven examples of $c.F_4(t)$ -geometries in Section 4. If $\mathcal{E} = \mathcal{E}(3 \cdot Fi_{22})$ then $\Gamma(x, y) \cong {}^1\Sigma$ and if $\mathcal{E} = \mathcal{E}(Fi_{22})$ then $\Gamma(x, y)$ consists of three connected components, each isomorphic to ${}^1\Sigma$. If $\mathcal{E} = \mathcal{E}(3 \cdot {}^2E_6(2))$ then $\Gamma(x, y) \cong {}^2\Sigma^a$ and if $\mathcal{E} = \mathcal{E}({}^2E_6(2))$ then $\Gamma(x, y)$ consists of three connected components, each isomorphic to ${}^2\Sigma^a$. If $\mathcal{E} = \mathcal{E}(E_6(2))$ then $\Gamma(x, y) \cong {}^2\Sigma^b$ while if $\mathcal{E} = \mathcal{E}(2^{26}.F_4(2))$ then $\Gamma(x, y) \cong {}^2\Sigma^c$. Finally if $\mathcal{E} = \mathcal{E}(B)$ then $\Gamma(x, y) \cong {}^4\Sigma$.

Let Σ be ${}^1\Sigma$, ${}^2\Sigma^a$, ${}^2\Sigma^b$ or ${}^4\Sigma$ as in (7.3) (not ${}^2\Sigma^c$) and T be isomorphic to $S_3 \times S_3 \times S_3 \times S_3$, $\Omega_8^-(2) : 2$, $\Omega_8^+(2) : 2$ or Fi_{22} , respectively, so that Σ is isomorphic to the commuting graph of a conjugacy class of 3-transpositions in T . This means that there is a bijection

$$j : \Sigma \rightarrow T,$$

such that $o(j(u)j(v)) = 2$ if $v \in \Sigma(u)$ and $o(j(u)j(v)) = 3$ if $v \in \Sigma_2(u)$. Notice that by (7.2 (ii)) u and v (considered as involutions in F) satisfy $o(uv) = 2$ if $v \in \Sigma(u)$ and $o(uv) = 3$ if $v \in \Sigma_2(u)$. For $t \in T$ and $x \in \Sigma$ the action $x^t = j^{-1}(j(x)^t)$ turns T into an automorphism group of Σ .

Lemma 7.4 *In the above terms (i.e. with $\Sigma \not\cong {}^2\Sigma^c$)*

- (i) *whenever $u, v \in \Sigma$ and $w = uvu$, we have $j(w) = j(u)j(v)j(u)$;*
- (ii) *if $u \in \Sigma$, then u (considered as an automorphism of Δ) stabilizes Σ as a whole.*

Proof. Let us proceed with (i) (which will immediately imply (ii)). If u and v are adjacent then $w = v$, $j(w) = j(v)$ and the assertion is obvious. If $v \in \Sigma_2(u)$ we define x so that $j(x) = j(u)j(v)j(u)$ (this is possible since $j(\Sigma)$ is a conjugacy class in T) and all we have to show is that $w := uvu$ coincides with x . Let $\Sigma(u, v)$ be the μ -graph in Σ (the subgraph induced by $\Sigma_1(u) \cap \Sigma_1(v)$). Since $j(u)$ fixes $\Sigma_1(u)$ elementwise and maps v onto x , we observe that $\Sigma(u, x) = \Sigma(u, v)$. Furthermore, $j(\Sigma(u, v)) = j(\Sigma) \cap T(u) \cap T(v)$. In view of the suborbit diagrams in (7.3) this shows, particularly, that $\Sigma(u, v)$ contains a pair of non-adjacent vertices, say p and q . Then $\{u, v, x\} \subset \Sigma_1(p) \cap \Sigma_1(q)$ and $q \in \Sigma_2(p)$. On the one hand $\Sigma(p, q)$ is isomorphic to the commuting graph of involutions in $j(\Sigma) \cap (T(p) \cap T(q))$ and on the other hand $\Sigma(p, q)$ is a subgraph in the commuting graph $\Delta(p, q)$ of

involutions in $F(p) \cap F(q)$. Thus the restriction σ of j^{-1} to $\Sigma(p, q)$ induces an isomorphic embedding of the commuting graph Π of involutions in $j(\Sigma(p, q))$ into the commuting graph Υ (as in (7.1)) of involutions in $\Delta(p, q)$ and it remains to show that the embedding σ is S_3 -consistent in the sense that it maps the set $\{u, v, x\}$ of involutions in an S_3 -subgroup in $K := T(p) \cap T(q)$ onto the set of involutions in an S_3 -subgroup in $H = F(p) \cap F(q)$. If $t = 1$ then $|\Pi| = |\Upsilon| = 9$, σ is an isomorphism and a triple of vertices in Π or in Υ generate an S_3 -subgroup if and only if these three vertices are pairwise non-adjacent, the result is trivial in this case.

In the case $t = 2$ the graph Π is the commuting graph of 3-transpositions in $K \cong \Omega_6^+(2) : 2 \cong S_8$ with the suborbit diagram

or the commuting graph of 3-transpositions in $K \cong \Omega_6^-(2) : 2 \cong U_4(2) : 2$ with the suborbit diagram

By (7.1 (iii) (b)) Υ is the collinearity graph of the polar space $\mathcal{P} = \mathcal{P}(Sp_6(2))$. The lines of this space can be identified with a class of triangles in Υ so that every edge is contained in a unique line and a vertex outside a line is adjacent to 1 or 3 vertices on the line. It is easy to see that for every triangle in Π there is a vertex adjacent to 2 vertices in the triangle. Hence $\sigma(\Pi)$ does not contain lines of $\mathcal{P}$ and since the valency of Π (which is 15) equals to the number of lines containing a given point, every line intersects $\sigma(\Pi)$ in 2 or 0 vertices. Hence $\sigma(\Pi)$ is the complement to a hyperplane in $\mathcal{P}$. All hyperplanes in $\mathcal{P}$ are known, in particular every hyperplane with complement of size 28, respectively 36 has stabilizer in $H \cong Sp_6(2)$ isomorphic to $\Omega_6^+(2) : 2$ or $\Omega_6^-(2) : 2$, which shows that σ is S_3 -consistent.

In the case $t = 4$ Π is the commuting graph of 3-transpositions in $K \cong U_4(3) : 2$ with the suborbit diagram

and locally it is the point graph of the generalized quadrangle of order $(4, 2)$ associated with $U_4(2)$. This shows that the vertices, edges and maximal cliques (of size 6) with respect to inclusion form a geometry $\mathcal{G}$ with the diagram $\begin{smallmatrix} & c & \\ \circ & \text{---} & \circ \\ 1 & & 2 \end{smallmatrix}$. Recall that in the considered situation Υ is the collinearity graph of the polar space $\mathcal{P} = \mathcal{P}(U_6(2))$ so that the lines and planes in $\mathcal{P}$ can be identified with certain complete subgraphs of size 5 and the maximal complete subgraphs (of size 21) in Υ . We extend σ to a morphism δ of $\mathcal{G}$ into $\mathcal{P}$ as follows. For $t \in \Pi$ put $\delta(t) = \sigma(t)$; for an edge e of Π define $\delta(e)$ to be the unique line in $\mathcal{P}$ containing $\sigma(e)$ and for a 6-clique X in Π define $\delta(X)$ to be the unique plane in $\mathcal{P}$ containing $\sigma(X)$. In the polar space $\mathcal{P}$ a point outside a line is adjacent to 1 or 5 points of the line, while in Π for every triangle there is a vertex adjacent to exactly 2 vertices in the triangle. It is an easy combinatorial exercise to check, using this observation, that δ is an isomorphic embedding. The distribution diagram of the graph on the elements of type 3 in $\mathcal{G}$ in

which two such elements are adjacent if they are incident to a common element of type 2 (cf. p. 202 in [199]) shows, that every edge in this graph is contained in a unique triangle. Hence the image under δ of the set of elements of type 3 from $\mathcal{G}$ is a hyperplane in the dual polar space associated with $\mathcal{P}$. Now by Proposition 2.1 in [C94] the stabilizer of $\delta(\mathcal{G})$ in $H \cong U_6(2) : S_3$ is isomorphic to $K \cong U_4(3) : 2$ and hence σ is S_3 -consistent. $\square$

Let us try to identify Σ as a subgraph in Δ . We proceed to this via identifying the stabilizer S of Σ in F . By the definition, S is the normalizer in F of Σ where the latter is considered as a set of central involutions in F . The following result is a consequence of (7.4 (ii)).

Corollary 7.5 *Let U be the subgroup in F generated by the involutions from Σ . If $\Sigma \not\cong^2 \Sigma^c$ then $S = N_F(U)$ and U is isomorphic to the group T as in the proof of (7.4). $\square$*

In order to identify S precisely we treat different values of t separately.

In the case $t = 1$ let $p \in \Delta$, $q \in \Psi_3(p)$ and r be the image of q under conjugation by p . Then by the proof of (7.4), up to conjugation in F we have ${}^1\Sigma = \{p, q, r\} \cup (\Delta(p) \cap \Delta(q))$. This specifies ${}^1\Sigma$ and in view of the list of maximal subgroups in 1F gives the following.

Lemma 7.6 *The stabilizer 1S of ${}^1\Sigma$ in ${}^1F \cong \Omega_8^+(2).S_3$ is isomorphic to $S_3 \wr S_4$. Furthermore 1S is a subgroup of index 3 in a maximal subgroup of 1F .*

Let us turn to the case $t = 2$. Recall that ${}^2F \cong F_4(2)$ contains two conjugacy classes of subgroups isomorphic to $Sp_8(2)$. The representatives P_1 and P_2 of these classes can be chosen in such way that the centralizer of a central involution in P_1 (isomorphic to $2^7.Sp_6(2)$) stabilizes a point in $\mathcal{F}$ while the similar centralizer in P_2 stabilizes a symplecton. Also 2F has two irreducible 26-dimensional $GF(2)$ -modules V_1 and V_2 such that P_i fixes a vector in V_i and acts irreducibly on V_{3-i} . The classes containing P_1 and P_2 as well as the modules V_1 and V_2 are permuted by the outer (diagram) automorphism of 2F . Under the choice of the classes, the class M of 255 central involutions in P_1 (which is a class of 3-transpositions) is a subset of Δ and the subgraph induced by M is clearly the commuting graph. For $\varepsilon = +$ or $-$ let Q^ε be (the unique up to conjugation) subgroup in P_1 isomorphic to $\Omega_8^\varepsilon(2) : 2$. Then the involutions from M contained in Q^ε induce a subgraph isomorphic to ${}^2\Sigma^a$ or ${}^2\Sigma^b$ if $\varepsilon = +$ or $-$, respectively.

Lemma 7.7 *For $\alpha = a$ or b the stabilizer ${}^2S^\alpha$ of ${}^2\Sigma^\alpha$ in 2F is a conjugate of Q^ε for $\varepsilon = +$ or $-$, respectively.*

Proof. By Theorem 4 in [LS98] every subgroup in $F_4(2)$ isomorphic to $\Omega_8^\pm(2) : 2$ stabilizes a vector either in V_1 or in V_2 and by Table 2 in [CC88] such a subgroup is contained either in a conjugate of P_1 or in a conjugate of P_2 . $\square$

Notice that $N_{2F}(O^2(Q^+)) \cong \Omega_8^+(2) : S_3$ and that for $u \in M$ the subgraph in Δ induced by the involutions from M which do not commute with u is isomorphic to ${}^2\Sigma^c$.

Finally consider the case $t = 4$. It has been shown by B. Fischer (cf. [Coo83]) that ${}^2E_6(2)$ contains at least three conjugacy classes of subgroups isomorphic to Fi_{22} and that these three classes are fused in ${}^4F \cong {}^2E_6(2).S_3$. By [N92] the subgroups in 4F isomorphic to Fi_{22} are all conjugate.

Lemma 7.8 *All subgroups in 4F isomorphic to Fi_{22} are conjugate and the stabilizer 4S of ${}^4\Sigma$ is the normalizer of such a subgroup, isomorphic to $Fi_{22} : 2$. $\square$*

Notice that 4S is contained in a subgroup of index 3 in 4F .

We have the following result (cf. [IS96] and Lemma 5.10.8 in [I99])

Lemma 7.9 *Let $S = {}^4S \cong Fi_{22} : 2$ be the stabilizer of $\Sigma = {}^4\Sigma$ in $F = {}^4F \cong {}^2E_6(2) : S_3$. Then*

- (i) S , acting on Δ has four orbits: $\Omega_1 = \Sigma$, Ω_2 , Ω_3 and Ω_4 ;
- (ii) $|\Omega_i| = 3\,510$, $142\,155$, $694\,980$ and $3\,127\,410$, for $i = 1, 2, 3$ and 4 , respectively;
- (iii) if $w_i \in \Omega_i$, then $S(w_i)$ is isomorphic, to $2 \cdot U_6(2).2$, $2^{10}.M_{22}.2$, $2^7 : Sp_6(2)$ and $2.2^9 : P\Sigma L_3(4)$ for $i = 1, 2, 3$ and 4 , respectively;
- (iv) for $i = 2, 3$ and 4 the vertex w_i is adjacent in Δ , respectively, to 22, 126 and 22 vertices from Σ ;
- (v) S acts transitively on the set of maximal cliques (of size 22) in Σ ; $\Delta(w_2) \cap \Sigma$ and $\Delta(w_4) \cap \Sigma$ are such cliques and $S(w_2) \cong 2^{10}.M_{22}.2$ is the stabilizer of $\Delta(w_2) \cap \Sigma$ in S ;
- (vi) $\Delta(w_2) \cap \Sigma \subset O_2(F(w_2)) \setminus \{w_2\}$ while $|\Delta(w_4) \cap \Sigma \cap O_2(F(w_4))| = 1$. □

Notice that $O_2(S(w_2))$ is the 10-dimensional Todd module for $S(w_2)/O_2(S(w_2))$.

Lemma 7.10 *In terms of (7.9) the subgroup $L = \langle \Delta(w_2) \cap \Sigma \rangle = O_2(S(w_2))$ is contained in exactly two conjugates of S in F , namely, in S and in w_2Sw_2 , moreover, $S \cap w_2Sw_2 = S(w_2) \cong 2^{10}.M_{22}.2$.*

Proof. By (7.9 (v)) $N_F(L)$ acts transitively on the set of conjugates of S in F , containing L . By (7.9 (vi)) w_2 is the unique vertex of Δ such that $L \leq O_2(F(w_2))$ and hence $N_F(L) \leq F(w_2)$. Since $O_2(F(w_2))$ is extraspecial of order 2^{21} , $N_{O_2(F(w_2))}(L) = L\langle w_2 \rangle$. We have

$$N_S(L)O_2(F(w_2))/O_2(F(w_2)) \cong M_{22}.2$$

and by the list of maximal subgroups in $U_6(2)$ [ATLAS] the inclusion $M_{22}.2 < N \leq U_6(2).S_3$ implies $N \geq U_6(2).2$. Hence $N_F(L) = N_S(L)\langle w_2 \rangle$ and the result follows. □

The next proposition is an immediate consequence of (7.9), (6.3) and (7.3 (iii)).

Proposition 7.11 *Suppose that $t = 4$ and let $y \in \Gamma_2^3(x)$. Then*

- (i) the subgraph $\Gamma(x, y)$ is connected with 3 510 vertices;
- (ii) if $u \in \Gamma(y) \setminus \Gamma(x, y)$, then $u \in \Gamma_2(x)$ and u is adjacent in Γ to a vertex in $\Gamma(x, y)$;
- (iii) $\Gamma_2^3(x) \cap \Gamma_2^4(x) = \emptyset$;
- (iv) $|\Gamma_2^3(x)| = |\Delta| \cdot |\Psi_3(x)|/3\,510$. □

The following lemma provides some further details on (6.10 (ii)).

Lemma 7.12 *Let e, f be distinct elements of type 2 in $\mathcal{E}$, such that $\Pi := \Xi^e \cap \Xi^f \neq \emptyset$. Let $\tilde{\Pi}$ be the image of Π under the natural mapping of Ξ^e onto $\tilde{\Xi}^e$. Assume without loss that $e = \{x, y\}$, $f = \{x, z\}$. Then the following assertions hold:*

- (i) if $i_x(z) \in \Psi_1(i_x(y))$ then $|\Pi| = 2 \cdot |\tilde{\Pi}| = 4(t^2 + t + 2)$ and $z \in \Xi^e$;

- (ii) if $i_x(z) \in \Psi_2^2(i_x(y))$ then $|\Pi| = |\tilde{\Pi}| = 3(2t+1) + 1$ and $|\Gamma(z) \cap \Xi^e| = 6(2t+1) + 2$;
- (iii) if $i_x(z) \in \Psi_2^4(i_x(y))$ then $|\Pi| = |\tilde{\Pi}| = 2$ and $|\Gamma(z) \cap \Xi^e| = 2(t^2 + t + 1) + 1$;
- (iv) if $i_x(z) \in \Psi_3(i_x(y))$ then $|\Pi| = |\tilde{\Pi}| = 1$ and $\Gamma(z) \cap \Gamma(x) \cap \Xi^e = \emptyset$; if in addition $t = 4$, then $\Gamma(z) \cap \Xi^e = \{x\}$.

In particular if $t = 4$ then Ξ^e is geodetically closed.

Proof. By (6.7) and (7.11) if z is adjacent to a vertex from $\Xi^e \setminus (\{x\} \cup \Gamma(x))$, then either $z \in \Xi^e$, or $t \neq 4$ and $i_x(z) \in \Psi_3(i_x(y))$. Now the result can be easily deduced from Fig. 2, Fig. 3, the fact that

$$\Gamma(x) \cap \Xi^e = \{y\} \cup \{u \mid i_x(u) \in \Psi_1(i_x(y))\}$$

and the definition of $\tilde{\Xi}^e$. □

Notice that if $t = 4$, ${}^4G \cong Co_2$ is the automorphism group of $\tilde{\Xi}^e$, $\tilde{\Pi}$ is as in (7.12 (i)) and if M is the stabilizer of $\tilde{\Pi}$ in 4G , then $M \cong 2^{10}.M_{22}.2$, where $O_2(M)$ is the Golay code module for $M/O_2(M)$.

8 A characterization of the Baby Monster graph

In this section (which consists of a few subsections) we prove Theorems 1 and 2 by showing that $\Gamma({}^4\mathcal{E}) \cong \Theta$ and that ${}^4\mathcal{E} \cong \mathcal{E}(B)$. Throughout the section we assume that $t = 4$.

8.1 x -equivalence on $\Gamma_2(x)$

Let $x \in \Gamma$, $y \in \Gamma_2^\alpha(x)$ for $\alpha = 3$ or 4 , $\Gamma(x, y)$ be the corresponding μ -graph, $\Sigma^y = \{i_x(u) \mid u \in \Gamma(x, y)\}$ and S^y be the stabilizer of Σ^y in $F \cong {}^2E_6(2).S_3$. Then by (6.4), (6.6), (7.8), (7.9), S^y is specified by α up to conjugation in F and S^y determines Σ^y uniquely. We summarize this in the following proposition.

Proposition 8.1 *The following assertions hold:*

- (i) if $y \in \Gamma_2^4(x)$, then $S^y \cong 2_+^{1+20} : U_4(3).2^2$ is contained in $F(p) \cong 2_+^{1+20} : U_6(2).S_3$ for a unique point p of $\mathcal{F}$ and Σ^y is the unique orbit of length 648 of S^y on Δ (contained in $\Psi_1(p)$);
- (ii) if $y \in \Gamma_2^3(x)$, then $S^y \cong Fi_{22} : 2$ and Σ^y is the unique orbit of length 3510 of S^y on Δ ;
- (iii) in either case if $H^y = S^y F^\infty$, then $H^y \cong {}^2E_6(2) : 2$ is a subgroup of index 3 in F . □

In the next subsection we show that H^y is independent on the choice of $y \in \Gamma_2(x)$. We need another preliminary result.

Lemma 8.2 *Let $F(p) \cong 2_+^{1+20} : U_6(2).S_3$ be the stabilizer in F of a point p of $\mathcal{F}$, $Z = \langle p \rangle$, $Q = O_2(F(p))$, $I = F(p)^\infty$. Then $\bar{I} := I/Z \cong 2^{20} : U_6(2)$ has four classes $\mathcal{U}_i$, $0 \leq i \leq 4$, of complements to Q/Z , such that $F(p)/I \cong S_3$ normalizes $\mathcal{U}_0$ and permutes transitively the remaining classes; the preimage in I of a subgroup from $\mathcal{U}_i$ splits over Z if and only if $i = 0$. □*

Let $p \in F$, M_1, M_2, M_3 be representatives of the conjugacy classes of Fi_{22} -subgroups in $F^\infty \cong {}^2E_6(2)$ (compare the paragraph before (7.8)) such that $p \in M_i$ for $1 \leq i \leq 3$ and $q \in \Psi_3(p)$. Then (within a suitable ordering) we have $(F(p) \cap F(q))^\infty \in \mathcal{U}_0$, $M_i(p)/\langle p \rangle \in \mathcal{U}_i$ for $1 \leq i \leq 3$.

Lemma 8.3 *For every $y \in \Gamma_2(x)$ there is exactly one vertex $y' \in \Gamma_2(x) \setminus \{y\}$ such that $\Sigma^y = \Sigma^{y'}$. Furthermore, $i_z(y') = i_z(x)i_z(y)i_z(x)$ for every $z \in \Gamma(x, y)$.*

Proof. Suppose first that $y \in \Gamma_2^4(x)$. Let $e = \{x, z\}$ be the element of type 2 in $\mathcal{E}$ such that $y \in \Xi^e$ and $\chi : \Xi^e \rightarrow \tilde{\Xi}^e$ be the natural mapping. Since Ξ^e is the 2-clique extension of $\tilde{\Xi}^e$, it is clear that $\Sigma^y = \Sigma^{y'}$ whenever $\chi(y) = \chi(y')$. On the other hand, it is easy to see from Fig. 2 and Fig. 3 that z is the only vertex in $\Gamma(x)$ adjacent to every vertex in $\Gamma(x, y)$. Hence $\Sigma^{y'} = \Sigma^y$ implies $y' \in \Xi^e$. Finally, different pairs of vertices at distance 2 in $\tilde{\Xi}^e$ have different μ -graphs, which gives $\chi(y') = \chi(y)$.

Now suppose that $y, y' \in \Gamma_2^3(x)$, $y \neq y'$, $\Sigma^y = \Sigma^{y'}$ and $z \in \Gamma(x, y)$. Put $p = i_z(x)$, $q = i_z(y)$, $q' = i_z(y')$. We claim that $q' = pqp$. The subgroup $U := O^\infty(F(p) \cap F(q)) \cong U_6(2)$ is generated by the involutions in $\Delta(p) \cap \Delta(q) = \Delta(p) \cap \Delta(q')$. Hence U commutes with the unique element $r \in O_2(F(p))$ which maps q onto q' . Since U acts irreducibly on $O_2(F(p))/\langle p \rangle$ and $q \neq q'$, we have $r = p$, hence the claim follows and proves the uniqueness statement in the lemma. Now let y and z be as above, and let y' be such that $i_z(y') = i_z(x)i_z(y)i_z(x)$. We claim that $S^y = S^{y'}$ (then (8.1 (ii)) will imply the equality $\Sigma^y = \Sigma^{y'}$). The subgroup K of F generated by

$$\{i_x(u) \mid u \in \{z\} \cup (\Gamma(x) \cap \Gamma(y) \cap \Gamma(z))\}$$

is isomorphic to $2 \cdot U_6(2)$ (a non-split extension) and $K = O^2(S^y(r)) = O^2(S^{y'}(r))$, where $r = i_x(z)$. By (7.8) there is $f \in F$ which conjugates S^y onto $S^{y'}$. Since S^y acts transitively on the set of vertices of Σ^y and r is one of these vertices, we can choose f to normalize K . By (7.9) r is the unique point in $\mathcal{F}$, stabilized by K . This implies that $N_F(K) \leq F(r)$. Since K does not split over its centre, by (8.2) $N_{F(r)}(K) \cong 2 \cdot U_6(2).2 \cong N_{S^y}(K)$. Hence $S^{y'} = S^y$. $\square$

As a consequence of the proof of (8.3) and in view of the fact that different pairs of vertices at distance 2 in $\tilde{\Xi}^e$ have different μ -graphs, we obtain the following.

Lemma 8.4 *Suppose that $y_1, y_2 \in \Gamma_2(x)$ and for some $z \in \Gamma(x, y_1) \cap \Gamma(x, y_2)$ we have $\Gamma(x) \cap \Gamma(y_1) \cap \Gamma(z) = \Gamma(x) \cap \Gamma(y_2) \cap \Gamma(z)$. Then $\Gamma(x, y_1) = \Gamma(x, y_2)$.* $\square$

As a direct consequence of (8.4) we obtain the following result (which can also be checked in Δ directly).

Lemma 8.5 *Let $p \in \Delta$, $q, q' \in \Delta_2(p)$ and suppose that $\Delta(p) \cap \Delta(q) = \Delta(p) \cap \Delta(q')$. Then either $q' = q$ or $q' = pqp$.* $\square$

To the end of this subsection we introduce some important notions.

Definition 8.6 *Let $x \in \Gamma$. Two vertices $y, y' \in \Gamma_2(x)$ are said to be x -equivalent if $\Gamma(x, y) = \Gamma(x, y')$. By π_x we denote the permutation of the vertex-set of Γ which fixes every vertex in $\Gamma \setminus \Gamma_2(x)$ and swaps the pairs of x -equivalent vertices in $\Gamma_2(x)$.*

By (8.3) each class of x -equivalent vertices is of size 2, so that π_x is well defined. In the Baby Monster graph Θ two vertices $b, b' \in \Theta_2(a)$ are a -equivalent if $\langle a, b \rangle = \langle a, b' \rangle$ (which happens exactly when $b' = aba$.) Furthermore, $a, b \in \Theta$ do not commute if and only if $b \in \Theta_2(a)$, so that π_a is nothing, but the conjugation by a , and clearly it is an automorphism of Θ .

Our ultimate goal is to prove the following.

Proposition 8.7 *For every $x \in \Gamma$ the permutation π_x is an automorphism of Γ .*

The proof of (8.7) will be achieved by showing that restrictions of π_x to various subsets of vertices of Γ are automorphisms of the subgraphs induced by that subsets. Our first result of this type follows directly from (8.3).

Lemma 8.8 *If $z \in \Gamma(x)$ and $y \in \Gamma(z)$, then $i_z(y^{\pi_x}) = i_z(x)i_z(y)i_z(x)$, in particular the restriction π_x^z of π_x to $\Gamma(z)$ is an automorphism of the subgraph in Γ induced by $\Gamma(z)$.* $\square$

The permutation π_x fixes every vertex in $\{x\} \cup \Gamma(x)$ and permutes the pairs of x -equivalent vertices in $\Gamma_2(x)$. Hence by (8.8) we also have the following.

Lemma 8.9 *For every $z \in \Gamma(x)$ the restriction of π_x to the set $\Gamma(x) \cup \Gamma(z)$ is an automorphism of the subgraph induced by this set.* $\square$

8.2 The class of μ -graphs

In this subsection we specify the family

$$\mathcal{S} = \{\Sigma^u \mid u \in \Gamma_2(x)\}$$

of subgraphs in Δ up to simultaneous conjugation by automorphisms of Δ .

For $y \in \Gamma_2^\alpha(x)$ let us adopt the notation introduced in (8.1) and in the paragraph before that lemma. By (8.3) there are exactly $|\Gamma_2^\alpha(x)|/2$ distinct μ -graphs $\Gamma(x, u)$ for $u \in \Gamma_2^\alpha(x)$ and by (6.10), (7.11), (8.1) and (8.3) we have

$$|\Gamma_2^\alpha(x)|/2 = [F : S^y]/3 = [H^y : S^y].$$

Thus exactly one third of the images of Σ^y under the elements $f \in F$ is contained in $\mathcal{S}$ and we are going to show that such an image is contained in $\mathcal{S}$ exactly when $f \in H^y$. First of all by the above equality we obtain the following.

Lemma 8.10 *For $\alpha = 3$ or 4 let $y \in \Gamma_2^\alpha(x)$. Then the following two statements are equivalent*

- (i) $H^u = H^y$ for every $u \in \Gamma_2^\alpha(x)$;
- (ii) for every $h \in H^y$ the image of Σ^y under h coincides with Σ^u for some $u \in \Gamma_2^\alpha(x)$. $\square$

Lemma 8.11 *The equivalent statements (i) and (ii) in (8.10) hold.*

Proof. Suppose first that $\alpha = 3$. For a vertex $z \in \Gamma(x, y)$ let $p = i_z(x)$, $q = i_z(y)$ (so that $q \in \Psi_3(p)$), $r \in \Psi_3(p) \cap \Psi_1(q)$ and $u = i_z^{-1}(r)$. Then $u \in \Gamma_2^3(x) \cap \Gamma(y)$. By (2.1) $F(p) \cap F(q) \cap F(r) \cong 2^9.L_3(4).S_3$ is a maximal parabolic in $F(p) \cap F(q) \cong U_6(2).S_3$. Hence $\Delta(p) \cap \Delta(q) \cap \Delta(r)$ consists of 21 3-transpositions from the set $\Delta(p) \cap \Delta(q)$ of 693 3-transpositions of $(F(p) \cap F(q))^\infty \cong U_6(2)$. This means that $\Gamma(x) \cap \Gamma(y) \cap \Gamma(u)$ contains a set of 22 pairwise adjacent vertices, namely

$$\{z\} \cup i_z^{-1}(\Delta(p) \cap \Delta(q) \cap \Delta(r)).$$

On the one hand by (8.3) $\Gamma(x, y) \neq \Gamma(x, u)$ (equivalently $S^y \neq S^u$) and on the other hand by (7.9 (v)) and (7.10) $S^y \cap S^u \cong 2^{10}.M_{22}.2$ and S^y is conjugate to S^u in F^∞ , which implies $H^u = H^y$, since $H^y = F^\infty S^y$. By the construction for every 22-vertex complete subgraph Υ in $\Gamma(x, y)$ there is (a unique) $u \in \Gamma_2^3(x) \cap \Gamma(y)$ such that $\Gamma(x, y) \cap \Gamma(u) = \Upsilon$. Since a Fi_{22} -subgroup is maximal in

$F^\infty \cong {}^2E_6(2)$, the graph on the set of F^∞ -conjugates of S^y in which two such subgroups are adjacent if they share a set of 22 pairwise commuting 3-transpositions, is connected. This implies (8.10 (ii)) for $\alpha = 3$.

Now let $y \in \Gamma_2^4(x)$, so that $y \in \Xi^e$ for some $e = \{x, z\}$ and $S^y \leq F(p)$ where $p = i_x(z)$. The stabilizer of e in the automorphism group Co_2 of $\tilde{\Xi}^e$ is isomorphic to $U_6(2) : 2$ (cf. (5.2 (iii))) and hence there is a unique subgroup D^z of index 3 in F such that for every $d \in D^z(p)$ the action of d on $\Psi_1(p)$ can be extended to an automorphism of Ξ^e . This shows that $H^u = D^z$ for every $u \in \Xi^e \cap \Gamma_2^4(x)$ and it remains to show that D^z is independent on z . Let $v \in \Gamma(x)$ be such that $i_x(v)$ is contained in $\Psi_1(p)$ and put $f = \{x, v\}$. Then by (7.12 (i)) $\Pi := \Xi^e \cap \Xi^f$ is a complete graph on 88 vertices. If R is the stabilizer of $i_x(\Pi \setminus \{x\})$ in F , then $R/O_2(R) \cong L_3(4).S_3$. In view of the remark after (7.12) for an element $r \in R$ its action on $\Psi_1(p) \cap \Psi_1(q)$ can be extended to an automorphism of Ξ^e if and only if it can be extended to an automorphism of Ξ^f . Hence $D^z = D^v$ and the result follows by the connectivity of Ψ . $\square$

Lemma 8.12 *In terms of (7.9) let $u_i \in \Gamma(x)$ be such that $i_x(u_i) = w_i$ for $i = 2, 3$ and 4. Then $u_2, u_3 \in \Gamma_2^3(y)$ and $u_4 \in \Gamma_2^4(y)$.* $\square$

Proof. For $i = 2$ and 4, in view of (7.9) and (7.10), the claim comes as a bi-product of the proof of (8.11) for the case $\alpha = 3$. By (7.9 (iv)) the subgraph induced by $\Gamma(x) \cap \Gamma(y) \cap \Gamma(u_3)$ has 126 vertices and it is not a complete graph by (7.9 (v)). By (7.12 (v)) this can not happen when $u_3 \in \Gamma_2^4(y)$. $\square$

We need the following result.

Lemma 8.13 *Let ${}^4\tilde{\Xi}^e$ be the graph on 2300 vertices for ${}^4G \cong Co_2$ as in (5.2), f be at distance 2 from e , Φ be the subgraph on 324 vertices induced by the common neighbours of e and f , Π be a complete 22-vertex subgraph in Φ and $D = {}^4G(e) \cap {}^4G(f) \cong U_4(3).2^2$. Then the setwise stabilizer of Π in D induces on Π an action of $P\Sigma L_3(4)$.*

Proof. Notice that if Λ is as in (5.1), then Φ is the subgraph in the distance 1-or-2 graph of Λ induced by the orbit X of length 324 of D . It is easy to see, using for instance (6.6) that the subgraph in Λ itself induced by X has the following intersection diagram:

By (6.1 (i)), or otherwise we know that the valency of Φ is 147 and $147 = 105 + 21 + 21$ is the only way to present the valency as a sum of suborbits. Now it is a standard fact about dual polar spaces that if Y is a maximal clique in the distance 1-or-2 graph of Λ , then either $|Y| = 27$ and Y is a quad, or $|Y| = 43$ and Y is a vertex together with its neighbours in Λ . By (6.5 (iii)) a quad

intersects X in at most 12 vertices, which implies that the clique Π is a vertex together with its neighbours in Λ which are contained in Φ . Hence the stabilizer of Π coincides with the stabilizer in D of a vertex of Φ and the result follows. $\square$

We have proved in (8.11) that for $\alpha \in \{3, 4\}$ H^y is independent on the choice of $y \in \Gamma_2^\alpha(x)$ and it remains to establish the following.

Lemma 8.14 *There are $y \in \Gamma_2^3(x)$ and $u \in \Gamma_2^4(x)$ such that $H^y = H^u$.*

Proof. By (8.12) we can choose $y \in \Gamma_2^3(x)$ and $u \in \Gamma_2^4(x)$ such that $\Pi := \Sigma^y \cap \Sigma^u$ contains a complete subgraph on 22 vertices. It is sufficient to show that $S^y[\Pi] \cap S^u[\Pi]$ is not contained in F^∞ . By the proof of (7.10) we have $F[\Pi] \cong 2 \times 2^{10}.M_{22}.2$, $S^y[\Pi] \cong 2^{10}.M_{22}.2$ and $F^\infty[\Pi] \cong 2 \times 2^{10}.M_{22}$. So it is sufficient to show that $S^u[\Pi]$ induces in Π the natural action of $P\Sigma L_3(4)$. Since $\Pi \subset \Sigma^y$, for any two distinct vertices p, q in Π we have $q \in \Psi_2^2(p)$ and hence Π maps bijectively onto its image in Ξ^e , where $e = \{x, z\}$ is the edge of Γ such that $u \in \Xi^e$. Hence the claim follows directly from (8.13). $\square$

8.3 The second neighbourhood of a vertex

In this subsection for a given $x \in \Gamma$ we analyze the adjacencies between the vertices in $\Gamma_2(x)$. In view of (2.5) the results established in the previous subsection can be summarized as follows.

Proposition 8.15 *Let x be a vertex of Γ , $F_x \cong^2 E_6(2).S_3$ be the automorphism group of the subgraph in Γ induced by $\Gamma(x)$. Then there is a unique subgroup $H_x \cong^2 E_6(2) : 2$ of index 3 in F_x such that for $\alpha = 3$ and 4 H_x acts transitively on the set*

$$M^\alpha = \{\Gamma(x, y) \mid y \in \Gamma_2^\alpha(x)\}$$

of μ -graphs as on the cosets of subgroups $Fi_{22}.2$ and $2^{1+20}.U_4(3).2^2$, respectively. If $X \in M^\alpha$ and $z \in \Gamma(x)$, then $z \in X$ if and only if

$$H_x(z) \cap H_x[X] \cong 2 \cdot U_6(2) : 2 \quad \text{and} \quad H_x(z) \cap H_x[X] \cong [2^{20}].P\Sigma L_3(4),$$

for $\alpha = 3$ and 4, respectively. $\square$

Notice that for $z \in \Gamma(x)$ the action π_z^x of π_z on $\Gamma(x)$ is an automorphism of the subgraph in Γ induced by $\Gamma(x)$. Furthermore, $\pi_z^x \in F^\infty$, particularly $\pi_z^x \in H_x$.

For $\alpha = 3$ or 4 and $X \in M^\alpha$ the subgroup $H_x(z) \cap H_x[X]$ in (8.15) is determined uniquely up to conjugation in $H_x(z) \cong 2_+^{1+20} : U_6(2).2$. For $X \in M^3$ the subgroup $(H_x(z) \cap H_x[X])' / Z(H_x(z)) \cong U_6(2)$ is a complement to $O_2(H_x(z) / Z(H_x(z)))$ in $H_x(z)' / Z(H_x(z))$ and $H_x(z)$ can be characterized as the stabilizer in $F_x(z)$ of the conjugacy class of $(H_x(z) \cap H_x[X])'$ in $F_x(z)^\infty$ (compare (8.2)). If $e = \{x, z\}$ and Ξ^e is the 4600-vertex subgraph as in Section 5, then $H_x(z)$ (resp. $H_z(x)$) consists of those elements of $F_x(z)$ (resp. of $F_z(x)$) whose actions on $\Xi^e \cap \Gamma(x)$ (resp. on $\Xi^e \cap \Gamma(z)$) can be extended to automorphisms of Ξ^e . Furthermore, $H_x = H_x(z)F_x^\infty$. This gives an alternative way to define the subgroups H_u consistently for all $u \in \Gamma$.

We need another preliminary result.

Lemma 8.16 *Let p be a vertex of Ψ , Π be the subgraph in Ψ induced by $\Psi_1(p) \cup \Psi_2^2(p)$ and J be the automorphism group of Π . Then*

$$2^{20} : U_6(2) \cong \bar{I} < J \leq \text{Aut } \bar{I} \cong 2^{20+2} : U_6(2) : S_3$$

where $\bar{I}$ is as in (8.2), so that $\text{Out } \bar{I} \cong S_4$ acts naturally on the classes of complements in $\bar{I}$ to $O_2(\bar{I})$.

Proof. By (2.1) the stabilizer of p in F induces on Π an action of $2^{20} : U_6(2).S_3$, so to prove the lemma it is sufficient to show that J is contained in $\text{Aut } \bar{I}$. Let Υ be a symplecton containing p and A be the action induced by $J[\Upsilon]$ on Υ . Then A is contained in the stabilizer of p in $\text{Aut } \Upsilon \cong \Omega_8^-(2).2$ and A is contained in the action induced on Υ by its stabilizer in $F(p)$. Hence $A \cong 2^6.\Omega_6^-(2).2$ and A acts faithfully on $\Upsilon \cap \Psi_1(p)$. By (2.4) Υ is uniquely determined by its intersection with $\Psi_1(p)$ and hence J acts faithfully on $\Psi_1(p)$. Let V be the kernel of the action of J on the set L of 891 lines containing p and $\tilde{J} = J/V$. In every symplecton containing p there are exactly 27 lines containing p . The collection of 693 such 27-element subsets of L defines on L the structure $\mathcal{D}$ of dual polar space of $U_6(2)$. This structure is preserved by J and hence $\tilde{J} \leq U_6(2).S_3$. Since every line in L contains besides p exactly two points, the kernel V is an elementary abelian 2-group. On every line from L the group V induces an action of order at most 2 and on the 27 lines in a symplecton Υ it induces an action of order at most $2^6 = |O_2(A)|$. Hence on a plane containing p the subgroup V induces an action of order at most 2^2 . This shows that the dual of V is a representation module of $\mathcal{D}$ and the latter was proved in [Y94] to be of order 2^{22} . Hence $J^\infty \cong \bar{I}$ and the result follows. $\square$

For $z \in \Gamma(x)$ put $\Omega = (\Gamma(x) \cap \Gamma(z)) \setminus \{x, z\}$. The mappings i_x and i_z restricted to Ω are bijections onto the vertex set of the graph Π as in (8.16). If $u \in \Omega$, then by (3.2) the type of $\{x, z, u\}$ is well defined and we can put

$$\begin{aligned}\Omega_1 &= \{u \in \Omega \mid \{x, z, u\} \text{ is short}\} \\ \Omega_2^2 &= \{u \in \Omega \mid \{x, z, u\} \text{ is long}\}.\end{aligned}$$

If $e = \{u, v\}$ is an edge of Γ contained in Ω , then the triangles $\{x, u, v\}$ and $\{z, u, v\}$ might or might not be of the same type. Define (τ_x, τ_z) to be the type of e , where

$$\tau_x = s \text{ or } l \text{ if } \{x, u, v\} \text{ is short or long, respectively}$$

and τ_z is defined similarly. By (3.6) we have the following.

Lemma 8.17 *Let $e = \{u, v\}$ be an edge of Γ contained in Ω and (τ_x, τ_z) be the type of e . Then*

- (i) *if $e \subset \Omega_1$ or $e \subset \Omega_2^2$, then $\tau_x = \tau_z$;*
- (ii) *if $u \in \Omega_1$ and $v \in \Omega_2^2$, then $\tau_x \neq \tau_z$.*

$\square$

Lemma 8.18 *Let K be the group of automorphisms of the subgraph in Γ induced by Ω , preserving the above defined types (τ_x, τ_z) of edges. Then $K \cong J = \text{Aut } \Pi$, where Π is as in (8.16).*

Proof. For $\gamma = x$ or z let Ω^γ be the graph on Ω , whose edges are the edges of Γ contained in Ω for which $\tau_\gamma = s$. Then i_γ induces an isomorphism of Ω^γ onto the graph Π . In view of (8.17) every automorphism of Π can be realized as a type-preserving automorphism of the subgraph induced by Ω . Hence the result. $\square$

Let ${}^x\mathcal{Y}$ be the set of x -equivalence classes of vertices in $\Gamma_2(x)$. By (8.15) we can define the action of H_x on ${}^x\mathcal{Y}$ by the following rule: if $\{y, y'\} \in {}^x\mathcal{Y}$, with $\Gamma(x, y) = \Gamma(x, y') = X$, then $\{y_1, y'_1\} = \{y, y'\}^h$ for $h \in H_x$ if and only if $\Gamma(x, y_1) = \Gamma(x, y'_1) = X^h$. By (8.10) and (8.11) this action is well defined. For $z \in \Gamma(x)$ let $H_x(z)$ be the stabilizer of z in H_x and ${}^x\mathcal{Y}_z$ be the set of classes from ${}^x\mathcal{Y}$ contained in $\Gamma(z)$. Since $\{y, y'\} \in {}^x\mathcal{Y}_z$ whenever $z \in \Gamma(x, y)$, $H_x(z)$ stabilizes ${}^x\mathcal{Y}_z$ as a whole and we can consider the action of $H_x(z)$ on ${}^x\mathcal{Y}_z$.

Lemma 8.19 *The above defined action of $H_x(z)$ on ${}^x\mathcal{Y}_z$ coincides with the natural action of $H_z(x)$ on ${}^x\mathcal{Y}_z$.*

Proof. With $\Omega = (\Gamma(x) \cap \Gamma(z)) \setminus \{x, z\}$ as above, let $\{y, y'\} \in {}^x\mathcal{Y}_z$ and $X = \Gamma(x, y)$. By (8.4) and (8.5) the stabilizer of $\{y, y'\}$ in $H_z(x)$ coincides with the setwise stabilizer of $\Gamma(y) \cap \Omega$ and the stabilizer of X in $H_x(z)$ coincides with the stabilizer of $X \cap \Omega$. Furthermore $\Gamma(y) \cap \Omega = X \cap \Omega$, which implies that the actions of $H_x(z)$ and $H_z(x)$ on ${}^x\mathcal{Y}_z$ are determined by their natural actions on Ω . Thus all we have to show is that the action A_x of $H_x(z)$ on Ω coincides with the action A_z of $H_z(x)$ on this set. Notice that $A_x \cong A_z \cong 2^{20} : U_6(2).2$ and both A_x and A_z are subgroups in the group K of type-preserving automorphisms of the subgraph in Γ induced by Ω (compare (8.18)). Let $t_0 \in \Gamma(x)$ with $i_x(t_0) \in \Psi_3(i_x(z))$, $t_1 \in \Gamma(z)$ with $i_z(t_1) \in \Psi_3(i_z(x))$. For $i = 0$ and 1 let U_i be the stabilizer of $\Gamma(t_i) \cap \Omega$ in A_x . Then $U_0 \cong U_1 \cong U_6(2).2$ and U_0 and U_1 belong to different classes of complements to $O_2(A_x)$. By the paragraph after (8.15) A_x is characterized as the stabilizer in $J \cong 2^{20} : U_6(2).S_4$ of the classes in J^∞ containing U'_0 and U'_1 . Since A_z is isomorphic to A_x and possesses the same properties, the result follows. $\square$

Now we can establish some further properties of the permutation π_x .

Lemma 8.20 *If $s, w \in \Gamma(x)$ and $s \in \Gamma(w)$ then the restriction of π_x to the set $\Gamma(x) \cup \Gamma(s) \cup \Gamma(w)$ is an automorphism of the subgraph in Γ induced by this set.*

Proof. In terms of (8.19) let $d \in \Gamma(x) \cap \Gamma(z)$. Then the action of π_d on $\Gamma(x)$ is an element in $H_x(z)$ while its action on $\Gamma(z)$ is an element of $H_z(x)$. Hence the result follows from (8.19). $\square$

By now with every vertex $z \in \Gamma$ we have associated various actions on Γ . On the one hand in (8.6) we have defined the permutation π_z of the vertex-set of Γ . On the other hand, if $x \in \Gamma(z)$, then the action π_z^x of π_z on the set $\Gamma(x)$ is an element of H_x and an action of π_z^x on the set ${}^x\mathcal{Y}$ of pairs of x -equivalent vertices in $\Gamma_2(x)$ is defined in the paragraph before (8.19). The natural question is whether these actions are consistent. A partial affirmative answer is given in the following lemma.

Lemma 8.21 *Let $x \in \Gamma$, $z \in \Gamma(x)$, $y \in \Gamma_2(x)$ and suppose that z is adjacent in Γ to a vertex $u \in \Gamma(x, y)$. Then $\Gamma(y^{\pi_z}, x)$ is the image of $\Gamma(x, y)$ under π_z^x , i.e. $\Gamma(x, y)^{\pi_z^x} = \Gamma(y^{\pi_z}, x)$.*

Proof. Under the hypothesis $x, y, z \in \Gamma(u)$. Hence y^{π_z} coincides with the image of y under π_z^u and $\Gamma(y^{\pi_z}, x) \cap \Gamma(u)$ is the image of $\Gamma(y, x) \cap \Gamma(u)$ under π_z^x (equivalently under π_z^u). By (8.4) $\Gamma(y^{\pi_z}, x)$ is uniquely determined by $\Gamma(y^{\pi_z}, x) \cap \Gamma(u)$ and hence the result. $\square$

8.4 π_x is an automorphism of Γ

In this subsection we prove Proposition 8.7. We need to show that whenever $\{y, u\}$ is an edge of Γ , its image under π_x is again an edge of Γ . Since π_x fixes every vertex in $\Gamma \setminus \Gamma_2(x)$, we can assume that at least one of the vertices on the edge, say y , is in $\Gamma_2(x)$. Then $y^{\pi_x} = y'$ is x -equivalent to y in $\Gamma_2(x)$. If there is $z \in \Gamma(x)$ such that $\{y, u\} \subset \{z\} \cup \Gamma(z)$, then $\{y^{\pi_x}, u^{\pi_x}\}$ is an edge by (8.8). Hence we assume that

$$\Gamma(x) \cap \Gamma(y) \cap \Gamma(u) = \emptyset.$$

By (7.11 (ii)) this implies that $y \notin \Gamma_2^3(x)$ and so $y \in \Gamma_2^4(x)$. Furthermore by the proof of (8.3) for $\alpha = 4$ we conclude that $x \in \Xi^e$ for the edge $e = \{y, y'\}$.

From now on we follow notation introduced in the paragraph after the proof of (6.3) (with the roles of x and y interchanged). As usual i_y is a fixed isomorphism of the subgraph in Γ induced by $\Gamma(y)$ onto the graph Δ . Let

$$\Omega = \{i_y(v) \mid v \in \Gamma(x, y)\},$$

$I \cong 2_+^{1+20} : U_4(3).2^2$ be the setwise stabilizer of Ω in $F = \text{Aut } \Delta$ and $\Omega_0, \Omega_1 = \Omega, \dots, \Omega_7$ be the orbits of I on the vertex-set of Δ , where $\Omega_0 = \{p\}$, $p = i_y(y')$, $\Omega_1 \cup \Omega_2 = \Psi_1(p)$, $\Omega_3 \cup \Omega_4 = \Psi_2^2(p)$, $\Omega_5 \cup \Omega_6 = \Psi_2^4(p)$, $\Omega_7 = \Psi_3(p)$ (the notation is as in (6.6)).

If $i_y(u) \in \Omega_\beta$ for $\beta \in \{0, 1, 2, 4, 5, 6\}$, then by (6.7) $i_y(u)$ is adjacent in Δ to a vertex from Ω which means that u is adjacent in Γ to a vertex from $\Gamma(x, y)$ and this is not the case we consider. Hence we assume that $i_y(u) \in \Omega_3 \cup \Omega_7$. For $\beta = 3$ or 7 put $\Phi_\beta^x(y) = \{w \mid w \in \Gamma(y), i_y(w) \in \Omega_\beta\}$.

Lemma 8.22 *The following assertions hold:*

- (i) if $q \in \Omega_3$ then $\Psi_3(q) \cap \Omega = \emptyset$;
- (ii) if $q \in \Omega_7$ then $|\Psi_3(q) \cap \Omega| = \frac{1}{2}|\Omega|$.

Proof. If $q \in \Omega_3$ and $r \in \Omega$, then by (2.1 (ii), (iii)) $r \in O_2(F(p))$ and $q \in F(p)$ which means that $\langle q, r \rangle$ has even order. If $q \in \Omega_7$ then the claim is by (2.1 (v)). $\square$

Lemma 8.23 *Suppose that $u \in \Phi_7^x(y)$. Then $u \in \Gamma_2^4(x)$ and $y \in \Phi_7^x(u)$.*

Proof. By (8.22) there is $w \in \Gamma(x, y)$, such that $u \in \Gamma_2^3(w)$. Then $x \in \Gamma(w) \setminus \Gamma(w, u)$ and by (7.11 (ii)) $u \in \Gamma_2(x)$. Since $\Gamma(x) \cap \Gamma(y) \cap \Gamma(u) = \emptyset$, we have $u \in \Gamma_2^4(x)$. By (7.11 (ii)) x is adjacent to a vertex $z \in \Gamma(u, w)$. Then both y and z are contained in $\Gamma(u, w)$ and are at distance 2 in Γ . By (6.3) this means that $y \in \Gamma_2^3(z)$ and by (8.22) $y \in \Phi_7^x(u)$. $\square$

Now we can easily handle the case $u \in \Phi_3^x(y)$.

Lemma 8.24 *Suppose that $u \in \Phi_3^x(y)$. Then the image of $\{y, u\}$ under π_x is an edge of Γ .*

Proof. The crucial observation is that in the considered situation $i_y(u)$ is adjacent in Δ to $p = i_y(y')$, which means that u is adjacent to y' in Γ . If $u \in \Gamma_3(x)$ (which is the case in the Baby Monster graph), then $u^{\pi_x} = u$ and hence $\{y', u\}$ is an edge by the above observation. Suppose that $u \in \Gamma_2(x)$. Then $u \in \Gamma_2^4(x)$ by (7.11 (ii)), since $\Gamma(x) \cap \Gamma(y) \cap \Gamma(u) = \emptyset$. Furthermore by (6.7) and (6.8) $y \in \Phi_3^x(u)$ or $y \in \Phi_7^x(u)$ and the latter case is impossible by (8.23). Hence $y \in \Phi_3^x(u)$ and by the obvious symmetry the subgraph in Γ induced by $\{y, y', u, u'\}$ (where u' is the x -equivalent to u) is complete. Hence the result. $\square$

Now we turn to the final case, when $u \in \Phi_7^x(y)$. We follow notation from the proof of (8.23), so that $w \in \Gamma(x, y)$, $u \in \Gamma_2^3(w)$, $z \in \Gamma(x) \cap \Gamma(w, u)$. Then $x, y, z \in \Gamma(w)$ and $u \in \Gamma_2(w)$. Hence π_x^w is an element of H_w , $y \in \Gamma(u, w)$ and x is adjacent to $z \in \Gamma(u, w)$. Then by (8.21) $\Gamma(u^{\pi_x}, w)$ is the image of $\Gamma(u, w)$ under π_x^w , in particular it contains y^{π_x} . Hence $\{y^{\pi_x}, u^{\pi_x}\}$ is an edge of Γ . This completes the proof of (8.7).

8.5 Proof of Theorems 1 and 2

By Proposition 8.7 for every $x \in \Gamma$ the permutation π_x is an automorphism of Γ . The actions π_z^x of the automorphisms π_z on $\Gamma(x)$ taken for all $z \in \Gamma(x)$ generate $F_x^\infty \cong {}^2E_6(2)$, hence Theorem 1 follows from Proposition 1.1. Now Theorem 2 follows from (3.1) and Theorem 1.

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