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A characterization of the Petersen-type geometry of the McLaughlin group

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Abstract

The McLaughlin sporadic simple group McL is the flag-transitive automorphism group of a Petersen-type geometry $\mathcal{G} = \mathcal{G}(\text{McL})$ with the diagram

where the edge in the middle indicates the geometry of vertices and edges of the Petersen graph. The elements corresponding to the nodes from the left to the right on the diagram P_3^3 are called points, lines, triangles and planes, respectively. The residue in $\mathcal{G}$ of a point is the P_3 -geometry $\mathcal{G}(\text{Mat}_{22})$ of the Mathieu group of degree 22 and the residue of a plane is the P^3 -geometry $\mathcal{G}(\text{Alt}_7)$ of the alternating group of degree 7. The geometries $\mathcal{G}(\text{Mat}_{22})$ and $\mathcal{G}(\text{Alt}_7)$ possess 3-fold covers $\mathcal{G}(3\text{Mat}_{22})$ and $\mathcal{G}(3\text{Alt}_7)$ which are known to be universal. In this paper we show that $\mathcal{G}$ is simply connected and construct a geometry $\tilde{\mathcal{G}}$ which possesses a 2-covering onto $\mathcal{G}$. The automorphism group of $\tilde{\mathcal{G}}$ is of the form 3^{23}McL ; the residues of a point and a plane are isomorphic to $\mathcal{G}(3\text{Mat}_{22})$ and $\mathcal{G}(3\text{Alt}_7)$, respectively. Moreover, we reduce the classification problem of all flag-transitive P_n^m -geometries with $n, m \geq 3$ to the calculation of the universal cover of $\tilde{\mathcal{G}}$.

1. Introduction

Our terminology and notation concerning geometries and their automorphism groups are fairly standard (cf. [Bu, P] and the next section).

We will use the following notation for diagrams of geometries related to the

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Petersen graph ($\begin{array}{c} \text{P} \\ \circ \text{---} \circ \\ 1 \quad 2 \end{array}$ denotes the geometry of vertices and edges of the Petersen graph):

A geometry with diagram D will be called a D -geometry or a geometry of type D . For a geometry with a string diagram the types on the diagram increase from the left to the right from 1 to the rank of the geometry. The elements of type 1, 2, 3 and 4 in a P_3^3 -geometry will be called points, lines, triangles and planes, respectively. We denote by $\mathcal{G}(G)$ a geometry possessing (a group) G as a flag-transitive automorphism group. The flag-transitive P_n -geometries are completely classified (cf. [IS2] and [Iv]). In the classification of flag-transitive P^m -geometries some partial results are available (cf. [IS1]). There exists a flag-transitive geometry P_3^3 -geometry $\mathcal{G}(\text{McL})$ for which the sporadic simple group of McLaughlin McL is the automorphism group. This geometry has been discovered by A. Neumaier (unpublished).

We formulate below the main results of the present paper.

THEOREM 1.1. *Let $n, m \geq 3$ and let $\mathcal{G}$ be a flag-transitive geometry of type P_n^m . Then $n = m = 3$.*

THEOREM 1.2. *Let $\mathcal{G}$ be a flag-transitive geometry of type P_3^3 . Then $\mathcal{G}$ is a quotient of the universal 2-cover of $\mathcal{G}(\text{McL})$.*

THEOREM 1.3. *The geometry $\mathcal{G}(\text{McL})$ is simply connected.*

THEOREM 1.4. *There exists a 3^{23} -fold 2-cover $\tilde{\mathcal{G}} \rightarrow \mathcal{G}(\text{McL})$. The geometry $\tilde{\mathcal{G}}$ is flag-transitive and its automorphism group $\tilde{G}$ is a non-split extension of McL by an elementary abelian 3-group of rank 23. As a $GF(3)$ -module for $\tilde{G}/O_3(\tilde{G}) \cong \text{McL}$, $O_3(\tilde{G})$ is isomorphic to a McL -submodule in the Leech lattice taken modulo 3.*

We construct the 2-cover $\tilde{\mathcal{G}}$ in Theorem 1.4 using a 3^{23} -fold 2-cover of the P_4 -geometry $\mathcal{G}(C_{O_2})$ constructed in [Sh3]. Notice that the centre of $\tilde{G}$ is of order 3 and by factorizing it out we obtain a flag-transitive 3^{22} -fold 2-cover of $\mathcal{G}(\text{McL})$.

In $\mathcal{G}(\text{McL})$ the residue of a point is a P_3 -geometry $\mathcal{G}(\text{Mat}_{22})$ and the residue of a plane is a P^3 -geometry $\mathcal{G}(\text{Alt}_7)$. Hence Theorems 1.2 and 1.3 imply the following.

COROLLARY 1.5. *$\mathcal{G}(\text{McL})$ is the unique flag-transitive geometry of type P_3^3 in which the residue of a point is isomorphic to $\mathcal{G}(\text{Mat}_{22})$ and the residue of a plane is isomorphic to $\mathcal{G}(\text{Alt}_7)$.*

In the geometry $\tilde{\mathcal{G}}$ the residue of a point is $\mathcal{G}(3\text{Mat}_{22})$, which is the universal cover of $\mathcal{G}(\text{Mat}_{22})$ and the residue of a plane is $\mathcal{G}(3\text{Alt}_7)$, which is the universal cover of $\mathcal{G}(\text{Alt}_7)$. It is clear that the residues of lines and triangles in $\mathcal{G}(\text{McL})$ are simply connected (in particular they are isomorphic to the corresponding residues in $\tilde{\mathcal{G}}$). Since $\tilde{\mathcal{G}}$ is a 2-cover of $\mathcal{G}(\text{McL})$ we have the following.

COROLLARY 1.6. *The universal cover of $\tilde{\mathcal{G}}$ is isomorphic to the universal 2-cover of $\mathcal{G}(\text{McL})$.*

The planes and triangles from $\mathcal{G}(\text{McL})$ can be realized by 1- and 2-dimensional subspaces in a 22-dimensional subspace of the Leech lattice modulo 2, so that the incidence relation is via inclusion. Thus $\mathcal{G}(\text{McL})$ possesses a natural (abelian) representation of dimension 22.

THEOREM 1.7. *The 22-dimensional natural representation of $\mathcal{G}(\text{McL})$ in a subspace of the Leech lattice modulo 2 is universal among abelian representations.*

The collinearity graph of $\mathcal{G}(\text{McL})$ contains a subgraph which is the collinearity graph of a GAB (geometry which is almost building) $\mathcal{G}(U_4(3))$ of type

$$\begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \\ 2 \quad \quad 2 \quad \quad 2 \end{array},$$

first described in [K1]. The collinearity graph of the 2-cover $\tilde{\mathcal{G}}$ as in Theorem 1.4 contains a subgraph which is the collinearity graph of a new flag-transitive tilde geometry $\tilde{\mathcal{X}}$ of type

$$\begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \\ \sim \quad \quad \sim \\ 2 \quad \quad 2 \quad \quad 2 \end{array},$$

which possesses a morphism onto $\mathcal{G}(U_4(3))$.

THEOREM 1.8. *There exists a 3^{23} -fold 1-cover $\tilde{\mathcal{X}} \rightarrow \mathcal{G}(U_4(3))$. The geometry $\tilde{\mathcal{X}}$ is flag-transitive and its automorphism group is a non-split extension of $U_4(3) \cdot 2^2$ by an elementary abelian 3-group of rank 23, isomorphic to a $U_4(3)$ -submodule in the Leech lattice taken modulo 3.*

The geometry $\tilde{\mathcal{X}}$ possesses a covering onto the geometry associated with $3^2 U_4(3) 2^2$ which appears as a residue in the minimal 2-local parabolic geometry of $3 \cdot M(24)$ (cf. [I]).

We pose the following further questions about $\mathcal{G}(\text{McL})$:

- (a) is there a flag-transitive proper 2-cover of $\mathcal{G}(\text{McL})$ in which the residue of a point is isomorphic to $\mathcal{G}(\text{Mat}_{22})$?
- (b) is there a flag-transitive proper 2-cover of $\mathcal{G}(\text{McL})$ in which the residue of a plane is isomorphic to $\mathcal{G}(\text{Alt}_7)$?
- (c) what is the universal (non-abelian) representation group of $\mathcal{G}(\text{McL})$ with respect to the planes and triangles?
- (d) what is the universal 2-cover of $\mathcal{G}(\text{McL})$ (equivalently the universal cover of $\tilde{\mathcal{G}}$)?

Let $\tilde{\mathbf{G}} = \{\tilde{G}_i \mid 1 \leq i \leq 4\}$ be the amalgam of maximal parabolics associated with the action of $\tilde{G}$ on $\tilde{\mathcal{G}}$, where $\tilde{G}_i$ is the stabilizer of an element of type i . Then for $i = 1$ and 4 the centre Z_i of $\tilde{G}_i$ is of order 3. Let U be the universal completion of $\tilde{\mathbf{G}}$ and K be the kernel of the homomorphism of U onto McL . By Theorem 1.3 K is generated by the union of the normal closures of U of Z_1 and Z_4 . The question (a) (respectively (b)) above is equivalent to the question whether or not K is generated by the normal closure of Z_1 (respectively of Z_4) in U . The question (d) asks for the structure of K . The quotient K_{ab} of K over its commutator subgroup is an elementary abelian 3-

group whose rank is at most the number of points and planes in $\mathcal{G}(\text{McL})$. This suggests the following restricted version of question (d):

(e) what is the rank of K_{ab} ?

The paper is organized as follows. In Section 2 we recall some basic terms and facts concerning geometries and their flag-transitive automorphism groups. In Section 3 (consisting of a few subsections) we review constructions of some known Petersen-type geometries, emphasising their mutual embeddings. In Section 4 we construct a 2-cover $\tilde{\mathcal{G}}$ of $\mathcal{G}(\text{McL})$. We consider $\mathcal{G}(\text{McL})$ as a subgeometry in $\mathcal{G}(\text{Co}_2)$ and construct $\tilde{\mathcal{G}}$ as the pre-image of $\mathcal{G}(\text{McL})$ in $\mathcal{G}(3^{23}\text{Co}_2)$. In Section 5 we describe in $\tilde{\mathcal{G}}$ a subgeometry $\hat{\mathcal{X}}$ of type $\circ \text{---} \circ \text{---} \circ$. In Sections 6 and 7 we prove Theorem 1.2 by studying the amalgam of maximal parabolics associated with a flag-transitive action on a P_3^3 -geometry. Finally, in Section 8 we prove Theorem 1.1.

2. Geometrical background

As mentioned in the introduction our terminology and notation are fairly standard. In this section we recall some basic definitions and facts.

An *i*-cover of a geometry $\mathcal{G}$ is a pair $(\tilde{\mathcal{G}}, \varphi)$, where $\tilde{\mathcal{G}}$ is a geometry and φ is a morphism of $\tilde{\mathcal{G}}$ onto $\mathcal{G}$ such that the restriction of φ to a residue in $\tilde{\mathcal{G}}$ of rank greater than or equal to *i* is an isomorphism. If φ is clear from the context or irrelevant we simply talk about an *i*-cover $\tilde{\mathcal{G}}$. An $(n-1)$ -cover of a geometry of rank *n* is called simply a *cover*.

An *i*-cover $(\tilde{\mathcal{G}}, \varphi)$ of $\mathcal{G}$ is called *universal*, if for every *i*-cover $(\bar{\mathcal{G}}, \psi)$ of $\mathcal{G}$ there is an *i*-cover $(\tilde{\mathcal{G}}, \theta)$ of $\mathcal{G}$ such that $\theta\psi = \varphi$.

Let *I* be a set of indices. An *amalgam* **G** (of type *I*) is a family $(G_J)_{J \subseteq I}$ of groups indexed by the subsets of *I* and a family of group homomorphisms $\delta_{JK}: G_J \rightarrow G_K$, one for each $J, K \subseteq I$ such that $J \subseteq K$, satisfying the following two conditions:

- (i) for all $J, K, L \subseteq I$ with $J \subseteq K \subseteq L$, the composite $\delta_{JK}\delta_{KL}$ equals δ_{JL} ;
- (ii) $\delta_{JJ} = \text{id}_J$ for each $J \subseteq I$.

A *completion* of **G** is a group *G* and a family of homomorphisms $\eta_J: G_J \rightarrow G$, one for each $J \subseteq I$, such that:

- (i) $\eta_J = \delta_{JK}\eta_K$ for all $J \subseteq K$;
- (ii) $G = \langle (G_J)\eta_J \mid J \subseteq I \rangle$.

Given two completions of **G**, say *G* and $\hat{G}$, with homomorphisms η_J and $\hat{\eta}_J$, then a *morphism of completions* is a homomorphism $\psi: G \rightarrow \hat{G}$ such that $\hat{\eta}_J = \eta_J\psi$ for each $J \subseteq I$. A completion is *faithful*, if for all $J \subseteq I$, the homomorphism η_J is injective. A completion of **G** is called *universal* if it possesses a morphism of completions onto any other completion of **G** (cf. [GLS, p. 162], [Bu] and [P]). To simplify the notation for an amalgam $(G_J)_{J \subseteq I}$ we will write $\{G_i \mid i \in I\}$ and for $J \subseteq I$ consider G_J as the intersection of the G_j for all $j \in J$.

Let $\mathcal{G}$ be a flag-transitive geometry of rank *n* with set of types *I* and *G* be a flag-transitive automorphism group of $\mathcal{G}$. For $\{x_i \mid i \in I\}$ being a maximal flag of $\mathcal{G}$, let G_i be the stabilizer of x_i in *G* and **G** = $\{G_i \mid i \in I\}$ be the amalgam of maximal parabolics associated with the action of *G* on $\mathcal{G}$. Then $\mathcal{G}$ is isomorphic to the coset geometry $\mathcal{G}(G, \mathbf{G})$, whose elements of type *i* are the cosets of G_i in *G*; two cosets $G_i a$ and $G_j b$ are adjacent if and only if their intersection is non-empty.

Set

$$G_J = \bigcap_{i \in J} G_i \quad \text{for } J \subseteq I \quad \text{and} \quad B = G_I.$$

We will omit brackets and commas in G_J , so writing for example G_{12} instead of $G_{\{1,2\}}$. Notice, if $\mathcal{G}$ is a geometry with a flag-transitive automorphism group G and x_i is an element of type i of $\mathcal{G}$, then the residue of x_i in $\mathcal{G}$ is isomorphic to the coset geometry $\mathcal{G}(G_i, \{G_{ij} | j \neq i\})$. Throughout the paper we assume that the Borel subgroup B is finite.

3. Some Petersen-type geometries

In this section we review constructions of some known Petersen-type geometries.

3.1. $\mathcal{G}(\text{Alt}_7)$

Let $\bar{T}(7)$ be the complement of the triangle graph of order 7, which means that $\bar{T}(7)$ is a graph on the set of 2-element subsets of a set of size 7 in which two subsets are adjacent if they are disjoint. The elements of type 1, 2 and 3 in $\mathcal{G}(\text{Alt}_7)$ are the vertices, edges and triangles in $\bar{T}(7)$ with the incidence relation via inclusion. The graph $\bar{T}(7)$ is a locally Petersen graph and this immediately shows that $\mathcal{G}(\text{Alt}_7)$ is of type P^3 . The geometry $\mathcal{G}(\text{Alt}_7)$ possesses two flag-transitive automorphism groups: Sym_7 and Alt_7 . Let $A \cong \text{Alt}_7$ and $\mathbf{A} = \{A_i | 1 \leq i \leq 3\}$ be the amalgam of maximal parabolics associated with the action of A on $\mathcal{G}(\text{Alt}_7)$. Then the structure of the maximal parabolics is as given on the diagram below.

The graph $\bar{T}(7)$ possesses a 3-fold cover $3\bar{T}(7)$ first described in [**Sm2**] (cf. section 13.2 in [**BCN**]). The vertices, edges and triangles in $3\bar{T}(7)$ form a geometry $\mathcal{G}(3\text{Alt}_7)$ which possesses a cover onto $\mathcal{G}(\text{Alt}_7)$. It follows from the classification of locally Petersen graphs in [**Ha**] that $\mathcal{G}(3\text{Alt}_7)$ is the universal cover of $\mathcal{G}(\text{Alt}_7)$. The geometry $\mathcal{G}(3\text{Alt}_7)$ can also be described as follows. Let $\tilde{A}$ be the unique non-split extension of Alt_7 by a centre of order 3. For $1 \leq i \leq 3$ let $\tilde{A}_i$ be the pre-image of A_i in $\tilde{A}$ and $\hat{A}_i = O^3(\tilde{A}_i)$. Then one can see from the above diagram that $\hat{A}_i \cong A_i$ for $1 \leq i \leq 3$ and hence $\mathcal{G}(3\text{Alt}_7) = \mathcal{G}(\tilde{A}; \{\hat{A}_i | 1 \leq i \leq 3\})$.

3.2. $\mathcal{G}(\text{Mat}_{22})$

A *Steiner system* $\mathcal{S} = (\mathcal{P}, \mathcal{B})$ of type $(5, 8, 24)$ is a set $\mathcal{P}$ of 24 *elements* and a collection $\mathcal{B}$ of 8-element subsets of $\mathcal{P}$ called *octads* such that every 5-element subset of $\mathcal{P}$ is contained in a unique octad. The Steiner system $\mathcal{S}$ is unique up to isomorphism. A 4-element subset of $\mathcal{P}$ is called a *tetrad*. Every tetrad S is contained in a unique *sextet*, the latter being a collection $\{S_1 = S, \dots, S_6\}$ of six pairwise disjoint tetrads such that $S_i \cup S_j$ is an octad whenever $1 \leq i < j \leq 6$. A *dodecad* is the symmetric difference of two octads intersecting in two elements.

Consider the set $2^{\mathcal{P}}$ of all subsets of $\mathcal{P}$ as a 24-dimensional $GF(2)$ -vector space with addition being the symmetric difference operation. The octads of $\mathcal{S}$ generate in $2^{\mathcal{P}}$ a 12-dimensional subspace $\mathcal{C}_{12}$ called the *Golay code*. The subspace $\mathcal{C}_{12}$ contains the

empty set, the whole set $\mathcal{P}$, 759 octads, 759 complements of octads and 2576 dodecads.

Let $Y = \{a, b\} \subseteq \mathcal{P}$ and let $\mathcal{B}(Y)$ be the set of 330 octads disjoint from Y . The elements of type 1, 2 and 3 in $\mathcal{G}(\text{Mat}_{22})$ are the octads from $\mathcal{B}(Y)$, pairs of disjoint octads from $\mathcal{B}(Y)$ and the 2-element subsets of $\mathcal{P} \setminus Y$. The incidence between elements of type 1 and 2 is via inclusion. If X is a 2-element subset in $\mathcal{P} \setminus Y$, then $X \cup Y$ is contained in a unique sextet $\{S_1 = X \cup Y, \dots, S_6\}$. Then X (as an element of type 3 in $\mathcal{G}(\text{Mat}_{22})$) is incident to $B \in \mathcal{B}(Y)$ if $B = S_i \cup S_j$ for some $2 \leq i < j \leq 6$. Finally X is incident to an element $\{B_1, B_2\}$ of type 2 if it is incident to B_1 and B_2 (which are elements of type 1). The *derived graph* Σ of $\mathcal{G}(\text{Mat}_{22})$ is a graph on the set of elements of type 1 in which two such elements are adjacent if they are incident to a common element of type 2. The elements of type 1, 2 and 3 in $\mathcal{G}(\text{Mat}_{22})$ can be identified with the vertices, edges and Petersen subgraphs in Σ so that the incidence relation is via inclusion. The geometry $\mathcal{G}(\text{Mat}_{22})$ possesses two flag-transitive automorphism groups: Mat_{22} and Aut Mat_{22} . The amalgam $\mathbf{M} = \{M_i \mid 1 \leq i \leq 3\}$ of maximal parabolics associated with the action of $M \cong \text{Mat}_{22}$ is as given on the diagram below.

The graph Σ possesses a triple cover 3Σ constructed in [FII]. The vertices, edges and Petersen subgraphs in 3Σ form a P_3 -geometry which is a 3-fold cover $\mathcal{G}(3\text{Mat}_{22})$ of $\mathcal{G}(\text{Mat}_{22})$. It has been proved in [Sh1] that $\mathcal{G}(3\text{Mat}_{22})$ is the universal cover of $\mathcal{G}(\text{Mat}_{22})$. The geometry $\mathcal{G}(3\text{Mat}_{22})$ can be also constructed in the following way. Let $\tilde{M} \cong 3\text{Mat}_{22}$ be the unique non-split extension of $M \cong \text{Mat}_{22}$ by a centre of order 3. For $1 \leq i \leq 3$ let $\tilde{M}_i$ be the pre-image of M_i in $\tilde{M}$ and $\hat{M}_i = O^3(\tilde{M}_i)$. Then $\hat{M}_i \cong M_i$ for $1 \leq i \leq 3$ and $\mathcal{G}(3\text{Mat}_{22}) = \mathcal{G}(\tilde{M}; \{\hat{M}_i \mid 1 \leq i \leq 3\})$.

3.3. Some embeddings

In this subsection we show that $\mathcal{G}(\text{Alt}_7)$ and $\mathcal{G}(3\text{Alt}_7)$ are (in a certain sense) subgeometries in $\mathcal{G}(\text{Mat}_{22})$ and $\mathcal{G}(3\text{Mat}_{22})$, respectively.

Let Δ be a graph on the set of elements of type 3 in $\mathcal{G}(\text{Mat}_{22})$ in which two elements are adjacent if they are incident to a common element of type 2. In other terms the vertices of Δ are the 2-element subsets of $\mathcal{P} \setminus \{a, b\}$ and two such subsets $\{c, d\}$ and $\{e, f\}$ are adjacent if there is an octad B which contains $\{a, b, c, d, e, f\}$. In the above terms if $\{g, h\} = B \setminus \{a, b, c, d, e, f\}$ then $T = \{\{c, d\}, \{e, f\}, \{g, h\}\}$ is a triangle in Δ called a *singular triangle*. Every edge is in a unique singular triangle. A maximal clique is of size 7 and it is $\star$ -closed in the sense that together with every edge it contains the singular triangle containing this edge. Then the elements of type 1, 2 and 3 in $\mathcal{G}(\text{Mat}_{22})$ are the maximal cliques, singular triangles and vertices of Δ with the incidence relation via inclusion.

If x is a vertex and Π is a subgraph in Δ , then x is said to be *adjacent to* Π if x is adjacent to every vertex in Π .

Let D be an octad such that $D \cap \{a, b\} = \{b\}$, A be the stabilizer of D in $M \cong \text{Mat}_{22}$ and for $0 \leq i \leq 2$ let $\Delta_i = \{\{c, d\} \subseteq \mathcal{P} \setminus \{a, b\} \mid |\{c, d\} \cap D| = i\}$. The following result is well known and easy to check.

LEMMA 3.1. $A \cong \text{Alt}_7$; Δ_2 , Δ_1 and Δ_0 are the orbits of A on Δ with lengths 21, 105 and 105, respectively.

LEMMA 3.2. Let $\mathcal{T}$ be the sets of singular triangles in Δ which intersect Δ_2 and let $\mathcal{R}$ be the sets of maximal cliques which intersect Δ_2 . Then

- (i) if $T \in \mathcal{T}$ then $|T \cap \Delta_i| = 1$ for $0 \leq i \leq 2$;
- (ii) if $R \in \mathcal{R}$ then $|R \cap \Delta_2| = 1$, $|R \cap \Delta_0| = |R \cap \Delta_1| = 3$ and $R \cap \Delta_1$ is a singular triangle.

Proof. We assume that $T = \{\{c, d\}, \{e, f\}, \{g, h\}\}$ (so that $B = \{a, b, c, d, e, f, g, h\}$ is an octad). If $T \in \mathcal{T}$ then we can assume that $\{c, d\} \in \Delta_2$. Since $D \cap \{a, b\} = \{b\}$ and any two octads intersect in 0, 2 or 4 elements, we may assume that $D \cap B = \{b, c, d, e\}$. In this case $\{e, f\} \in \Delta_1$, $\{g, h\} \in \Delta_0$.

Suppose that $R \in \mathcal{R}$ and let $x \in R \cap \Delta_2$. Since R is a $\star$ -closed clique of size 7, it is a union of three singular triangles (from $\mathcal{T}$) containing x . Hence by (i) we have $|R \cap \Delta_0| = |R \cap \Delta_1| = 3$. It is clear that the vertices and singular triangles contained in R form a projective plane of order 2. On the other hand $A(x) \cong \text{Sym}_5$ permutes transitively the 10 maximal cliques containing x and hence the stabilizer of R in $A(x)$ induces on R an action whose order is divisible by 3. Furthermore $R \cap \Delta_0$ and $R \cap \Delta_1$ are the orbits of this action and hence one of them is a singular triangle. If $R \cap \Delta_0$ would be a singular triangle, then the corresponding octad (containing $\{a, b\}$) would intersect D in the element b only which is impossible and hence $R \cap \Delta_1$ is a singular triangle.

Let $\mathcal{X}$ be a rank 3 geometry whose elements of type 1 are the vertices in Δ_2 ; the elements of type 2 are the singular triangles contained in Δ_1 ; the elements of type 3 are the vertices in Δ_1 ; the incidence between the elements of type 2 and 3 is via inclusion and the incidence between the elements of type 1 and the other elements is via adjacency in Δ .

LEMMA 3.3. The geometry $\mathcal{X}$ is isomorphic to $\mathcal{G}(\text{Alt}_7)$ and $A \geq \text{Alt}_7$ induces on it a flag-transitive action.

Proof. A vertex $x \in \Delta_2$ can be considered as an element of type 1 in $\mathcal{X}$ and as an element of type 3 in $\mathcal{G}(\text{Mat}_{22})$. By Lemma 3.2 there is a natural bijection between the singular triangles containing x and the vertices in Δ_1 adjacent to x and also between the maximal cliques containing x and the singular triangles contained in Δ_1 and adjacent to x . Thus both $\text{res}_{\mathcal{X}}(x)$ and $\text{res}_{\mathcal{G}(\text{Mat}_{22})}(x)$ are isomorphic to the geometry of vertices and edges in the Petersen graph.

For any triangle T in Δ_1 the elements of type 1 and of type 3 in $\text{res}_x(T)$ form a generalized 2-gon.

Next let $y \in \Delta_1$, so that $y = \{c, d\}$ where $c \in D$, $d \notin D$ and $\{a, b\} \cap \{c, d\} = \emptyset$. Let $\{S_1 = \{a, b, c, d\}, S_2, \dots, S_6\}$ be a sextet. Since any two octads have 0, 2 or 4 common elements, we may assume that for $2 \leq i \leq 4$ the set $S_i \cap D$ is of size 2 which means that $S_i \cap D$ is an element from Δ_2 . Hence the elements from Δ_2 incident with y are precisely $S_i \cap D$, $2 \leq i \leq 4$. Let O be the octad $S_1 \cup S_2$ and let $S_2 \cap D = \{s_1, s_2\}$. Then the octad O_i which is generated by $\{a, b, s_i\} \cup (S_i \cap D)$ intersect S_2 in two points, say s_i and t_i , where $t_i \notin D$, for $i = 1, 2$. Hence $T = \{\{c, d\}, \{s_1, t_1\}, \{s_2, t_2\}\}$ is the unique triangle in Δ_1 which is incident to $S_3 \cap D$ and $S_4 \cap D$. This shows that in $\text{res}_x(y)$ any

two vertices in Δ_2 are incident to exactly one triangle in Δ_1 . Since an incident pair (x, y) with $y \in \Delta_1$ and $x \in \Delta_2$ corresponds to a flag of type $\{2, 3\}$ in $\mathcal{G}(\text{Mat}_{22})$, any such pair is contained in exactly two maximal cliques of Δ . By Lemma 3.2(ii), $\text{res}_x(y)$ is isomorphic to the geometry of points and edges of a triangle. This proves the lemma.

Let 3Δ be a graph on the set of elements of type 3 in $\mathcal{G}(\text{Mat}_{22})$ in which two such elements are adjacent if they are incident to a common element of type 2. The covering $\mathcal{G}(\text{Mat}_{22}) \rightarrow \mathcal{G}(\text{Mat}_{22})$ induces a morphism $\rho: 3\Delta \rightarrow \Delta$ which is a covering of graphs in the sense that for every vertex $x \in 3\Delta$ the restriction of ρ to the subgraph in 3Δ induced by the neighbours of x is an isomorphism onto the subgraph in Δ induced by the neighbours of $\rho(x)$. A triangle in 3Δ is called singular if ρ maps it onto a singular triangle in Δ .

Let $\tilde{A}$ be the pre-image of A (as in Lemma 3.1) in $\tilde{M} \cong 3\text{Mat}_{22}$. Then $\tilde{A}$ does not split over $Z(\tilde{M})$ (since A contains a Sylow 3-subgroup of M). It is an immediate consequence of Lemma 3.1 that $\tilde{A}$ has three orbits, $\tilde{\Delta}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_0$ on 3Δ with lengths 63, 315 and 315 such that $\rho(\tilde{\Delta}_i) = \Delta_i$ for $0 \leq i \leq 2$.

Let $\mathcal{Y}$ be a rank 3 geometry whose elements of type 1 are the vertices in $\tilde{\Delta}_2$; the elements of type 2 are the singular triangles contained in $\tilde{\Delta}_1$; the elements of type 3 are the vertices in $\tilde{\Delta}_0$; the incidence relation is similar to that in $\mathcal{X}$. Because of the fact that $\tilde{A}$ is the unique non-split extension of Alt_7 by a centre of order 3, we have the following.

LEMMA 3.4. *The geometry $\mathcal{Y}$ is isomorphic to $\mathcal{G}(\text{Alt}_7)$ and $\tilde{A} \cong 3\text{Alt}_7$ induces on it a flag-transitive action.*

3.4. $\mathcal{G}(\text{Co}_2)$

We follow the combinatorial description of the P_4 -geometry $\mathcal{G}(\text{Co}_2)$ given in [Sh3]. Consider the set $\mathcal{P}$ of elements of the Steiner system $\mathcal{S}$ from the previous subsection as a basis of a 24-dimensional vector space $\mathbb{R}^{24}$ over real numbers and introduce a symmetric bilinear form $(\ , \)$ on $\mathbb{R}^{24}$ by setting $(a, b) = \frac{1}{8}\delta_{a,b}$ for $a, b \in \mathcal{P}$. Let Λ be the set of vectors $\lambda \in \mathbb{R}^{24}$ whose coordinates $\{\lambda(a) \mid a \in \mathcal{P}\}$ in the basis $\mathcal{P}$ are integral and for a fixed $m = 0$ or 1 the following three conditions hold:

- (A1) $\lambda(a) \equiv m \pmod{2} \quad \text{for } a \in \mathcal{P};$
- (A2) $\{a \mid \lambda(a) \equiv m \pmod{4}\} \in \mathcal{C}_{12};$
- (A3) $\sum_{i=1}^{24} \lambda(a) \equiv 4m \pmod{8}.$

Then Λ is the *Leech lattice* which is the unique (up to isomorphism) even unimodular lattice of dimension 24 without roots. If p is a prime then the form $(\ , \)$ taken modulo p is a non-degenerate symmetric $GF(p)$ -form $(\ , \)_p$ on $\Lambda/p\Lambda$. In what follows isotropic, non-isotropic vectors, orthogonal complements etc. in $\Lambda/p\Lambda$ shall mean those with respect to $(\ , \)_p$.

The group C of automorphisms of Λ preserving the origin is the Conway group Co_0 and the quotient of C over its centre of order 2 is the largest sporadic simple group of Conway denoted by Co_1 . Let $\Lambda_i = \{\lambda \in \Lambda \mid (\lambda, \lambda) = 2i\}$, let $\bar{\Lambda} = \Lambda/2\Lambda$ be the Leech lattice taken modulo 2 and let $\bar{\Lambda}_i$ be the image of Λ_i in $\bar{\Lambda}$. Then $\bar{\Lambda}$ is a disjoint union of the $\bar{\Lambda}_i$ taken for $0 \leq i \leq 4$ (notice that Λ_1 is empty since there are no roots in the Leech lattice).

Let Λ_4^8 be the set of Leech vectors λ such that $N(\lambda) = (80^{23})$. It is of size 48 and the setwise stabilizer B of Λ_4^8 in C can be represented in the basis $\mathcal{P}$ by monomial matrices. Namely $B \cong 2^{12}\text{Mat}_{24}$ is a semidirect product of an elementary abelian subgroup Q of order 2^{12} consisting of the transformations:

$$\tau(E) : a \mapsto \epsilon(a, E) \cdot a, \quad \text{for } a \in \mathcal{P}, \quad E \in \mathcal{C}_1,$$

Let μ_0 be a vector in Λ_2 such that $N(\mu_0) = (4^{20}2^{22})$, $P_4(\mu_0) = \{a, b\}$, $X(\mu_0) = \emptyset$. Let

$$\Theta = \{\lambda \in \Lambda, \mid (\lambda, \mu_0) = 0\}$$

Let $\bar{\Theta}$ also denote a graph on $\bar{\Theta}$ invariant under the action of H in which $\bar{v}_0$ is adjacent to the images of the vectors $\mu \in \Theta$ such that $N(\mu) = (4^{20}2^2)$ and $P_4(\mu) \cap P_4(v_0) = \emptyset$. Then $\bar{\Theta}$ is of valency 2×231 and the action of H on $\bar{\Theta}$ is edge-transitive.

There is a mapping σ from the set $\bar{\Theta}(\bar{\nu}_0)$ of vertices adjacent to $\bar{\nu}_0$ in $\bar{\Theta}$ onto the set of elements of type 3 in $\mathcal{G}(\text{Mat}_{22})$, commuting with the action of $H(\bar{\nu}_0)$ such that whenever $\bar{\lambda}$ and $\bar{\mu}$ are distinct vertices in $\bar{\Theta}(\bar{\nu}_0)$ then (a) $\sigma(\bar{\lambda}) = \sigma(\bar{\mu})$ if and only if $\{\bar{\nu}_0, \bar{\lambda}, \bar{\mu}\}$ is a singular triangle and (b) $\bar{\lambda}$ and $\bar{\mu}$ are adjacent in $\bar{\Theta}$ if and only if $\sigma(\bar{\lambda})$ and $\sigma(\bar{\mu})$ are either equal or adjacent in the graph Δ (as in Subsection 3.3).

The elements of type 4, 3, 2 and 1 in $\mathcal{G}(\text{Co}_2)$ are the vertices, singular triangles, $\star$ -closed cliques of size 7 and $\star$ -closed cliques of size 15 in $\bar{\Theta}$ respectively; the incidence relation is via inclusion. In fact every maximal clique in $\bar{\Theta}$ is an element of type 4 in $\mathcal{G}(\text{Co}_2)$ (cf. proposition 4.2 in [Sh3]).

If $\mathbf{H} = \{H_i \mid 1 \leq i \leq 4\}$ is the amalgam of maximal parabolics associated with the action of $H \cong \text{Co}_2$ on $\mathcal{G}(\text{Co}_2)$ then the structures of the H_i are as presented on the diagram below.

Notice that $\bar{\Theta}$ is a graph on the set of elements of type 4 in $\mathcal{G}(\text{Co}_2)$ in which two such elements are adjacent if they are incident to a common element of type 3.

Define a mapping ψ from $\mathcal{G}(\text{Co}_2)$ into the set of subspaces in $\bar{\Lambda}$ by the following rule. If x is of type 4, then $\psi(x) = \langle x + \bar{\mu}_0 \rangle$; otherwise $\psi(x)$ is the span of the images of the elements of type 4 incident to x . Then $\dim \psi(x) = 5 - i$ for x of type i . Hence ψ defines a natural representation of $\mathcal{G}(\text{Co}_2)$ with respect to the elements of type 4 and 3. The image of ψ generates the orthogonal complement of $\bar{\mu}_0$ in $\bar{\Lambda}$ of dimension 23 and ψ was proved in [IS3] to be the universal natural representation of $\mathcal{G}(\text{Co}_2)$.

3.5. $\mathcal{G}(\text{McL})$

We start with following standard result.

LEMMA 3.5. *The set $\Phi = \{\omega \mid \omega \in \Lambda_2, (\mu_0, \omega) = 1\}$ is of size 47 104 and $H = C(\mu_0) \cong \text{Co}_2$ acts on Φ transitively with stabilizer isomorphic to the sporadic simple McLaughlin group McL .*

Put $\Phi_i = \{\omega \mid \omega \in \Phi, (\nu_0, \omega) = \pm i\}$.

LEMMA 3.6. *The sets Φ_0, Φ_1, Φ_2 are the orbits of $H_4 = H(\bar{\nu}_0) \cong 2^{10} \text{Aut Mat}_{22}$ on Φ with lengths 22 528, 22 528 and 2048 and with stabilizers isomorphic to $P\Omega L_3(4)$, 2^4Alt_7 and Mat_{22} , respectively.*

Proof. Since $\mu_0 \pm \nu_0 \in \Lambda_4^8$, the subgroup $H(\bar{\nu}_0)$ is contained in the stabilizer B of Λ_4^8 in C . Let ω_0, ω_1 and ω_2 be the following Leech vectors:

$$\begin{aligned} N(\omega_0) &= (31^{23}), \quad P_3(\omega_0) = X(\omega_0) \subseteq \mathcal{P} \setminus \{a, b\}; \\ N(\omega_1) &= (2^8 0^{16}), \quad P_2(\omega_1) \cap \{a, b\} = \{a\}, \quad X(\omega_1) = \emptyset; \\ N(\omega_2) &= (31^{23}), \quad P_3(\omega_2) = \{a\}, \quad X(\omega_2) = P_1(\omega_2). \end{aligned}$$

Then it is easy to see that $\omega_i \in \Phi_i$ for $0 \leq i \leq 2$ and since $H(\bar{\nu}_0) \leq B$, it is easy to check that the structures of the stabilizers $H(\bar{\nu}_0) \cap H(\omega_i)$ are as stated in the lemma. By Lemma 3.5 the result follows by the order reason.

By Lemma 3.5 we also have the following dual version of Lemma 3.6.

LEMMA 3.7. *If $\omega \in \Phi$ then $H(\omega) \cong \text{McL}$ has three orbits $\bar{\Theta}_0, \bar{\Theta}_1$ and $\bar{\Theta}_2$ on $\bar{\Theta}$ with lengths 22 275, 22 275 and 2025 and with stabilizers isomorphic to $P\Omega L_3(4)$, 2^4Alt_7 and Mat_{22} , respectively. If $\lambda \in \bar{\Theta}$ then $\bar{\lambda} \in \bar{\Theta}_i$ if and only if $(\omega, \lambda) = \pm i$.*

Let us discuss the graph structure $\bar{\Theta}$ with respect to the orbits $\bar{\Theta}_i$.

LEMMA 3.8. *If R is a $\star$ -closed clique of size $2^m - 1$ ($m = 2, 3$ or 4) in $\bar{\Theta}$ which intersects $\bar{\Theta}_2$ then $|R \cap \bar{\Theta}_0| = |R \cap \bar{\Theta}_1| = 2^{m-1} - 1$.*

Proof. Since in $\bar{\Theta}$ any edge is in exactly one singular triangle, it suffices to show the assertion for $m = 2$. Let T be a singular triangle and let $\omega \in \Phi$ with $N(\omega) = (31^{23})$, $P_3(\omega) = \{a\}$ and $X(\omega) = \{b\}$. Then $(\omega, \mu_0) = 2$. Thus $\mu_0 \in \bar{\Theta}$ and we may assume $\mu_0 \in T$. The result follows then by straightforward calculations.

Define $\mathcal{G}(\text{McL})$ to be a geometry of rank 4 whose elements of type 1 are the vertices in $\bar{\Theta}_2$, for $m = 2$ and 3 the elements of type m are the $\star$ -closed cliques of size $2^{5-m} - 1$ contained in $\bar{\Theta}_1$, the elements of type 4 are the vertices in $\bar{\Theta}_1$. The incidence between elements of type 2, 3 and 4 is via inclusion and the incidence between the elements

of type 1 with all other elements is via adjacency. Then by Lemma 3·8 the residue in $\mathcal{G}(\text{McL})$ of an element of type 1 is isomorphic to the residue in $\mathcal{G}(\text{Co}_2)$ of an element of type 4. Let $\omega \in \Phi$ and $G = H(\omega)$. The stabilizer in G (resp. in H) of a vertex $y \in \bar{\Theta}_2$ is isomorphic to 2^4Alt_7 (resp. $2^{10} \text{Aut Mat}_{22}$). Therefore $G(y)$ induces on $\text{res}_{\mathcal{G}(\text{McL})}(y)$ an action isomorphic to Alt_7 . By Lemmas 3·1 and 3·7, G_4 has three orbits Δ_0, Δ_1 and Δ_2 , on the set of singular triangles in $\bar{\Theta}$ which contain y ; they consist of those singular triangles $\{y, x_1, x_2\}$ with $x_1, x_2 \in \bar{\Theta}_0$; $x_1, x_2 \in \bar{\Theta}_1$; and $x_1 \in \bar{\Theta}_0$ and $x_2 \in \bar{\Theta}_2$, respectively, and are of size 105, 105 and 21, respectively. Since each edge in Θ is contained in exactly one singular triangle, we may identify the singular triangle $\{y, x_1, x_2\}$ in Δ_2 with the edges (y, x_2) , where $x_2 \in \bar{\Theta}_2$. Hence according to Lemma 3·3 the residue of an element of type 4 in $\mathcal{G}(\text{McL})$ is isomorphic to $\mathcal{G}(\text{Alt}_7)$. It follows basically by the definition that $\mathcal{G}(\text{McL})$ belongs to a string diagram, which shows that $\mathcal{G}(\text{McL})$ is a P_3^3 -geometry. By Lemma 3·7 the group $G = \text{McL}$ acts flag-transitively on $\mathcal{G}(\text{McL})$ with the amalgam $\mathbf{G} = \{G_i \mid 1 \leq i \leq 4\}$ of maximal parabolics as given in the diagram below.

Let us discuss the collinearity graph of $\mathcal{G}(\text{McL})$. Let $\omega = \omega_2$ in the above terms and $\Omega = \{\nu \mid \nu \in \Lambda_2, (\nu, \mu_0) = 0, (\nu, \omega) = 2\}$. Then it is clear that Ω maps bijectively onto $\bar{\Theta}_2$. Furthermore, $\nu_0 \in \Omega$ and $G_1 := G(\nu_0)$ is isomorphic to Mat_{22} . Using the characteristic properties (A1)–(A3) of Leech vectors, one can check (see also [I]) that G_1 has the following three orbits on $\Omega \setminus \{\nu_0\}$:

$$\begin{aligned} \Omega_1 &= \{\nu \mid N(\nu) = (2^8 0^{16}), \quad X(\nu) = P_2(\nu_2), \quad P_2(\nu) \cap \{a, b\} = \emptyset\}; \\ \Omega_2 &= \{\nu \mid N(\nu) = (2^8 0^{16}), \quad \{a, b\} \subseteq P_2(\nu), \quad |X(\nu)| = 6, \quad X(\nu) \cap \{a, b\} = \{b\}\}; \\ \Omega_3 &= \{\nu \mid N(\nu) = (31^{23}), \quad \nu(a) = 1, \quad \nu(b) = -1, \quad P_3(\nu) \subseteq X(\nu), \quad |X(\nu)| = 17\}. \end{aligned}$$

Let Ω also denote the graph on this set invariant under the action of G with Ω_1 the set of neighbours of ν_0 . In other terms two vectors from Ω are adjacent if they are perpendicular. Then the suborbit diagram of Ω with respect to the action of Ω is the following.

The graph Ω is the collinearity graph of $\mathcal{G}(\text{McL})$. Furthermore the elements of type 1, 2, 3 and 4 in $\mathcal{G}(\text{McL})$ can be identified with the vertices, edges, triangles and subgraphs isomorphic to $\bar{T}(7)$ with the incidence relation via inclusion. One can see directly from the suborbit diagram that Ω is locally the derived graph Σ of $\mathcal{G}(\text{Mat}_{22})$.

By definition the set X of elements of type 4 and 3 in $\mathcal{G}(\text{McL})$ is a subset of the set of elements of type 4 and 3 in $\mathcal{G}(\text{Co}_2)$ which means that the restriction to X of the mapping ψ defined at the end of Subsection 3.4 is a natural representation of $\mathcal{G}(\text{McL})$ with respect to elements of type 4 and 3. If $l \in \Theta$ and $\bar{\lambda} \in \bar{\Theta}_1$ then $(\omega, \lambda) = 1$. Since $(\omega, \mu_0) = 1$ as well, we conclude that

$$\psi(X) = \{\bar{\lambda} + \mu_0 \mid \bar{\lambda} \in \bar{\Theta}_1\}$$

is contained in the orthogonal complement V of $\langle \bar{\mu}_0, \bar{\omega} \rangle$ in $\bar{\Lambda}$. This complement V is 22-dimensional and it is well known (cf. [W]) that $G \cong \text{McL}$ acts on V irreducibly which implies that V is generated by the image of ψ .

The *universal (abelian) natural representation* of $\mathcal{G}(\text{McL})$ is supported by a vector space U defined as follows (here $\mathcal{G}^4$ is the set of elements of type 4 in $\mathcal{G}$).

$$U = \langle v_\xi, \xi \in \mathcal{G}^4 \mid v_{\xi_1} + v_{\xi_2} + v_{\xi_3} = 0 \text{ whenever } \xi_1, \xi_2, \xi_3 \in \mathcal{G}^4 \text{ are incident to a common element of type 3} \rangle.$$

Thus v is a quotient of U . Using the computer package **me** by Linton [L] we have established the following result which implies the equality $V = U$ and hence Theorem 1.7 as well.

LEMMA 3.9. $\dim U = 22$.

3.6. $\mathcal{G}(3^{23}\text{Co}_2)$

A flag-transitive 2-cover $\mathcal{G}(3^{23}\text{Co}_2)$ of $\mathcal{G}(\text{Co}_2)$ has been constructed in [Sh3]. The group $H \cong \text{Co}_2$ contains a subgroup $U \cong U_6(2)2$ of index 2300. There is a non-split extension $\tilde{U} \cong 3U_6(2).2$ such that every element from $\tilde{U} \setminus O^2(\tilde{U})$ inverts $O_3(\tilde{U})$, which means that $O_3(\tilde{U})$ is a non-trivial 1-dimensional $GF(3)$ -module for $\tilde{U}/O_3(\tilde{U}) \cong U$. By the Eckman-Shapiro lemma (cf. Shapiro's lemma in [Br]) there exists a non-split extension $\tilde{H} \cong 3^{2300}\text{Co}_2$ where $O_3(\tilde{H})$ (as a module for $\tilde{H}/O_3(\tilde{H}) \cong \text{Co}_2$) is induced from the 1-dimensional non-trivial $GF(3)$ -module for U . Let φ denote the natural homomorphism of $\tilde{H}$ onto H .

Let $\mathbf{H} = \{H_i \mid 1 \leq i \leq 4\}$ be the amalgam of maximal parabolics associated with the action of H on $\mathcal{G}(\text{Co}_2)$. For $J \subset I := \{1, 2, 3, 4\}$ let H_J be the intersection of the H_j taken for all $j \in J$ and let $Q_J = O_2(H_J)$. Then $D := H_I$ is the Borel subgroup which is a subgroup of index 2 in a Sylow 2-subgroup of H and contains all the Q_J . Let $\tilde{D}$ be a Sylow 2-subgroup of the pre-image $\varphi^{-1}(D)$ of D in $\tilde{H}$. Then the restriction φ_D of φ to $\tilde{D}$ is an isomorphism onto D . For $J \subseteq I$ put $\tilde{Q}_J = \varphi_D^{-1}(Q_J)$ which is the isomorphic preimage of Q_J in $\tilde{D}$. The crucial step in the construction was to show that if $|J| = 2$ then $C_{O_3(\tilde{H})}(\tilde{Q}_J) = 1$, which implies that $\tilde{H}_J := N_{\tilde{H}}(\tilde{Q}_J)$ is isomorphic to $H_J = N_H(Q_J)$. Next it was shown that the rank 2 parabolics $\tilde{H}_J$ generate in $\tilde{H}$ a subgroup $\hat{H}$ of the form 3^{23}Co_2 (a non-split extension). This was achieved by bounding the order of the automorphism group of a 2-cover of $\mathcal{G}(\text{Co}_2)$. The 2-cover $\mathcal{G}(3^{23}\text{Co}_2)$ has been defined as the coset geometry $\mathcal{G}(\hat{H}, \hat{\mathbf{H}})$. The outcome is the following.

PROPOSITION 3.10. [Sh3]. *There exists a P_4 -geometry $\mathcal{G}(3^{23}\text{Co}_2)$ with a flag-transitive automorphism group $\hat{H}$ of the form 3^{23}Co_2 (a non-split extension) such that*

there is a 2-cover $\mathcal{G}(3^{23}\text{Co}_2) \rightarrow \mathcal{G}(\text{Co}_2)$ and a group homomorphism $\kappa: \hat{H} \rightarrow H$. If $\hat{\mathbf{H}} = \{\hat{H}_i \mid 1 \leq i \leq 4\}$ is the amalgam of maximal parabolics associated with the action of $\hat{H}$ on $\mathcal{G}(3^{23}\text{Co}_2)$ then κ maps $\hat{\mathbf{H}}$ onto $\mathbf{H}$; the restrictions of κ to the $\hat{H}_i$ for $1 \leq i \leq 3$ and to the $\hat{H}_i \cap \hat{H}_j$ for $1 \leq i < j \leq 4$ are isomorphisms onto the H_i and the $H_i \cap H_j$, respectively; $\hat{H}_4 \cong 3 \cdot 2^{10} \text{Aut Mat}_{22}$ and the restriction of κ to $\hat{H}_4$ has kernel of order 3. The residue in $\mathcal{G}(3^{23}\text{Co}_2)$ of an element of type 4 is isomorphic to $\mathcal{G}(3\text{Mat}_{22})$. As a $GF(3)$ -module for $\text{Co}_2 \cong H \cong \hat{H}/O_3(\hat{H})$ the subgroup $O_3(\hat{H})$ is the orthogonal complement in the Leech lattice modulo 3 of a non-zero vector stabilized by H .

Let $\tilde{\Theta}$ be a graph on the set of elements of type 4 in $\mathcal{G}(3^{23}\text{Co}_2)$ in which two such elements are adjacent if they are incident to a common element of type 3. Then by Proposition 3.10 the 2-cover $\mathcal{G}(3^{23}\text{Co}_2) \rightarrow \mathcal{G}(\text{Co}_2)$ induces a morphism $\delta: \tilde{\Theta} \rightarrow \bar{\Theta}$ of graphs. In this case δ is not a covering of graphs, since the valences of $\tilde{\Theta}$ and $\bar{\Theta}$ are different, but for any adjacent vertices x and y in $\bar{\Theta}$ the restriction of δ to the subgraph induced by the common neighbours of x and y is an isomorphism onto the subgraph in $\bar{\Theta}$ induced by the common neighbours of $\delta(x)$ and $\delta(y)$. This enables one to introduce in $\tilde{\Theta}$ the notion of a singular triangle and of a $\star$ -closed clique. Then the elements of $\mathcal{G}(3^{23}\text{Co}_2)$ are vertices, singular triangles and $\star$ -closed cliques in $\tilde{\Theta}$. Notice that a fibre of δ is of size 3^{23} and $O_3(\hat{H})$ induces on such a fibre a regular action.

Let $\tilde{\nu}_0$ be a preimage in $\tilde{\Theta}$ of $\bar{\nu}_0$. Then there is a mapping $\tilde{\sigma}$ from the set $\tilde{\Theta}(\tilde{\nu}_0)$ of vertices adjacent to $\tilde{\nu}_0$ onto the set of elements of type 3 in $\text{res}_{\mathcal{G}(\text{Co}_2)}(\nu_0) \cong \mathcal{G}(3\text{Mat}_{22})$, commuting with the action of $H(\bar{\nu}_0)$ such that whenever $\tilde{\lambda}$ and $\tilde{\mu}$ are distinct vertices in $\tilde{\Theta}(\tilde{\nu}_0)$ then (a) $\tilde{\sigma}(\tilde{\lambda}) = \tilde{\sigma}(\tilde{\mu})$ if and only if $\{\tilde{\nu}_0, \tilde{\lambda}, \tilde{\mu}\}$ is a singular triangle and (b) $\tilde{\lambda}$ and $\tilde{\mu}$ are adjacent in $\tilde{\Theta}$ if and only if $\tilde{\sigma}(\tilde{\lambda})$ and $\tilde{\sigma}(\tilde{\mu})$ are either equal or adjacent in the graph 3Δ (as in Subsection 3.3).

4. A 2-cover of McL

In this section we construct a proper 2-cover of $\mathcal{G}(\text{McL})$ as a subgeometry (in a broad sense) in $\mathcal{G}(3^{23}\text{Co}_2)$. We follow notation as in Subsections 3.5 and 3.6. Let $\tilde{G}$ be the pre-image (with respect to the natural homomorphism) in $\hat{H} \cong 3^{23}\text{Co}_2$ of the subgroup $G \cong \text{McL}$ in H . Then $\tilde{G} \cong 3^{23}\text{McL}$ and since McL contains a Sylow 3-subgroup of H (cf. Lemma 3.5) $\tilde{G}$ does not split over $O_3(\tilde{G})$. For $0 \leq i \leq 2$ let $\tilde{\Theta}_i$ be the pre-image of $\bar{\Theta}_i$ with respect to δ . Then by Lemma 3.7 and Proposition 3.10 we have the following.

LEMMA 4.1. *The sets $\tilde{\Theta}_0$, $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ are the orbits of $\tilde{G}$ on $\tilde{\Theta}$ with stabilizers isomorphic to $3P\Omega L_3(4)$, 32^4Alt_7 and 3Mat_{22} .*

The following result is analogous to Lemma 3.8.

LEMMA 4.2. *If R is a $\star$ -closed clique of size $2^m - 1$ ($m = 2, 3$ or 4) in $\tilde{\Theta}$ which intersects $\tilde{\Theta}_2$ then $|R \cap \tilde{\Theta}_0| = |R \cap \tilde{\Theta}_1| = 2^{m-1} - 1$.*

Define $\tilde{\mathcal{G}}$ to be a geometry of rank 4 whose elements of type 1 are the vertices in $\tilde{\Theta}_2$, for $m = 2$ and 3 the elements of type m the $\star$ -closed cliques of size $2^{5-m} - 1$ contained in $\tilde{\Theta}_1$, the elements of type 4 are the vertices in $\tilde{\Theta}_1$; the incidence between elements of type 2, 3 and 4 is via inclusion and the incidence between the elements of type 1 with all other elements is via adjacency. Then by Lemma 4.2 the residue

in $\tilde{\mathcal{G}}$ of an element of type 1 is isomorphic to the residue in $\mathcal{G}(3^{23}\text{Co}_2)$ of an element of type 4 and by Lemma 3.4 the residue of an element of type 4 in $\tilde{\mathcal{G}}$ is isomorphic to $\mathcal{G}(3\text{Alt}_7)$. It is also clear that $\tilde{\mathcal{G}}$ belongs to a string diagram, which shows that $\tilde{\mathcal{G}}$ is a P_3^3 -geometry. By Lemma 4.1 the group $\tilde{G}$ acts flag-transitively on $\tilde{\mathcal{G}}$. To complete the construction it is sufficient to prove the following.

LEMMA 4.3. *The geometry $\tilde{\mathcal{G}}$ constructed above is connected.*

Proof. Since $\mathcal{G}(\text{McL})$ is connected and $\tilde{G}$ does not split over $O_3(\tilde{G})$, to prove the lemma it is sufficient to show that for $\tilde{\lambda} \in \tilde{\Theta}_2$ the subgroup $R := O_3(\tilde{G}(\tilde{\lambda}))$ (of order 3) is not contained in a proper $G \cong \tilde{G}/O_3(\tilde{G})$ -submodule of $P := O_3(\tilde{G})$. Recall that P is a submodule in the Leech lattice $\hat{\Lambda} = \Lambda/3\Lambda$ modulo 3. We claim that P possesses a unique composition series

$$1 < P_1 < P_2 < P,$$

where P_1 is the span of the image of $\omega + \mu_0$ in $\hat{\Lambda}$ and P_2 is the orthogonal complement of P_1 . The claim follows from the following facts: (1) the symmetric bilinear form $(\cdot, \cdot)_3$ on $\hat{\Lambda}$ is nonsingular; (2) up to a scalar multiplication $\omega + \mu_0$ is the only vector in P stabilized by G (basically since G is maximal in H); (3) $21 = \dim P_2/P_1$ is the smallest dimension of a faithful $GF(3)$ -representation of G (cf. [ModAt]). Now the result is immediate from the fact that R is not normalized by G .

Using a program of Holt [Ho] we could calculate that $H^2(G, O_3(\tilde{G})/Z(\tilde{G}))$ is isomorphic to $\mathbb{Z}_3$. Notice that in terms of the proof of Lemma 4.3 the quotient $\tilde{G}/P_2$ is the extension of McL by its Schur multiplier of order 3.

5. A subgeometry in $\tilde{\mathcal{G}}$

In this section we show that $\tilde{\mathcal{G}}$ contains a new tilde geometry. Consider $\nu_1 \in \Lambda_2$ with $N(\nu_1) = (2^{8016})$; $\{a, b\} \subseteq P_2(\nu_1)$ and $X(\nu_1) = \emptyset$. Put $X = G(\nu_1)$ and $X_1 = X \cap G_1$. According to [I] or [Atlas] $X \cong U_4(3)$. It is easy to see that X_1 is the stabilizer in $G_1 \cong \text{Mat}_{22}$ of the octad $P_2(\nu_1)$, so that $X_1 \cong 2^4\text{Alt}_6$. In particular, the set Ψ of images of ν_0 under X is of size 567 and in fact

$$\Psi = \{\mu \mid \mu \in \Omega, (\mu, \nu_1) = 0\}.$$

The group X_1 has the following 4 orbits on $\Psi \setminus \{\nu_0\}$:

$$\begin{aligned} \Psi_1 &= \{\mu \mid \mu \in \Omega_1, P_2(\mu) \cap P_2(\nu_1) = \emptyset\}; \\ \Psi_2 &= \{\mu \mid \mu \in \Omega_2, P_2(\mu) \cap P_2(\nu_1) = \{b\}\}; \\ \Psi_3 &= \{\mu \mid \mu \in \Omega_2, |P_2(\mu) \cap P_2(\nu_1)| = 3, X(\mu) \cap P_2(\nu_1) = \emptyset\}; \\ \Psi_4 &= \{\mu \mid \mu \in \Omega_3, P_3(\mu) \cap P_2(\nu_1) = \{b\}\} \end{aligned}$$

of length 30, 96, 120, 320, respectively.

Let Ξ be a subgraph in Ω isomorphic to $\bar{T}(7)$ and containing ν_0 . Then Ξ represents an element of type 4 in the residue of $\tilde{\nu}_0$ in $\tilde{\mathcal{G}}$. Then Ξ corresponds to a sextet $\{S_1, S_2, \dots, S_6\}$ such that $\{a, b\} \subseteq S_1$. Let us assume that $P_2(\nu_1) = S_1 \cup S_2$. Then $\Xi \cap \Psi = (\Xi_1 \cap \Psi) \cup (\Xi_2 \cap \Psi) \cup \{\nu_0\}$, where

$$\Xi_i \cap \Psi = \{\mu \mid \mu \in \Omega_i, P_2(\mu) = S_i \cup S_j, 3 < i < j \leq 6\}$$

and

$$\Xi_2 \cap \Psi = \{\mu \mid \mu \in \Omega_2, \quad P_6(\mu) \cup P_2(\mu) = S_1 \cup S_i, \quad 3 \leq i \leq 6 \quad \text{and} \quad X(\mu) \subset S_1\}$$

are of size 6 and 8, respectively. This shows that the vertices and triangles in $\Xi \cap \Psi$ form a generalized quadrangle $\mathcal{G}(\text{Sp}_4(2))$. Let $\mathcal{G}(U_4(3))$ be the geometry whose elements of type 1, 2 and 3 are the vertices, triangles of Ψ and the images of $\Xi \cap \Psi$ under X ; the incidence relation is via inclusion. Then, as explained above, the residue of an element of type 3 is a generalized quadrangle $\mathcal{G}(\text{Sp}_4(2))$. The elements of type 3 and 4 incident to the element ν_0 are in terms of the Steiner system related to $\text{res}_{\mathcal{G}(\text{McL})}(\nu_0)$ the pairs of octads $\{B_1, B_2\}$ which do not intersect $S_1 \cup S_2$ and the 2-element subsets in $(S_1 \cup S_2) \setminus \{a, b\}$, respectively. This shows that the residue of a point is also isomorphic to the generalized quadrangle $\mathcal{G}(\text{Sp}_4(2))$ (see also the definition of $\mathcal{G}(\text{Mat}_{22})$). Thus $\mathcal{G}(U_4(3))$ is of type

with $X \cong U_4(3)$ acting flag-transitively.

We have $X_2 = G_3(\nu_1) = G_3$ and $X_3 = G_4(\nu_1) \cong 2^4 \text{Alt}_6$. Since $\tilde{G}_1 \cong 3\text{Mat}_{22}$, if $\tilde{X}_1$ is the preimage of X_1 in $\tilde{G}_1$, then $\tilde{X}_1 \cong 3 \cdot 2^4 \text{Alt}_6$. Similarly, if $\tilde{X}_3$ is the preimage of X_3 in $\tilde{G}_4$, then $\tilde{X}_3 \cong 3 \cdot 2^4 \text{Alt}_6$. Put $\tilde{X}_2 = \tilde{G}_3$.

LEMMA 5.1. *Let $\tilde{X} = \langle \tilde{X}_1, \tilde{X}_2, \tilde{X}_3 \rangle$ and let $\mathcal{X}$ be the coset geometry $\mathcal{X} = \mathcal{G}(\tilde{X}, \{\tilde{X}_i \mid i = 1, 2, 3\})$. Then $\mathcal{X}$ is of type*

$\tilde{X}$ is a non-split extension $3^{23}U_4(3)$ which acts flag-transitively on $\mathcal{X}$.

Proof. Since $\tilde{\mathcal{G}}$ is a 2-cover of $\mathcal{G}$ in which the residue of a point is isomorphic to $\mathcal{G}(3\text{Mat}_{22})$ and the residue of a plane is isomorphic to $\mathcal{G}(3\text{Alt}_7)$, it is easy to see that the diagram of $\mathcal{X}$ is as stated. Thus it remains to show that $O_3(\tilde{X}) = O_3(\hat{H}) \cong 3^{23}$.

Recall that $P := O_3(\hat{H})$ is the orthogonal complement $\langle \hat{\mu}_0 \rangle^\perp$ in $\hat{\Lambda} = \Lambda/3\Lambda$ of the image $\hat{\mu}_0$ of μ_0 in $\hat{\Lambda}$. Put $\alpha = \mu_0 + \nu_1$. Then $\hat{\alpha}$ and $\hat{\omega}$ are totally isotropic vectors in $\hat{\Lambda}$ and we have the following composition series of P (as an X -module):

$$1 < \langle \hat{\omega} \rangle < \langle \hat{\alpha}, \hat{\omega} \rangle < \langle \hat{\alpha}, \hat{\omega} \rangle^\perp < \langle \hat{\omega} \rangle^\perp < P.$$

Since $C_P(X) = \langle \hat{\alpha}, \hat{\omega} \rangle$, the module $P/\langle \hat{\alpha}, \hat{\omega} \rangle$ is indecomposable. Since $\langle \hat{\omega} \rangle^\perp$ and $\langle \hat{\alpha} \rangle^\perp$ have no X -invariant complements in P and since P is a self-dual module, the only X -invariant submodule of P which contains a vector in $P \setminus \langle \hat{\omega} \rangle^\perp$, is P itself. Since $(\xi, \omega) = -1$ for $\langle \hat{\xi} \rangle = O_3(\tilde{X}_3)$, the assertion follows.

Remark. Since $O_3(\tilde{X}_1)$ and $O_3(\tilde{X}_3)$ are not contained in $\langle \hat{\alpha}, \hat{\omega} \rangle^\perp$ and $O_3(\tilde{X}_1) \leq \hat{\alpha}^\perp$, but $O_3(\tilde{X}_3) \not\leq \langle \hat{\alpha} \rangle^\perp$, $\tilde{X}/(\langle \hat{\alpha}, \hat{\omega} \rangle^\perp \cong 3^2 U_4(3)$ defines a quotient of $\mathcal{X}$.

6. The amalgams of McL-type

Let $\mathcal{F}$ be a geometry of type P_3^3 , let F be a flag-transitive automorphism group of $\mathcal{F}$, and let $\Phi = \{x_i \mid 1 \leq i \leq 4\}$ be a maximal flag in $\mathcal{F}$ where x_i is of type i . For $J \subseteq \{1, 2, 3, 4\}$ let $\Phi_J = \{x_i \mid i \in J\}$ be the flag of type J contained in Φ and $F_J = F(\Phi_J)$

be the (elementwise) stabilizer of Φ_J . As usual we write F_1 instead of $F_{\{1\}}$ etc. so that $\mathbf{F} = \{F_i \mid 1 < i \leq 4\}$ is the amalgam of maximal parabolics associated with the action of F on $\mathcal{F}$. Let K_J be the kernel of the action of F_J on the residue $\text{res}_{\mathcal{F}}(\Phi_J)$ of Φ_J in $\mathcal{F}$ and $\bar{F}_J = F_J/K_J$ for $J \subseteq \{1, 2, 3, 4\}$. The classification of flag-transitive P_3 -geometries in [Sh1] implies the following.

LEMMA 6.1. *The subgroup F_{14} induces Sym_5 on $\text{res}_{\mathcal{F}}(\{x_1, x_4\})$ and one of the following holds:*

- (i) $\text{res}_{\mathcal{F}}(x_1) \cong \mathcal{G}(\text{Mat}_{22})$ and $\bar{F}_1$ is isomorphic either to Mat_{22} or to Aut Mat_{22} ;
- (ii) $\text{res}_{\mathcal{F}}(x_1) \cong \mathcal{G}(3\text{Mat}_{22})$ and $\bar{F}_1$ is isomorphic either to 3Mat_{22} or to 3Aut Mat_{22} .

Proof. The first statement follows immediately from Section 3.2 while (i) and (ii) follow from [Sh1].

LEMMA 6.2. *One of the following holds:*

- (i) $\text{res}_{\mathcal{F}}(x_4) \cong \mathcal{G}(\text{Alt}_7)$ and $\text{Alt}_7 \leq \bar{F}_4 \leq \text{Sym}_7$;
- (ii) $\text{res}_{\mathcal{F}}(x_4) \cong \mathcal{G}(3\text{Alt}_7)$ and $3\text{Alt}_7 \leq \bar{F}_4 \leq 3\text{Sym}_7$;
- (iii) $\text{res}_{\mathcal{F}}(x_4) \cong \mathcal{G}(L_2(25))$ and $L_2(25) \leq \bar{F}_4 \leq P\Omega L_2(25)$.

Furthermore, if $\bar{E}$ is the stabilizer of x_1 in $\bar{F}_4$ then $\text{Sym}_5 \leq \bar{E} \leq \text{Sym}_5 \times 2$.

Proof. The residue $\text{res}_{\mathcal{F}}(x_4)$ is a flag-transitive P^3 -geometry. It was shown in [IS1] that there are exactly four such geometries: $\mathcal{G}(\text{Alt}_7)$, $\mathcal{G}(3\text{Alt}_7)$, $\mathcal{G}(L_2(25))$ and $\mathcal{G}(L_2(11))$. The only flag-transitive automorphism group of the latter geometry is $L_2(11)$ and the stabilizer in $L_2(11)$ of an element of type 1 in $\mathcal{G}(L_2(11))$ is isomorphic to Alt_5 . By Lemma 6.1 this means that $\text{res}_{\mathcal{F}}(x_4)$ cannot be isomorphic to $\mathcal{G}(L_2(11))$ and the result follows.

Definition 6.3. *An amalgam $\mathbf{F}$ of maximal parabolics associated with a flag-transitive action on a geometry of type P_3^3 is said to be of McL-type if $K_1 = 1$.*

In the remainder of the section we assume that $\mathbf{F}$ is of McL-type which implies in particular that $\bar{F}_1 = F_1$. Let

$$\delta = |F_1 : O_2(F_1)|$$

so that by Lemma 6.1, $\delta = 1$ or 2 .

We start by analysing the structure of F_4 . From the basic properties of P_3 -geometries we have the following:

LEMMA 6.4. *If $\delta = 1$ then $F_{14} \cong 2^4\text{Sym}_5$ where $O_2(F_{14})$ is a faithful and irreducible module for $F_{14}/O_2(F_{14}) \cong \text{Sym}_5$. If $\delta = 2$ then $F_{14} \cong 2^5\text{Sym}_5$ where $O_2(F_{14})$ is a faithful module for $F_{14}/O_2(F_{14})$ which contains a unique proper submodule and the dimension of this submodule is 4.*

We have $K_4 \leq F_{14}$ and by the last sentence in Lemma 6.2 if $\bar{E}$ is the stabilizer of x_1 in the action $\bar{F}_4$ induced by F_4 on $\text{res}_{\mathcal{F}}(x_4)$, then

$$\text{Sym}_5 \leq \bar{E} \leq \text{Sym}_5 \times 2.$$

LEMMA 6.5. *The following assertions hold:*

- (i) $\text{res}_{\mathcal{F}}(x_4)$ is isomorphic to $\mathcal{G}(\text{Alt}_7)$ or $\mathcal{G}(3\text{Alt}_7)$ and $\bar{F}_4 \cong \text{Alt}_7$ or 3Alt_7 , respectively;
- (ii) $K_4 = O_2(F_{14})$;
- (iii) $\delta = 1$.

Proof. By Lemma 6.4 and the paragraph before the lemma we have either $K_4 = O_2(F_{14})$ or $\delta = 2$ and K_4 has index 2 in $O_2(F_{14})$. Consider the natural homomorphism ξ of F_4 onto $F_4/C_{F_4}(K_4)$. Since K_4 is abelian, it is in the kernel of ξ and the image of ξ is a subgroup of $L_n(2) \cong \text{Aut } K_4$ (where $n = 4$ or 5) which contains $\xi(F_{14}) \cong \text{Sym}_5$. Since Sym_7 , 3Alt_7 , 3Sym_7 and $L_2(25)$ are not subgroups of $L_5(2)$ we obtain the isomorphism $F_4/C_{F_4}(K_4) \cong \text{Alt}_7$ which implies (i). Since F_4 does not involve Sym_7 we also obtain (ii). If $\delta = 2$ then by Lemma 6.4 K_4 is a 5-dimensional indecomposable $GF(2)$ -module for $F_4/C_{F_4}(K_4) \cong \text{Alt}_7$, but it is well known and easy to check that there are no such modules. Hence (iii) holds.

By Lemmas 6.1 and 6.5 F_1 is isomorphic either to Mat_{22} or to 3Mat_{22} . One can see from Subsection 3.5 that the amalgam of maximal parabolics associated with the action of McL on $\mathcal{G}(\text{McL})$ is of McL -type which explains our terminology.

PROPOSITION 6.6. *Let $\mathbf{F}$ be an amalgam of McL -type and let $\mathbf{D} = \{F_{ij} \mid 1 \leq i < j \leq 4\}$. Then the isomorphism type of $\mathbf{D}$ is uniquely determined.*

We are going to show that up to equivalence there is a unique way to write down a presentation for the universal completion of $\mathbf{D}$ (we will use this presentation to prove Theorem 1.3).

By Lemmas 6.1 and 6.5(iii) the subamalgam $\{F_{ij} \mid 2 \leq j \leq 4\}$ is isomorphic to the amalgam of maximal parabolics associated with the action of Mat_{22} on $\mathcal{G}(\text{Mat}_{22})$ (equivalently of 3Mat_{22} on $\mathcal{G}(3\text{Mat}_{22})$). In particular (cf. [Sh1]) $F_{12} \cong \text{AGL}_3(2)$ which is the semidirect product (with respect to the natural action) of an elementary abelian subgroup $V = O_2(F_{12})$ of order 2^3 and a subgroup $K \cong L_3(2)$.

We start by writing down a presentation for F_{12} . Let (a_1, a_2, a_3) be a linear basis of V and a_i , $4 \leq i \leq 8$ be elements from K presented in the above basis by the following matrices (the elements of V are row-vectors and the action is from the right):

$$\begin{aligned} a_4 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} & a_5 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} & a_6 &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ a_7 &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & a_8 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Then it is easy to see that the following (Steinberg type) relations R_1 provide a presentation of F_{12}

$$\begin{aligned} R_1: & a_i^2 = 1 \quad \text{for } 1 \leq i \leq 8, \quad [a_i, a_j] = 1 \quad \text{for } 1 \leq i < j \leq 3; \\ & [a_1, a_4] = [a_2, a_4] = 1, \quad a_3^{a_4} = a_2 a_3, \quad [a_4, a_5] = 1, \quad [a_4, a_6] = a_5, \quad [a_4, a_7] = 1, \\ & (a_4 a_8)^3 = 1, \quad [a_1, a_5] = [a_2, a_5] = 1, \quad a_3^{a_5} = a_1 a_3, \quad [a_5, a_6] = 1, \quad a_5^{a_7} = a_5 a_4, \quad a_5^{a_8} = a_5 a_6, \\ & [a_1, a_6] = 1, \quad a_2^{a_6} = a_1 a_2, \quad [a_3, a_6] = 1, \quad (a_6 a_7)^3 = 1, \quad [a_6, a_8] = 1, \quad a_1^{a_7} = a_1 a_2, \\ & [a_7, a_2] = [a_3, a_7] = 1, \quad (a_7 a_8)^4 = 1, \quad [a_1, a_8] = [a_3, a_8] = 1, \quad a_2^{a_8} = a_2 a_3. \end{aligned}$$

Furthermore, without loss of generality we assume that $F_{123} = \langle a_i \mid 1 \leq i \leq 8, i \neq 7 \rangle$, $F_{124} = \langle a_i \mid 1 \leq i \leq 7 \rangle$ and $B := F_{1234} = \langle a_i \mid 1 \leq i \leq 6 \rangle$.

It is not difficult to check that $U := \langle a_1, a_2, a_4, a_5 \rangle$ is the only elementary abelian subgroup of order 16 in B and since $F_{14} \cong 2^4 \text{Sym}_5$ and $O_2(F_{14}) \leq B$, we must have $O_2(F_{14}) = U$. Notice that the element $a_6 a_7$ (of order 3) acts fixed point freely on U . Therefore there is a complement L to U in F_{14} which contains $\langle a_3, a_6, a_7 \rangle \cong \text{Sym}_3 \times 2$. Since a_3 is central in $\langle a_3, a_6, a_7 \rangle$, this element corresponds to a field automorphism of $L \cong \text{P}\Gamma\text{L}_2(4)$ and therefore we can see by the action of $\langle a_3, a_6 \rangle$ on U that a_6 is central in $L \cap F_{134} \cong D_8$. Then $L' \cap F_{134}$ contains an element a_9 satisfying the following relations:

$$\begin{aligned} R_2: a_9^2 &= [a_1, a_9] = [a_5, a_9] = [a_6, a_9] = 1, & a_2^{a_9} &= a_2 a_5, & a_4^{a_9} &= a_1 a_5 a_4, \\ a_3^{a_9} &= a_3 a_6, & (a_7 a_9)^5 &= 1, & a_8^{a_9} &= a_1 a_6 a_9. \end{aligned}$$

LEMMA 6.7. *The generators a_i , $1 \leq i \leq 9$ together with the relations $R_1 \cup R_2$ provide a presentation of the universal completion of the amalgam $\{F_{1j} \mid 2 \leq i \leq 4\}$.*

Proof. One can either apply the coset enumeration (on a computer) to see that the presentation defines the group 3Mat_{22} , or appeal to the main result of [Sh1] which gives the classification of the amalgams of maximal parabolics associated with flag-transitive actions on P_3 -geometries.

Next consider F_2 . From the diagram P_3^3 we observe that F_2 contains $F_{12} \cong \text{AGL}_3(2)$ with index 2. It is well known that the outer automorphism group of $\text{AGL}_3(2)$ is of order 2. Thus either $F_2 \cong F_{12} \times 2$ or $F_2 = \text{Aut } \text{AGL}_3(2)$. In the latter case $O_2(F_2)$ is an indecomposable module for $F_2/O_2(F_2) \cong L_3(2)$. Consider the subgroup F_{24} which is the full pre-image in F_4 of a subgroup of the form $(3 \times 2^2) \cdot 2$ from $\bar{F}_4$ and $O_2(F_4) \leq F_{24}$. Since $O_2(F_4)$ is the natural module for $F_{14}/O_2(F_4) \cong L_3(2)$ and since $|F_{24} : F_{124}| = 2$, any element of order 3 of F_{24} acts fixed point freely on $O_2(F_4)$. As furthermore, $F_4/O_2(F_4)$ acts faithfully on $O_2(F_4)$, the centre of F_{24} is trivial which shows the following.

LEMMA 6.8. $F_2 \cong \text{Aut } \text{AGL}_3(2)$.

From Lemma 6.8 we deduce that F_{234} contains an involution a_{10} which satisfies the following relations:

$$\begin{aligned} R_3: a_{10}^2 &= 1, & [a_i, a_{10}] &= 1 \quad \text{for } 1 \leq i \leq 3, & a_5^{a_{10}} &= a_2 a_5, & a_4^{a_{10}} &= a_1 a_2 a_4, \\ a_6^{a_{10}} &= a_3 a_6, & a_7^{a_{10}} &= a_3 a_7, & a_8^{a_{10}} &= a_1 a_3 a_8, \end{aligned}$$

so that the generators $\{a_i \mid 1 \leq i < 10, i \neq 9\}$ together with the relations $R_1 \cup R_3$ provide a presentation of F_2 .

Thus it remains to find a relation between a_9 and a_{10} . Such a relation can be found in F_3 . By analysing the relations in R_2 and R_3 it is easy to see that $a_9 a_{10}$ induces on $K_4 = U = \langle a_1, a_2, a_4, a_5 \rangle$ an action of order 3. Thus $(a_9 a_{10})^3$ is an involution and hence must be centralised by a_9 and a_{10} . Again looking at the relations R_2 and R_3 we obtain the following:

$$C_U(a_9) \cap C_U(a_{10}) = \langle a_1, a_3 \rangle \cap \langle a_1, a_2 \rangle = \langle a_1 \rangle.$$

Thus we have two possibilities R_3^0 and R_3^1 for the presentation of F_3 :

$$R_3^k: (a_9 a_{10})^3 = a_1^k \quad \text{for } k = 0 \text{ or } 1.$$

LEMMA 6.9. *For $k = 0$ and 1 let X_k be the group with generators $\{a_i \mid 1 \leq i \leq 10\}$ and relations $R_1 \cup R_2 \cup R_3^k$. Then $X_0 \cong X_1$.*

Proof. One can see directly from the relations $R_1 \cup R_2$ that the group 3Mat_{22} (which is the universal completion of the amalgam $\{F_{1j} \mid 2 \leq j \leq k\}$) has an (outer) automorphism α which centralizes F_{12} and maps a_9 onto $a_9 a_1$. Let $\alpha'_i = a_i^\alpha$ for $1 \leq i \leq 9$ and $\alpha'_{10} = a_{10}$. Then the elements α'_i generate X_0 and since $(\alpha'_9 \alpha'_{10})^3 = (a_9 a_1 a_{10})^3 = a_1$, they satisfy the defining relations of X_1 . So we have an isomorphism of X_0 onto X_1 . This proves the assertion and also completes the proof of Proposition 6.6.

PROPOSITION 6.10. *Up to isomorphism there are at most 4 amalgams $\mathbf{F}^i$, $1 \leq i \leq 4$ of McL-type, and the amalgam $\mathbf{F}^i$ is characterized by the conditions (i) below:*

- (1) $Z(F_1) \cong Z(F_4) \cong 3$;
- (2) $Z(F_1) \cong 1$ and $Z(F_4) \cong 3$;
- (3) $Z(F_1) \cong 3$ and $Z(F_4) \cong 1$;
- (4) $Z(F_1) \cong Z(F_4) \cong 1$.

Proof. Let $\mathbf{F}$ be an amalgam of McL-type. Then by Proposition 6.6 the subamalgam $\mathbf{D}$ is uniquely determined up to isomorphism. The parabolics F_i are completions of the subamalgams $\mathbf{B}_i = \{F_{ij} \mid 1 \leq j \leq 4, j \neq i\}$ contained in $\mathbf{D}$. Hence the isomorphism type of $\mathbf{F}$ is determined by the choice of completions of the $\mathbf{B}_i$. Since F_2 is factorized by F_{12} and F_{24} while F_3 is factorized by F_{13} and F_{34} , it is easy to see that each of $\mathbf{B}_2$ and $\mathbf{B}_3$ has a unique completion. On the other hand there are two completions of $\mathbf{B}_1$, namely 3Mat_{22} and Mat_{22} and two completions of $\mathbf{B}_4$, namely $3 \cdot 2^4\text{Alt}_7$ and 2^4Alt_7 and the result follows.

The amalgam $\mathbf{F}^4$ is the amalgam of maximal parabolics associated with the action of McL on $\mathcal{G}(\text{McL})$. The amalgam $\mathbf{F}^1$ is the amalgam of maximal parabolics associated with the action of $\tilde{G}$ on $\tilde{\mathcal{G}}$ as in Section 4. The amalgams $\mathbf{F}^2$ and $\mathbf{F}^3$ exist as abstract amalgams and questions (b) and (c) posed in the Introduction ask whether or not they are amalgams of maximal parabolics associated with flag-transitive actions on P_3^3 -geometries (equivalently whether or not they have faithful completions).

We have found by a computer that the centres of the universal completions of $\mathbf{B}_1$ and $\mathbf{B}_4$ are generated by the elements $(a_7 a_4 a_8 a_1 a_4 a_8 a_9)^{11}$ and $(a_5 a_6 a_7 a_{10} a_9)^7$, respectively (assuming the relation R_3^0). Thus the generators $\{a_i \mid 1 \leq i \leq 10\}$ and the relations $R_1 \cup R_2 \cup R_3^0 \cup R_4$, where

$$R_4: (a_7 a_4 a_8 a_1 a_4 a_8 a_9)^{11} = (a_5 a_6 a_7 a_{10} a_9)^7 = 1$$

provide a presentation for the universal completion U_4 of $\mathbf{F}^4$.

Clearly, G is a factor group of U_4 . In the previous section we saw that there is a subgroup $X \cong U_4(3)$ in G with $X_1 \cong 2^4\text{Alt}_6$ and with $X = \langle X_1, G_3 \rangle$. Therefore, it is not difficult to see that in G , the group X is generated by

$$S = \{a_i \mid i \in \{1, \dots, 10\} \setminus \{7\}\} \cup \{y\}$$

where

$$y = (a_7 a_9)^{-2} (a_6 a_7) (a_7 a_9)^2.$$

Let F_0 be the subgroup of U_4 generated by S . Using coset enumeration implemented in **GAP** we obtain that F_0 is of index 275 in U_4 and that U_4 acts as a rank 3 permutation group on the cosets of F_0 in U_4 with intersection numbers $\lambda = 30$ and $\mu = 56$. Therefore, there is a normal subgroup N in U_4 contained in F_0 such that $U_4/N \cong \text{McL}$

and $F_0/N \cong U_4(3)$ [Sm1]. We follow [St, p. 2830] (where the simple connectness of a different geometry for McL has been proved) to show that $N = 1$. Since $X_1 \leq G_1$, $X_2 = G_3$ and $X_3 \leq G_4$, the group F_0 defines a 2-cover of $\mathcal{G}(U_4(3))$. According to [K1] the universal 2-cover of $\mathcal{G}(U_4(3))$ is the affine building belonging to $\Omega^-(Q_2, f)$ where

$$f = \sum_{i=1}^6 x_i^2$$

and the universal completion of the amalgam $\{X_1, X_2, X_3\}$ is isomorphic to $\Omega(\mathbb{Z}[\frac{1}{2}], f)$ which means that F_0 is a factor group of the latter group. Stroth first considers the situation when the universal cover of his geometry is finite and his result can be summarized as follows.

LEMMA 6.11 [St, p. 2831]. *Let G be a finite group possessing a normal subgroup N and a subgroup X containing N such that $G/N \cong \text{McL}$, $X/N \cong U_4(3)$ and X acts faithfully and flag-transitively on a 2-cover of $\mathcal{G}(U_4(3))$. Then $N = 1$.*

Hence we may assume that U_4 (or equivalently F_0) is infinite. Again we follow the argumentation in [St]. By a result due to [M, (2.4.6)(i)] any normal subgroup of infinite index in $\Omega(\mathbb{Z}[\frac{1}{2}], f)$ is contained in the centre. Therefore, there is a normal subgroup U in F_0 contained in N such that $F_0/U \cong \Omega(6, m)$, where m is an integer not equal to 3. In particular F_0/U is finite. Set $V = \text{Cor}_{U_4}(U)$. As $|U_4 : U|$ is finite also U_4/V is finite. Since V is contained in N , it has trivial intersection with the parabolics of F_0 . As X_2 equals F_3 and as $X_1 \cong 2^4\text{Alt}_6 \leq F_1 \cong \text{Mat}_{22}$, it follows that V also intersects trivially the parabolics F_i , $1 \leq i \leq 4$. Hence $|U_4/V : F_0/V| = 275$ and by Lemma 6.11 $V = 1$ which is not possible as we assumed U_4 to be infinite. This shows the following result which immediately implies Theorem 1.3.

LEMMA 6.12. *The group McL is the universal completion of the amalgam $\mathbf{F}^4$ and the geometry $\mathcal{G}(\text{McL})$ is simply connected.*

Originally we proved Lemma 6.12 using Havas's 'Adaptive Coset Enumerator' [Hav]. With its help we were able to show that the index of $F_1 \cong \text{Mat}_{22}$ in U_4 is 2025 which is exactly the index of Mat_{22} in McL . We thank George Havas for making his program available for us.

7. The flag-transitive geometries of type P_3^3

In this section we complete the proof of Theorem 1.2, by showing that every amalgam of maximal parabolics associated with a flag-transitive action on a P_3^3 -geometry is of McL -type (i.e. it satisfies $K_1 = 1$).

We follow notation introduced in the first paragraph of Section 6, so that $\mathbf{F} = \{F_i \mid 1 \leq i \leq 4\}$ is the amalgam of maximal parabolics associated with a flag-transitive action on a P_3^3 -geometry. By Proposition 5.4 in order to prove Theorem 1.2 all we have to show is that $\mathbf{F}$ is of McL -type.

LEMMA 7.1. *K_1 is a 2-group.*

Proof. Clearly, K_1 fixes the neighbours of x_1 elementwise. If K_1 fixes also the second neighbourhood of x_1 elementwise, then by the connectivity, K_1 would fix all the

points of $\mathcal{F}$ which would imply $K_1 = 1$. Therefore, we may assume that K_1 acts non-trivially on the neighbourhood of y_1 , where y_1 is a point incident to x_2 and $y_1 \neq x_1$. Since K_1 fixes $\text{res}_{\mathcal{F}}(x_1)$ elementwise, it also fixes $\text{res}_{\mathcal{F}}(\Phi_{12}) = \text{res}_{\mathcal{F}}(\{y_1, x_2\})$ which is a projective plane of order 2. We have seen in Subsection 3.2 that the stabilizer of an element of type 1 in a flag-transitive automorphism group of a P_3 -geometry is of the form $2^3L_3(2)$ or $2^4L_3(2)$; it induces $L_3(2)$ on the residual projective plane and hence K_1 induces a 2-group on the neighbours of y_1 . Thus, if K_1 contains an element k of odd order, then by the connectivity of $\mathcal{F}$, k fixes all points of $\mathcal{F}$ which shows $k = 1$ and the result follows.

LEMMA 7.2. *Suppose $K_1 \neq 1$. Then $C_{F_1}(K_1) \leq O_{2,3}(K_1)$.*

Proof. Assume $C_{F_1}(K_1) \not\leq O_{2,3}(F_1)$. Then $F'_1 \leq K_1 C_{F_1}(K_1)$ and $(K_1 C_{F_1}(K_1) \cap K_1 F'_{14})/K_1 \cong 2^4\text{Alt}_5$. Let s be an element of order 3 in $C_{F_1}(K_1) \cap F_{14}$. By Lemma 6.2 $F'_4/O_{2,3}(F_4)$ is isomorphic to one of Alt_7 or $L_2(25)$. Therefore, we may assume that s is chosen so that there exists an involution t in $F_4 \setminus F_{14}$ which commutes with s modulo K_4 . Then $F_4 = \langle F_{14}, t \rangle$ and as K_1 is not normal in F_4 , it follows $K_1 K_1^t/K_1 \neq 1$. Recall that $F_{14}/K_1 \cong 2^4\text{Sym}_5$ or 2^5Sym_5 , that $F_{14}/K_4 \cong \text{Sym}_5$ or $2 \times \text{Sym}_5$ and that $O_2(F_{14}/K_1)$ contains only one irreducible $F_{14}/O_2(F_{14})$ -submodule, which is of dimension 4 and which is the natural $SL_2(4)$ -module for $F_{14}/O_2(F_{14}) \cong \text{Sym}_5 \cong P\Gamma L_2(4)$. Now the fact that $[t, s]$ is in K_4 and the fact that K_4/K_1 is abelian, implies $[s, K_1 K_1^t/K_1] = 1$ in contradiction to the fact that $O_2(F'_{14}/K_1)$ contains the natural $SL_2(4)$ -module for $F_{14}/O_2(F_{14})$. Hence the assertion holds.

LEMMA 7.3. *Suppose that $K_1 \neq 1$. Set $V = \Omega_1(Z(K_1))$. Then $E(F_1/C_{F_1}(V)) \cong 3\text{Mat}_{22}$ or Mat_{22} .*

Proof. Assume that the assertion is false. Since F_1/K_1 is isomorphic to Mat_{22} or to the non-split extension 3Mat_{22} , it follows $O^2(F_1) \leq C_{F_1}(V)$, and, as $O^2(F_1)$ is generated by elements of odd order, that $[O^2(F_1), Z(K_1)] = 1$.

Since any line of our geometry is incident with exactly two points, K_1 is contained in K_2 . Therefore, and as $K_2 \leq F_1$, it follows that $Z(K_2)$ is contained in $C_{F_1}(K_1)$. Hence, according to Lemma 7.2, $Z(K_2) \leq O_{2,3}(F_1)$ which implies $Z(K_2) \leq K_1$ and therefore $Z(K_2) \leq Z(K_1)$. As $[O^2(F_1), Z(K_1)] = 1$, the subgroup $Z(K_2)$ is normal in $O^2(F_1)$. Hence $Z(K_2)$ is normal in $\langle O^2(F_1), F_2 \rangle = F$. Thus our assumption was false and the assertion holds.

To show that $K_1 = 1$ we prove that otherwise V would be an F -module for $F_1/C_{F_1}(V)$. A module V is an F -module for a group G , if there exists a non-trivial elementary abelian subgroup A of G (the *offender*) such that $|V/C_V(A)| \leq |A|$ (see for instance [MS]). Then we apply the following well-known result.

LEMMA 7.4 [MS]. *Let G be a group with $E(G)/Z(E(G)) \cong \text{Mat}_{22}$. Then G has no F -module.*

Proof. This is part of the statement of lemma 4.2 in [MS].

LEMMA 7.5. $K_1 = 1$.

Proof. Assume $K_1 \neq 1$ and set $P_1 = F_2 \cap F_3 \cap F_4$. Since B is a 2-group and $|P_1 : B| = 2$, P_1 is also a 2-group and we may define $\mathcal{A}(P_1)$ and $\mathcal{A}(K_1)$ to be the set of elementary abelian subgroups of P_1 and K_1 of maximal rank, respectively.

We claim that $\mathcal{A}(P_1) = \mathcal{A}(K_1)$. Let $A \in \mathcal{A}(P_1)$ and assume $A \notin \mathcal{A}(K_1)$. Notice that $A \cap K_1 = C_A(V)$ by Lemma 7.3 and $A \cap V = C_V(A)$, since $A \in \mathcal{A}(P_1)$. Since $VC_A(V)$ is elementary abelian, we have

$$|A| \geq |VC_A(V)| = |V| |C_A(V)| / |C_V(A)|,$$

which implies

$$|A/A \cap K_1| = |A/C_A(V)| \geq |V : C_V(A)|.$$

Thus V is an F -module for $F_1/C_{F_1}(V)$ with offender $AC_{F_1}(V)/C_{F_1}(V)$ in contradiction with Lemma 7.4. Hence the assumption $A \notin \mathcal{A}(K_1)$ was false. Therefore, $\mathcal{A}(P_1) \leq \mathcal{A}(K_1)$ and, as $K_1 \leq P_1$, in fact $\mathcal{A}(P_1) = \mathcal{A}(K_1)$. In particular the Thompson subgroup $J(P_1) = \langle \mathcal{A}(P_1) \rangle$ of P_1 coincides with the Thompson subgroup $J(K_1) = \langle \mathcal{A}(K_1) \rangle$ of K_1 and hence it is normal in $F = \langle P_1, F_1 \rangle$ which is not possible, as $\bigcap_{g \in F} F_{1234}^g = 1$. This proves the lemma.

Originally we have proved Lemma 7.1 using the main result of [PR].

8. The flag-transitive geometries of type P_m^n

In this section we prove Theorem 1.1 by showing that there are no flag-transitive geometries of type P_3^4 or P_4^3 . Theorem 1.2 implies the following:

COROLLARY 8.1. *Let $\mathcal{F}$ be a P_3^3 -geometry, F be a flag-transitive automorphism group of $\mathcal{F}$ and $\mathbf{F} = \{F_i \mid 1 \leq i \leq 4\}$ be the amalgam of maximal parabolics associated with the action of F on $\mathcal{F}$. Then F_1 induces on the corresponding residue an action isomorphic to Mat_{22} or 3Mat_{22} and F_4 induces on the corresponding residue an action isomorphic to Alt_7 or 3Alt_7 .*

We start with the following observation which can be checked directly.

LEMMA 8.2. *Let A_2 be the stabilizer of an edge in the natural action of $A \cong \text{Alt}_7$ on $T(7)$. Then a Sylow 2-subgroup in A_2 is isomorphic to D_8 .*

The following elementary result about P^4 -geometries will be sufficient for our purposes.

LEMMA 8.3. *Let $\mathcal{Y}$ be a P^4 -geometry, Y be a flag-transitive automorphism group of $\mathcal{Y}$ and $\mathbf{Y} = \{Y_i \mid 1 \leq i \leq 4\}$ be the amalgam of maximal parabolics associated with the action of Y on $\mathcal{Y}$. Suppose that the residue of an element of type 1 in $\mathcal{Y}$ is isomorphic either to $\mathcal{G}(\text{Alt}_7)$ or to $\mathcal{G}(3\text{Alt}_7)$. Then Y_1 induces on the corresponding residue the action isomorphic to Sym_7 or 3Sym_7 , respectively.*

Proof. Let $\{y_i \mid 1 \leq i \leq 4\}$ be the maximal flag associated with $\mathbf{Y}$. By Lemma 6.2 all we have to show is that the action of Y_1 on $\text{res}_{\mathcal{Y}}(y_1)$ can not be isomorphic to Alt_7 or 3Alt_7 in the respective cases. The arguments in the two cases are almost identical and we assume that Y_1 induces Alt_7 . In this case it is easy to see that the action of Y_1 on $\text{res}_{\mathcal{Y}}(y_1)$ is faithful and hence $Y_1 \cong \text{Alt}_7$. It follows directly from the diagram that $Y_2 \cong \text{Sym}_5 \times 2$ and hence $Y_{23} \cong \text{Sym}_3 \times 2 \times 2$. There is exactly one element, say z_1 of type 1 which is incident to y_3 and which is not incident to y_2 . Clearly z_1 is stabilized by Y_{23} and hence Y_{23} is a conjugate of the stabilizer in Y_1 of an element of type 3 in $\text{res}_{\mathcal{Y}}(y_1) \cong \text{Alt}_7$. This contradicts Lemma 8.2.

PROPOSITION 8.4. *There are no flag-transitive geometries of type P_3^4 .*

Proof. Let $\mathcal{X}$ be a P_3^4 -geometry, X be a flag-transitive automorphism group of X , $\{x_i \mid 1 \leq i \leq 5\}$ be a maximal flag and $X_i = X(x_i)$, $1 \leq i \leq 5$ be the corresponding maximal parabolics. By Lemma 8.1 $\text{res}_x(\{x_1, x_2\})$ is isomorphic to $\mathcal{G}(\text{Alt}_7)$ or $\mathcal{G}(3\text{Alt}_7)$. Furthermore, by Lemma 8.1, F_{15} induces on this residue Alt_7 or 3Alt_7 and, by Lemma 8.3, F_{15} induces Sym_7 or 3Sym_7 , a contradiction.

In order to prove the non-existence of flag-transitive P_4^3 -geometries we will use the following consequence of the classification of the flag-transitive P_n -geometries announced in [IS2] (this corollary follows already from the main result of [Sh2]).

COROLLARY 8.5. *Let $\mathcal{Y}$ be a P_4 -geometry, Y be a flag-transitive automorphism group of $\mathcal{Y}$ and suppose that Y_5 induces on the corresponding residue Mat_{22} or 3Mat_{22} . Then $Y_1 \cong L_4(2)$.*

Notice that the only P_4 -geometry which satisfies the conditions in the above corollary is the geometry $\mathcal{G}(\text{Mat}_{23})$ associated with the Mathieu group of degree 23.

PROPOSITION 8.6. *There are no flag-transitive geometries of type P_4^3 .*

Proof. Let $\mathcal{X}$ be a P_4^3 -geometry, X be a flag-transitive automorphism group of X , $\{x_i \mid 1 \leq i \leq 5\}$ be a maximal flag and $X_i = X(x_i)$, $1 \leq i \leq 5$ be the corresponding maximal parabolics. By Lemma 8.5, X_{12} induces on $\text{res}_x(\{x_1, x_2\})$ the action of $L_4(2)$. It is easy to show that the action is faithful and hence $X_{12} \cong L_4(2)$. It follows directly from the diagram of $\mathcal{X}$ that X_2 contains X_{12} with index two. By Lemma 6.8, X_2 contains $\text{Aut AGL}_3(2)$, but it well known and easy to check that the latter group is not contained in $\text{Aut } L_4(2)$, a contradiction.

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REFERENCES

- [Bu] F. BUEKENHOUT. Foundation of incidence geometry, in *Handbook of incidence geometry* (ed. F. Buekenhout), chap. 3 (North-Holland, 1995).
- [BCN] A. E. BROUWER, A. M. COHEN and A. NEUMAIER. *Distance-regular graphs* (Springer Verlag, 1989).
- [Br] K. S. BROWN. *Cohomology of groups* (Springer Verlag, 1982).
- [Atlas] J. H. CONWAY, R. T. CURTIS, S. P. NORTON, R. A. PARKER and R. A. WILSON. *Atlas of finite groups* (Clarendon Press, 1985).
- [C] J. H. CONWAY. Three lectures on exceptional groups: In: *Finite groups* (eds M. B. Powell and G. Higman), pp. 215–247 (Academic Press, 1971).
- [FII] I. A. FARADJEV, A. A. IVANOV and A. V. IVANOV. Distance-transitive graphs of valency 5, 6 and 7. *Europ. J. Combin.* **7** (1986), 303–319.
- [GLS] D. GORENSTEIN, R. LYONS and R. SOLOMON. *The Classification of the Finite Simple Groups, Number 2*, Math. Surveys and Monographs, AMS, vol. 40, no. 2 (Providence, Rhode Island, 1996).
- [Ha] J. HALL. Locally Petersen graphs. *J. Graph Theory* **4** (1980), 173–187.
- [Hav] G. HAVAS. Coset enumeration strategies. In: *Proc. ISSAC-1991*, pp. 191–199 (ACM Press 1991).
- [Ho] D. F. HOLT. The mechanical computation of first and second cohomology groups. *J. Symbolic Comput.* **1** (1985) (4), 351–361.
- [I] A. A. IVANOV. *Geometry of sporadic groups I. Petersen and tilde geometries* Cambridge Univ. Press, Cambridge, 1999.

- [IS1] A. A. IVANOV and S. V. SHPECTOROV. Geometries for sporadic groups related to the Petersen graph. I. *Comm. Algebra* **16** (1988), 925–954.
- [IS2] A. A. IVANOV and S. V. SHPECTOROV. Flag-transitive tilde and Petersen type geometries are all known. *Bull. Amer. Math. Soc.* **31** (1994), 173–184.
- [IS3] A. A. IVANOV and S. V. SHPECTOROV. Natural representations of the P -geometries of Co_2 -type. *J. Algebra* **164** (1994), 718–749.
- [K1] W. M. KANTOR. Some geometries which are almost buildings. *Europ. J. Comb.* **2** (1981) 239–247.
- [K2] W. M. KANTOR. 2-adic buildings. *J. Algebra* **92** (1985), 208–223.
- [L] S. LINTON. On vector enumeration. *Linear Algebra Appl.* **192** (1993), 235–248.
- [M] G. A. MARGULIS. Finiteness of quotient groups of discrete subgroups. *Functional Analysis and Applications* **13** (1979), 178–187.
- [MS] U. MEIERFRANKENFELD and G. STROTH. F -modules for finite simple groups, preprint (Halle, 1998).
- [ModAt] CH. JANSEN, K. LUX, R. PARKER and R. WILSON. *An Atlas of Brauer Characters* (Clarendon Press, 1995).
- [P] A. PASINI. *Diagram geometries* (Clarendon Press, 1994).
- [PR] C. PARKER and P. ROWLEY. Amalgams which involve sporadic simple groups. I. *J. Algebra* **193** (1997), 441–491.
- [Sh1] S. V. SHPECTOROV. A geometric characterization of the group M_{22} . In: *Investigations in algebraic theory of combinatorial objects*, pp. 112–123, VNIISI, Moscow, 1985 [Russian, English translation: (eds I. A. Faradzev *et al.*), pp. 443–457, Kluwer Acad. Publ., 1994].
- [Sh2] S. V. SHPECTOROV. On geometries with diagram P^n , preprint (VNIISI, Moscow, 1988) [Russian].
- [Sh3] S. V. SHPECTOROV. The universal 2-cover of the P -geometry $\mathcal{G}(Co_2)$. *Europ. J. Comb.* **13** (1992), 291–312.
- [Sm1] M. S. SMITH. On rank 3 permutation groups. *J. Algebra* **33** (1975), 22–42.
- [Sm2] S. D. SMITH. Non-associative commutative algebras for triple covers of 3-transposition groups. *Michigan Math. J.* **24** (1977), 273–287.
- [St] G. STROTH. Some geometry for McL . *Comm Algebra* **17** (11) (1989), 2825–2833.
- [W] R. A. WILSON. Vector stabilizers and subgroups of Leech lattice groups. *J. Algebra* **127** (1989), 387–408.