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LARGE SIEVE INEQUALITY WITH CHARACTERS FOR POWERFUL MODULI

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Abstract. In this paper we aim to generalize the results in [1],[2],[19] and develop a general formula for large sieve with characters to powerful moduli that will be an improvement to the result in [19].

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1. Introduction

Throughout this paper, we reserve the symbols c_i $(i = 1, 2, ...)$ for absolute positive constants. Large sieve was an idea originated by J. V. Linnik [10] in 1941 while studying the distribution of quadratic non-residues. Refinements of this idea were made by many. In this paper, we develop a large sieve inequality for powerful moduli. More in particular, we aim to have an estimate for the following sum

$$\sum_{q\le Q} \sum_{\substack{a=1\\(a,q)=1}}^{q^k} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^k} n\right) \right|^2, \tag{1.1}$$

where $k \ge 2$ is a natural number. In the sequel, let

$$Z := \sum_{n=M+1}^{M+N} |a_n|^2.$$

With $k = 1$ in (1.1), it is

$$\ll (Q^2 + N) Z. \tag{1.2}$$

This is in fact the consequence of a more general result first introduced by H. Davenport and H. Halberstam [7] in which the Farey fractions in the outer sums of (1.1) can be replaced by any set of well-spaced points. Applying the said more general result, (1.1) is bounded above by

$$\ll (Q^{k+1} + QN) Z, \text{ and } \ll (Q^{2k} + N) Z \tag{1.3}$$

(see [19]). Literature abound on the subject of the classical large sieve. See [3], [6], [7], [8], [10], [12], [13] and [14]. In [19] it was proved that the sum (1.1) can be estimated by

$$\ll \left(Q^{k+1} + \left(NQ^{1-1/\kappa} + N^{1-1/\kappa} Q^{1+k/\kappa} \right) N^{\varepsilon} \right) Z, \tag{1.4}$$

where $\kappa := 2^{k-1}$ and the implied constant depends on ε and k. Furthermore, when appropriate, some of the constants c_i's and the implied constants in $\ll$ in the remainder of this paper will depend on ε or both ε and k. In [1] and [2] this bound was improved for $k = 2$. Extending the elementary method in [1] to higher power moduli, we here establish the following bound for (1.1).

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Theorem 1: *We have*

$$\sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q^k}\left|\sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^k}n\right)\right|^2 \ll (\log\log 10NQ)^{k+1}(Q^{k+1}+N+N^{1/2+\varepsilon}Q^k)Z. \tag{1.5}$$

For $k\ge 3$ Theorem 1 improves the classical bounds (1.3) as well as Zhao's bound (1.4) in the range $N^{1/(2k)+\varepsilon}\ll Q\ll N^{(\kappa-2)/(2(k-1)\kappa-2k)-\varepsilon}$. In particular, for $k=3$ we obtain an improvement in the range $N^{1/6+\varepsilon}\ll Q\ll N^{1/5-\varepsilon}$. We note that for a large k the exponent $(\kappa-2)/(2(k-1)\kappa-2k)$ is close to $1/(2(k-1))$.

Extending the Fourier analytic methods in [2], [19], we establish another bound for cubic moduli which improves the bounds (1.3), (1.4) in the range $N^{7/25+\varepsilon}\ll Q\ll N^{1/3-\varepsilon}$.

Theorem 2: *Suppose that* $1\le Q\le N^{1/2}$. *Then we have*

$$\sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q^3}\left|\sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^3}n\right)\right|^2 \ll \begin{cases} N^{\varepsilon}(Q^4+N^{9/10}Q^{6/5})Z, & \text{if } N^{7/24}\le Q\le N^{1/2},\\ NQ^{6/7+\varepsilon}Z, & \text{if } 1\le Q< N^{7/24}.\end{cases} \tag{1.6}$$

Unfortunately, our Fourier analytic method does not yield any improvement if $k\ge 4$.

2. Proof of Theorem 1

Let $\mathcal{S}$ be the set of k-th powers of natural numbers. Let $Q_0\ge\sqrt{N}$. Set

$$\mathcal{S}(Q_0)=\mathcal{S}\cap(Q_0,2Q_0].$$

We first note, by classical large sieve, setting $Q=\sqrt{N}$ in (1.2),

$$\sum_{q\le\sqrt{N}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q}n\right)\right|^2\le 2NZ. \tag{2.1}$$

Let

$$\mathcal{S}_t(Q_0)=\{q\in\mathbb{N}\ :\ tq\in\mathcal{S}(Q_0)\}.$$

Let $t=p_1^{v_1}\cdots p_n^{v_n}$ be the prime decomposition of t. Furthermore, let

$$u_i:=\left\lceil\frac{v_i}{k}\right\rceil,$$

where for $x\in\mathbb{R}$, $\lceil x\rceil=\min\{k\in\mathbb{Z}\ :\ k\ge x\}$ is the ceiling of x. Moreover, set

$$f_t=p_1^{u_1}\cdots p_n^{u_n}.$$

Therefore, for all $q_0^k=q\in\mathcal{S}$, q is divisible by t if and only if q_0 is divisible by f_t. Therefore, we have

$$\mathcal{S}_t(Q_0)=\{q_1^kg_t\ :\ Q_0^{1/k}/f_t<q_1\le(2Q_0)^{1/k}/f_t\},$$

where

$$g_t:=\frac{f_t^k}{t}.$$

Moreover we note that

$$\mathcal{S}_t(Q_0)\subset(Q_0/t,2Q_0/t]$$

and that

$$|\mathcal{S}_t(Q_0)|\le\frac{(2Q_0)^{1/k}}{f_t}. \tag{2.2}$$

We set for $m\in\mathbb{N}$, $l\in\mathbb{Z}$ with $(m,l)=1$

$$A_t(u,m,l)=\max_{Q_0/t\le y\le 2Q_0/t}|\{q\in\mathcal{S}_t(Q_0)\cap(y,y+u]\ :\ q\equiv l \bmod m\}|. \tag{2.3}$$

Let $\delta_t(m,l)$ be the number of solutions x to the congruence

$$x^k g_t \equiv l \bmod m.$$

We now use Theorem 2 in [1] with $Q_0 \geq \sqrt{N}$:

Theorem 3: *Assume that for all* $t \in \mathbb{N}$, $m \in \mathbb{N}$, $l \in \mathbb{Z}$, $u \in \mathbb{R}$ *with* $t \leq \sqrt{N}$, $m \leq \sqrt{N}/t$, $(m,l) = 1$, $mQ_0/\sqrt{N} \leq u \leq Q_0/t$ *the conditions*

$$A_t(u,m,l) \leq C\left(1 + \frac{|\mathcal{S}_t(Q_0)|/m}{Q_0/t} \cdot u\right) \delta_t(m,l), \tag{2.4}$$

$$\sum_{\substack{l=1 \\ (m,l)=1}}^{m} \delta_t(m,l) \leq m, \tag{2.5}$$

$$\delta_t(m,l) \leq X \tag{2.6}$$

hold for some suitable positive numbers C *and* X. *Then,*

(2.7)

$$\sum_{q \in \mathcal{S}(Q_0)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q} n\right) \right|^2 \leq c_0 C(\min\{Q_0 X, N\} + Q_0) \left(\sqrt{N} \log\log 10N + \max_{r \leq \sqrt{N}} \sum_{t|r} |\mathcal{S}_t(Q_0)| \right) Z.$$

First, we have to check the validity of the conditions (2.4), (2.5) and (2.6). Conditions (2.4) and (2.5) are obviously satisfied with C absolute. We further suppose that $(g_t, m) = 1$ for otherwise $\delta_t(m,l) = 0$ since $(m,l) = 1$. Therefore, we must estimate the number of solutions to

$$x^k \equiv \overline{g_t} l \bmod m, \tag{2.8}$$

where $\overline{g_t}$ is the multiplicative inverse of g_t modulo m. By the virtue of the Chinese remainder theorem, it suffices to estimate the number of solutions to (2.8) with m as a prime power, say $m = p^e$, for $p \in \mathbb{P}$ and $e \in \mathbb{N}$. Note that the function

$$\sigma_k \ : \ (\mathbb{Z}/p^e\mathbb{Z})^* \longrightarrow (\mathbb{Z}/p^e\mathbb{Z})^* \ : \ x \longrightarrow x^k$$

is an endomorphism. Hence it is enough to estimate the size of its kernel $\ker(\sigma_k)$. If $k = \pi_1^{a_1} \cdots \pi_h^{a_h}$ is the prime decomposition of k, then

$$\sigma_k = \prod_{i=1}^{h} \sigma_{\pi_i}^{a_i}.$$

Thus,

$$|\ker\, \sigma_k| \leq \prod_{i=1}^{h} |\ker\, \sigma_{\pi_i}|^{a_i}. \tag{2.9}$$

Hence, it suffices to estimate the size of $|\ker\, \sigma_\pi|$ for prime numbers π.

For $p \in \mathbb{P}$,

$$x^\pi - 1 \equiv 0 \bmod p$$

has at most π solutions. By elementary result (see [15], for example), a solution, $a \bmod p^e$ with $e \geq 1$, of the congruence

$$x^\pi - 1 \equiv 0 \bmod p^e \tag{2.10}$$

lifts to more than one solution to

$$x^\pi - 1 \equiv 0 \bmod p^{e+1}$$

only when $p|\pi a^{\pi-1}$ and $p^{e+1}|a^\pi - 1$. If $p \neq \pi$, $p|\pi a^{\pi-1}$ implies $p|a$, but it is not possible that $p^{e+1}|a^\pi - 1$ as $(a^\pi - 1, a) = 1$. Thus, in this case (2.10) has at most π solutions for all $e \geq 1$. In the following, we consider the case $p = \pi$.

By Fermat's little theorem, there exists only one solution of the congruence

$$x^{\pi}-1\equiv 0 \bmod \pi,$$

namely 1 mod π. This solution lifts to exactly π solutions to

$$x^{\pi}-1\equiv 0 \bmod \pi^2,$$

namely

$$1,\ 1+\pi,\ 1+2\pi,\ ...,1+(\pi-1)\pi \bmod \pi^2.$$

More generally, if a mod π^e is a solution to

$$x^{\pi}-1\equiv 0 \bmod \pi^e, \tag{2.11}$$

then, if a lifts to solutions to

$$x^{\pi}-1\equiv 0 \bmod \pi^{e+1},$$

they are of the form

$$a,\ a+\pi^e,\ a+2\pi^e,\ ...,\ a+(\pi-1)\pi^e \bmod \pi^{e+1}. \tag{2.12}$$

Assume there are $j_1,j_2\in\{0,...,\pi-1\}$, $j_1\neq j_2$ such that both $a+j_1\pi^e$ and $a+j_2\pi^e$ lift to solutions modulo π^{e+2}. Then $\pi^{e+2}|(a+j_1\pi^e)^{\pi}-1$ and $\pi^{e+2}|(a+j_2\pi^e)^{\pi}-1$, hence

$$(a+j_1\pi^e)^{\pi}-(a+j_2\pi^e)^{\pi}=(j_1-j_2)\pi^e\sum_{i=0}^{\pi-1}(a+j_1\pi^e)^{\pi-1-i}(a+j_2\pi^e)^i$$

is divisible by π^{e+2}. If $e\geq 2$, this implies $a\equiv 0 \bmod \pi$, but then a cannot be a solution to (2.11). Therefore, if $e\geq 2$, only one of the solutions (2.12) lifts to a solution modulo π^{e+2}. From this we infer that the number of solutions to (2.11) never exceeds π^2, *i.e.*

$$|\ker\ \sigma_{\pi}|\leq \pi^2.$$

Combining this with (2.9), we get

$$|\ker\ \sigma_k|\leq k^2.$$

Therefore, by the Chinese remainder theorem, we obtain

$$\delta_t(m,l)\leq k^{2\omega(m)},$$

where $\omega(m)$ is the number of distinct prime divisors of m. Since $2^{\omega(m)}$ is the number of square-free divisors of m, we have

$$k^{2\omega(m)}\leq \tau(m)^{2\log_2 k}\ll m^{\varepsilon},$$

where $\tau(m)$ is the number of divisors of m. Thus, if $m\leq\sqrt{N}$, (2.6) holds with

$$X\ll N^{\varepsilon}.$$

Now, by Theorem 3,

$$\sum_{q\in\mathcal{S}(Q_0)}\ \sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\sum_{n=M+1}^{M+N}a_n e\left(\frac{a}{q}n\right)\right|^2 \tag{2.13}$$

is majorized by

$$\ll (\min\{Q_0N^{\varepsilon},N\}+Q_0)\left(\sqrt{N}\log\log(10N)+\max_{r\leq\sqrt{N}}\sum_{t|r}Q_0^{1/k}f_t^{-1}\right)Z.$$

The function

$$G(r)=\sum_{t|r}\frac{1}{f_t}$$

is clearly multiplicative. If r is a prime power p^v, then

$$G(r)\leq 1+k\left(\frac{1}{p}+\frac{1}{p^2}+...\right)=1+\frac{k}{p-1}\leq\left(1+\frac{1}{p-1}\right)^k=\left(\frac{p^v}{\varphi\left(p^v\right)}\right)^k.$$

Hence, for all $r \in \mathbb{N}$ we have

$$G(r) \le \left(\frac{r}{\varphi(r)}\right)^k \ll (\log\log 10r)^k. \tag{2.14}$$

Hence (2.13) is

$$\ll (\log\log 10NQ_0)^{k+1}(\sqrt{N}+Q_0^{1/k})(\min\{Q_0N^{\varepsilon},N\}+Q_0).$$

The above is always majorized by

$$\ll (\log\log 10NQ_0)^{k+1}\left(Q_0^{1+1/k}+N^{1/2+\varepsilon}Q_0\right).$$

Summing over all relevant dyadic intervals and combining with (2.1), we see that (1.1) is majorized by

$$\ll (\log\log 10NQ)^{k+1}(Q^{k+1}+N+N^{1/2+\varepsilon}Q^k)Z.$$

Therefore, our result follows. □

3. Proof of Theorem 2

3.1. **Reduction to Farey fractions in short intervals.** As in [1], [2], our starting point is the following general large sieve inequality.

Lemma 1: *Let $(\alpha_r)_{r\in\mathbb{N}}$ be a sequence of real numbers. Suppose that $0<\Delta\le 1/2$ and $R\in\mathbb{N}$. Put*

$$K(\Delta) := \max_{\alpha\in\mathbb{R}} \sum_{\substack{r=1\\ \|\alpha_r-\alpha\|\le\Delta}}^{R} 1, \tag{3.1}$$

where $\|x\|$ denotes the distance of a real x to its closest integer. Then

$$\sum_{r=1}^{R} |S(\alpha_r)|^2 \le c_1K(\Delta)(N+\Delta^{-1})Z.$$

In the sequel, we suppose that $\mathcal{S}$ is the set of cubes of natural numbers and that $\alpha_1,...,\alpha_R$ is the sequence of Farey fractions a/q with $q\in\mathcal{S}(Q_0)$, $1\le a\le q$ and $(a,q)=1$, where $Q_0\ge 1$. We further suppose that $\alpha\in\mathbb{R}$ and $0<\Delta\le 1/2$. Put

$$I(\alpha) := [\alpha-\Delta,\alpha+\Delta] \quad \text{and} \quad P(\alpha) := \sum_{\substack{q\in\mathcal{S}\cap(Q_0,2Q_0]\\ (a,q)=1\\ a/q\in I(\alpha)}} 1.$$

Then we have

$$K(\Delta) = \max_{\alpha\in\mathbb{R}} P(\alpha).$$

Therefore, the proof of Theorem 2 reduces to estimating $P(\alpha)$.

As in [1] and [2], we begin with an idea of D. Wolke [18]. Let τ be a positive number with

$$1\le\tau\le\frac{1}{\sqrt{\Delta}}. \tag{3.2}$$

In [1] and [2] we put $\tau := 1/\sqrt{\Delta}$, but in fact our method works for all τ satisfying (3.2). We will later fix τ in an optimal manner. In the said earlier papers, $\tau = 1/\sqrt{\Delta}$ was the optimal choice.

By Dirichlet's approximation theorem, α can be written in the form

$$\alpha = \frac{b}{r}+z,$$

where

$$r\le\tau,\ (b,r)=1,\ |z|\le\frac{1}{r\tau}. \tag{3.3}$$

Thus, it suffices to estimate $P(b/r+z)$ for all b,r,z satisfying (3.3).

We further note that we can restrict ourselves to the case when

$$z \geq \Delta. \tag{3.4}$$

If $|z| < \Delta$, then

$$P(\alpha) \leq P\left(\frac{b}{r} - \Delta\right) + P\left(\frac{b}{r} + \Delta\right).$$

Furthermore, we have

$$\Delta \leq \frac{1}{\tau^2} \leq \frac{1}{r\tau}.$$

Therefore this case can be reduced to the case $|z| = \Delta$. Moreover, as $P(\alpha) = P(-\alpha)$, we can choose z positive. So we can assume (3.4), without any loss of generality.

Summarizing the above observations, we deduce

Lemma 2: *We have*

$$K(\Delta) \leq 2 \max_{\substack{r\in\mathbb{N} \\ r\leq\tau}} \max_{\substack{b\in\mathbb{Z} \\ (b,r)=1}} \max_{\Delta\leq z\leq 1/(\tau r)} P\left(\frac{b}{r} + z\right). \tag{3.5}$$

3.2. **Estimation of $P(b/r+z)$ - first way.** We now prove a first estimate for $P\left(b/r + z\right)$ by using some results in [1]. In the sequel, we suppose that the conditions (3.2), (3.3) and (3.4) are satisfied.

By inequality (41) in [1], we have

$$P\left(\frac{b}{r} + z\right) \leq 1 + 6 \sum_{t|r} \sum_{\substack{0<m\leq 4rzQ_0/t \\ (m,r/t)=1}} A_t\left(\frac{\Delta Q_0}{tz}, \frac{r}{t}, -\overline{b}m\right), \tag{3.6}$$

where $A_t(u,m,l)$ is defined as in (2.3) and $\overline{b}$ is the multiplicative inverse of b modulo r. By the results of section 2, for $\mathcal{S}$ the set of cubes, the conditions (2.4), (2.5) and (2.6) with $X = \Delta^{-\varepsilon}$ are satisfied for all $t \in \mathbb{N}$, $m \in \mathbb{N}$, $l \in \mathbb{Z}$, $u \in \mathbb{R}$ with $t \leq \tau$, $m \leq \tau/t$, $(m,l) = 1$, $mQ_0/\tau \leq u \leq Q_0/t$. Conditions (2.4) and (2.6) imply

$$\sum_{\substack{0<m\leq 4rzQ_0/t \\ (m,r/t)=1}} A_t\left(\frac{\Delta Q_0}{tz}, \frac{r}{t}, -\overline{b}m\right) \leq C\left(1 + \frac{\Delta t|\mathcal{S}_t(Q_0)|}{rz}\right)\frac{4rzQ_0X}{t} \tag{3.7}$$

From (3.6), (3.7) and

$$\sum_{t|r} \frac{1}{t} \leq \prod_{p|r}\left(1 + \frac{1}{p} + \frac{1}{p^2} + ...\right) = \prod_{p|r} \frac{p}{p-1} = \frac{r}{\varphi(r)} \leq c_2 \log\log 10r,$$

we derive

$$P\left(\frac{b}{r} + z\right) \leq 1 + c_3 Q_0 X\left(rz\log\log 10r + \Delta\sum_{t|r}|\mathcal{S}_t(Q_0)|\right). \tag{3.8}$$

Furthermore, by (2.2) and (2.14), we have

$$\sum_{t|r} |\mathcal{S}(Q_0)| \ll (\log\log 10r)^3 Q_0^{1/3}.$$

Thus, from (3.8) and the fact that $r \leq \tau = \Delta^{-1/2}$, we obtain

Proposition 1: *Let $\mathcal{S}$ be the set of cubes of natural numbers. Suppose that the conditions (3.2), (3.3) and (3.4) are satisfied. Then we have*

$$P\left(\frac{b}{r} + z\right) \leq 1 + c_4\Delta^{-\varepsilon}\left(Q_0^{4/3}\Delta + Q_0 rz\right). \tag{3.9}$$

3.3. Estimation of $P(b/r+z)$ - second way. We now prove a second estimate for $P(b/r+z)$ by extending the Fourier analytic methods in [2], [19] to cubic moduli. The following bound for $P(b/r+z)$ can be proved in the same manner as Lemma 2 in [2].

Lemma 3: *Let $\mathcal{S}$ be the set of cubes of natural numbers. Suppose that*

$$\frac{Q_0\Delta}{z} \le \delta \le Q_0. \tag{3.10}$$

Then,

$$P\left(\frac{b}{r}+z\right) \le c_5 \left(1+\frac{1}{\delta}\int\limits_{Q_0}^{2Q_0} \Pi(\delta,y)\,\mathrm{d}y\right), \tag{3.11}$$

where $I(\delta,y)=[y^{1/3}-c_6\delta/Q_0^{2/3}, y^{1/3}+c_6\delta/Q_0^{2/3}]$, $J(\delta,y)=[(y-4\delta)rz,(y+4\delta)rz]$ and

$$\Pi(\delta,y) = \sum_{q\in I(\delta,y)} \sum_{\substack{m\in J(\delta,y)\\ m\equiv -bq^3 \bmod r\\ m\neq 0}} 1. \tag{3.12}$$

We shall prove the following

Proposition 2: *Let $\mathcal{S}$ be the set of cubes of natural numbers. Suppose that the conditions (3.2), (3.3) and (3.4) are satisfied. Then we have*

$$P\left(\frac{b}{r}+z\right) \le c_7\Delta^{-\varepsilon}\left(Q_0^{4/3}\Delta + Q_0^{1/3}\Delta r^{-1/3}z^{-1}+\Delta^{-1/2}(rz)^{1/2}\right). \tag{3.13}$$

To derive Proposition 2 from Lemma 3, we need the following standard results from Fourier analysis.

Lemma 4: (Poisson summation formula, [5]) *Let $f(x)$ be a complex-valued function on the real numbers that is piecewise continuous with only finitely many discontinuities and for all real numbers a satisfies*

$$f(a) = \frac{1}{2}\left(\lim_{x\to a^-} f(x) + \lim_{x\to a^+} f(x)\right).$$

Moreover, suppose that $f(x)\le c_8(1+|x|)^{-c}$ for some $c>1$. Then,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n), \text{ where } \hat{f}(x) := \int\limits_{-\infty}^{\infty} f(y)e(xy)\mathrm{d}y,$$

the Fourier transform of $f(x)$.

Lemma 5: (see [19], for example) *For $x\in\mathbb{R}\setminus\{0\}$ define*

$$\phi(x) := \left(\frac{\sin \pi x}{2x}\right)^2, \text{ and } \phi(0) := \lim_{x\to 0}\phi(x) = \frac{\pi^2}{4}.$$

Then $\phi(x)\ge 1$ for $|x|\le 1/2$, and the Fourier transform of the function $\phi(x)$ is

$$\hat{\phi}(s) = \frac{\pi^2}{4}\max\{1-|s|,0\}.$$

Lemma 6: (see Lemma 3.1. in [9]) *Let $F:[a,b]\to\mathbb{R}$ be twice differentiable. Assume that $|F'(x)|\ge u>0$ for all $x\in[a,b]$. Then,*

$$\left|\int\limits_a^b e^{iF(x)}\mathrm{d}x\right| \le \frac{c_9}{u}.$$

Lemma 7: (see Lemma 4.3.1. in [4]) *Let $F : [a,b] \to \mathbb{R}$ be twice continuously differentiable. Assume that $|F''(x)| \ge u > 0$ for all $x \in [a,b]$. Then,*

$$\left| \int_a^b e^{iF(x)} \mathrm{d}x \right| \le \frac{c_{10}}{\sqrt{u}}.$$

We shall also need the following estimates for cubic exponential sums.

Lemma 8: (see [11], [17]) *Let $c \in \mathbb{N}$, $k, l \in \mathbb{Z}$ with $(k,c) = 1$. Then,*

$$\sum_{d=1}^{c} e\left(\frac{kd^3 + ld}{c}\right) \le c_{11} c^{1/2+\varepsilon} (l, c).$$

Furthermore,

$$\sum_{d=1}^{c} e\left(\frac{kd^3}{c}\right) \le c_{11} c^{2/3}.$$

Proof of Proposition 2: We put

$$\delta := \frac{Q_0 \Delta}{z}. \tag{3.14}$$

By Lemma 5, (3.12) can be estimated by

$$\Pi(\delta, y) \le \sum_{q \in \mathbb{Z}} \phi\left(\frac{q - y^{1/3}}{2c_6 \delta / Q_0^{2/3}}\right) \sum_{\substack{m \in \mathbb{Z} \\ m \equiv -bq^3 \bmod r}} \phi\left(\frac{m - yrz}{8\delta rz}\right). \tag{3.15}$$

Using Lemma 4 after a linear change of variables, we transform the inner sum on the right-hand side of (3.15) into

$$\sum_{\substack{m \in \mathbb{Z} \\ m \equiv -bq^3 \bmod r}} \phi\left(\frac{m - yrz}{8\delta rz}\right) = 8\delta z \sum_{j \in \mathbb{Z}} e\left(\frac{jbq^3}{r} + jyz\right) \hat{\phi}(8j\delta z).$$

Therefore, we get for the double sum on the right-hand side of (3.15)

$$\begin{aligned} &\sum_{q \in \mathbb{Z}} \phi\left(\frac{q - y^{1/3}}{2c_6 \delta / Q_0^{2/3}}\right) \sum_{\substack{m \in \mathbb{Z} \\ m \equiv -bq^3 \bmod r}} \phi\left(\frac{m - yrz}{8\delta rz}\right) \\ &= 8\delta z \sum_{j \in \mathbb{Z}} e(jyz) \hat{\phi}(8j\delta z) \sum_{d=1}^{\tilde{r}} e\left(\frac{\tilde{j} b d^3}{\tilde{r}}\right) \sum_{\substack{k \in \mathbb{Z} \\ k \equiv d \bmod \tilde{r}}} \phi\left(\frac{k - y^{1/3}}{2c_6 \delta / Q_0^{2/3}}\right), \end{aligned} \tag{3.16}$$

where $\tilde{r} := r/(r,j)$ and $\tilde{j} := j/(r,j)$. Again using Lemma 4 after a linear change of variables, we transform the inner sum on the right-hand side of (3.16) into

$$\sum_{\substack{k \in \mathbb{Z} \\ k \equiv d \bmod \tilde{r}}} \phi\left(\frac{k - y^{1/3}}{2c_6 \delta / Q_0^{2/3}}\right) = \frac{2c_6 \delta}{\tilde{r} Q_0^{2/3}} \sum_{l \in \mathbb{Z}} e\left(l \cdot \frac{d - y^{1/3}}{\tilde{r}}\right) \hat{\phi}\left(\frac{2c_6 l \delta}{\tilde{r} Q_0^{2/3}}\right). \tag{3.17}$$

From (3.16) and (3.17), we obtain

$$\frac{1}{\delta}\int\limits_{Q_0}^{2Q_0}\sum_{q\in\mathbb{Z}}\phi\left(\frac{q-y^{1/3}}{2c_6\delta/Q_0^{2/3}}\right)\sum_{\substack{m\in\mathbb{Z}\\ m\equiv -bq^3 \bmod r}}\phi\left(\frac{m-yrz}{8\delta rz}\right)\mathrm{d}y$$
$$\le \frac{16c_6\delta z}{Q_0^{2/3}}\sum_{j\in\mathbb{Z}}\frac{\hat{\phi}(8j\delta z)}{\tilde{r}}\sum_{l\in\mathbb{Z}}\hat{\phi}\left(\frac{2c_6 l\delta}{\tilde{r}Q_0^{2/3}}\right)\left|\sum_{d=1}^{\tilde{r}}e\left(\frac{\tilde{j}bd^3+ld}{\tilde{r}}\right)\int\limits_{Q_0}^{2Q_0}e\left(jyz-l\cdot\frac{y^{1/3}}{\tilde{r}}\right)\ \mathrm{d}y\right|. \tag{3.18}$$

Applying the Lemmas 5 and 8 to the right-hand side of (3.18), and taking $r\le 1/\sqrt{\Delta}$ by (3.2) and (3.3) into account, we deduce

$$\frac{1}{\delta}\int\limits_{Q_0}^{2Q_0}\sum_{q\in\mathbb{Z}}\phi\left(\frac{q-y^{1/3}}{c_6\delta/Q_0^{2/3}}\right)\sum_{\substack{m\in\mathbb{Z}\\ m\equiv -bq^3 \bmod r}}\phi\left(\frac{m-yrz}{8\delta rz}\right)\mathrm{d}y$$
$$\le \frac{c_{12}\delta z\Delta^{-\varepsilon}}{Q_0^{2/3}}\left(\sum_{|j|\le 1/(8\delta z)}\frac{1}{\sqrt{\tilde{r}}}\sum_{\substack{|l|\le(\tilde{r}Q_0^{2/3})/(2c_6\delta)\\ l\neq 0}}(l,\tilde{r})\left|\int\limits_{Q_0}^{2Q_0}e\left(jyz-l\cdot\frac{y^{1/3}}{\tilde{r}}\right)\ \mathrm{d}y\right|+\sum_{|j|\le 1/(8\delta z)}\frac{1}{\sqrt[3]{\tilde{r}}}\left|\int\limits_{Q_0}^{2Q_0}e\left(jyz\right)\ \mathrm{d}y\right|\right). \tag{3.19}$$

If $j\neq 0$, then

$$\left|\int\limits_{Q_0}^{2Q_0}e\left(jyz\right)\ \mathrm{d}y\right|\le\frac{1}{|j|z}.$$

If $j=0$ and $l\neq 0$, then

$$\left|\int\limits_{Q_0}^{2Q_0}e\left(jyz-l\cdot\frac{y^{1/3}}{\tilde{r}}\right)\ \mathrm{d}y\right|\le\frac{c_{13}Q_0^{2/3}}{|l|}$$

by Lemma 6 (take into account that $\tilde{r}=1$ if $j=0$). If $j\neq 0$ and $l\neq 0$, then Lemma 7 yields

$$\left|\int\limits_{Q_0}^{2Q_0}e\left(jyz-l\cdot\frac{y^{1/3}}{\tilde{r}}\right)\ \mathrm{d}y\right|\le\frac{c_{14}\sqrt{\tilde{r}}Q_0^{5/6}}{\sqrt{|l|}}.$$

Therefore, the right-hand side of (3.19) is majorized by

$$\le c_{15}\delta\Delta^{-\varepsilon}\left(zQ_0^{1/3}+\frac{1}{Q_0^{2/3}}\sum_{1\le j\le 1/(8\delta z)}\frac{1}{j\sqrt[3]{\tilde{r}}}+z\sum_{1\le l\le Q_0^{2/3}/(2c_6\delta)}\frac{1}{l}+zQ_0^{1/6}\sum_{1\le j\le 1/(8\delta z)}\ \sum_{1\le l\le \tilde{r}Q_0^{2/3}/(2c_6\delta)}\frac{(l,\tilde{r})}{\sqrt{l}}\right). \tag{3.20}$$

Now, we estimate the sums in the last line of (3.20). Using (3.2), (3.3) and (3.14), we obtain

$$\sum_{1\le l\le Q_0^{2/3}/(2c_6\delta)}\frac{1}{l}\le c_{16}\Delta^{-\varepsilon}. \tag{3.21}$$

Using the definition of $\tilde{r}$, (3.2), (3.3) and (3.14), we obtain

$$\sum_{1\le j\le 1/(8\delta z)}\frac{1}{j\sqrt[3]{\tilde{r}}}=\frac{1}{\sqrt[3]{r}}\sum_{t|r}\sqrt[3]{t}\sum_{\substack{1\le j\le 1/(8\delta z)\\ (r,j)=t}}\frac{1}{j}\le\frac{c_{17}\Delta^{-\varepsilon}}{\sqrt[3]{r}}\sum_{t|r}t^{-2/3}\le c_{18}\Delta^{-2\varepsilon}r^{-1/3}. \tag{3.22}$$

For $A \geq 1$, we have

$$\sum_{1 \leq l \leq A} \frac{(l, \tilde{r})}{\sqrt{l}} \leq \sum_{t | \tilde{r}} t \sum_{1 \leq l \leq A/t} \frac{1}{\sqrt{lt}} \ll \sqrt{A} \sum_{t | \tilde{r}} 1 \ll \tilde{r}^{\varepsilon} \sqrt{A}.$$

Therefore,

$$\sum_{1 \leq j \leq 1/(8\delta z)} \sum_{1 \leq l \leq \tilde{r} Q_0^{2/3}/(2c_6\delta)} \frac{(l, \tilde{r})}{\sqrt{l}} \leq \frac{c_{19} \Delta^{-\varepsilon} Q_0^{1/3}}{\sqrt{\delta}} \sum_{1 \leq j \leq 1/(8\delta z)} \sqrt{\tilde{r}}. \tag{3.23}$$

Using the definition of $\tilde{r}$, we obtain

$$\sum_{1 \leq j \leq 1/(8\delta z)} \sqrt{\tilde{r}} = \sqrt{r} \sum_{t|r} \frac{1}{\sqrt{t}} \sum_{\substack{1 \leq j \leq 1/(8\delta z) \\ (r,j) = t}} 1 \leq \frac{\sqrt{r}}{8\delta z} \sum_{t|r} \frac{1}{t^{3/2}} \leq \frac{c_{20}\sqrt{r}}{\delta z}. \tag{3.24}$$

Combining Lemma 3 and (3.19-3.24), we obtain

$$P\left(\frac{b}{r} + z\right) \leq c_7 \Delta^{-3\varepsilon} \left(1 + \delta z Q_0^{1/3} + \delta Q_0^{-2/3} r^{-1/3} + \delta^{-1/2} Q_0^{1/2} \sqrt{r}\right). \tag{3.25}$$

From (3.14) and (3.25), we infer the desired estimate. Note that the first term in the right-hand side of (3.25) can be absorbed into the last term on the right-hand side of (3.13) by (3.4). □

3.4. **Final proof of Theorem 2.** Combining Propositions 1,2 and (3.3), we obtain

$$P\left(\frac{b}{r} + z\right) \leq c_{21} \Delta^{-\varepsilon} \left(Q_0^{4/3} \Delta + \min\left\{Q_0 r z, Q_0^{1/3} \Delta r^{-1/3} z^{-1}\right\} + \Delta^{-1/2} \tau^{-1/2}\right). \tag{3.26}$$

If

$$z \leq \Delta^{1/2} Q_0^{-1/3} r^{-2/3},$$

then

$$\min\left\{Q_0 r z, Q_0^{1/3} \Delta r^{-1/3} z^{-1}\right\} = Q_0 r z \leq Q_0^{2/3} \Delta^{1/2} r^{1/3}.$$

If

$$z > \Delta^{1/2} Q_0^{-1/3} r^{-2/3},$$

then

$$\min\left\{Q_0 r z, Q_0^{1/3} \Delta r^{-1/3} z^{-1}\right\} = Q_0^{1/3} \Delta r^{-1/3} z^{-1} \leq Q_0^{2/3} \Delta^{1/2} r^{1/3}.$$

From the above inequalities and (3.3), we deduce

$$\min\left\{Q_0 r z, Q_0^{1/3} \Delta r^{-1/2} z^{-1}\right\} \leq Q_0^{2/3} \Delta^{1/2} r^{1/3} \leq Q_0^{2/3} \Delta^{1/2} \tau^{1/3}. \tag{3.27}$$

Combining (3.26) and (3.27), we get

$$P\left(\frac{b}{r} + z\right) \leq c_{22} \Delta^{-\varepsilon} \left(Q_0^{4/3} \Delta \tau^{\varepsilon} + Q_0^{2/3} \Delta^{1/2} \tau^{1/3+\varepsilon} + \Delta^{-1/2} \tau^{-1/2}\right). \tag{3.28}$$

Now we choose

$$\tau := \begin{cases} N^{6/5} Q_0^{-4/5}, & \text{if } N^{7/8} \leq Q_0 \leq N^{3/2}, \\ Q_0^{4/7}, & \text{if } 1 \leq Q_0 < N^{7/8}, \end{cases} \quad \text{and } \Delta := \begin{cases} N^{-1}, & \text{if } N^{7/8} \leq Q_0 \leq N^{3/2}, \\ Q_0^{-8/7}, & \text{if } 1 \leq Q_0 < N^{7/8}. \end{cases}$$

Then the condition (3.2) is satisfied in each case, and from (3.28) and Lemmas 1,2, we obtain

$$\sum_{Q_0^{1/3} \leq q \leq (2Q_0)^{1/3}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q^3} \left| S\left(\frac{a}{q^3}\right) \right|^2 \ll \begin{cases} N^{\varepsilon} \left(Q_0^{4/3} + N^{9/10} Q_0^{2/5}\right) Z, & \text{if } N^{7/8} \leq Q_0 \leq N^{3/2}, \\ N Q_0^{2/7+\varepsilon} Z, & \text{if } 1 \leq Q_0 < N^{7/8}. \end{cases} \tag{3.29}$$

We can divide the interval $[1, Q]$ into $O(\log Q)$ subintervals of the form $\left[Q_0^{1/3}, (2Q_0)^{1/3}\right]$, where $1 \leq Q_0 \leq Q^3$. Hence, the result of Theorem 2 follows from (3.29). □

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References

[1] S. Baier, *On the large sieve with a sparse set of moduli*, preprint.
[2] S. Baier, *The large sieve with square moduli*, preprint.
[3] E. Bombieri and H. Davenport, *Some inequalities involving trigonometrical polynomials*, Annali Scuola Normale Superiore - Pisa 23 (1969) 223-241.
[4] J. Brüdern, *Einführung in die analytische Zahlentheorie*, Springer-Verlag, Berlin ect., 1995.
[5] D. Bump, *Automorphic Forms and Representations*, Cambridge Stud. Adv. Math. 55, Cambridge Univ. Press, Cambridge, 1996.
[6] H. Davenport, *Multiplicative Number Theory*, Third Edition, Graduate Texts in Mathematics, 74, Springer-Verlag, New York, etc., 2000.
[7] H. Davenport and H. Halberstam, *The values of a trigonometric polynomial at well spaced points*, Mathematika 13 (1966) 91-96, *Corrigendum and addendum*, Mathematika 14 (1967) 299-232.
[8] P. X. Gallagher, *The large sieve*, Mathematika 14 (1967) 14-20.
[9] S.W. Graham, G. Kolesnik, *Van der Corput's Method of Exponential Sums*, Cambridge University Press, Cambridge ect., 1991.
[10] J. V. Linnik, *The large sieve*, Doklady Akad. nauk SSSR 36 (1941) 119-120.
[11] L. K. Hua, *On exponential sums*, Sci. Record 1 (1957) 1-4.
[12] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Lecture Notes in Mathematics, 227, Spring-Verlag, Berlin, etc., 1971.
[13] H. L. Montgomery, *The analytic principles of large sieve*, Bull. Amer. Math. Soc., 84 (1978) 547-567.
[14] H. L. Montgomery and R. C. Vaughan, *The Large Sieve*, Mathematika 20 (1973) 119-134.
[15] I. Niven, H.S. Zuckerman, H.L. Montgomery, *An introduction to the Theory of Numbers*, John Wiley & Sons, New York, 1991.
[16] R.C. Vaughan, *The Hardy-Littlewood Method*, Cambridge University Press, Cambridge, 1997.
[17] I.M. Vinogradov, *The Method of Trigonometrical Sums in the Theory of Numbers*, Trav. Inst. Math. Stekloff 23, 1947.
[18] D. Wolke, *On the large sieve with primes*, Acta Math. Acad. Sci. Hungar. 22 (1971/72) 239-247.
[19] L. Zhao, *Large sieve inequality with characters to square moduli*, Acta Arith. 112 (2004) 297-308.

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