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Lattice Code Design for the Rayleigh Fading Wiretap Channel

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Abstract—It has been shown recently that coding for the Gaussian Wiretap Channel can be done with nested lattices. A fine lattice intended to the legitimate user must be designed as a usual lattice code for the Gaussian Channel, while a coarse lattice is added to introduce confusion at the eavesdropper, whose theta series must be minimized. We present a design criterion for both the fine and coarse lattice to obtain wiretap lattice codes for the Rayleigh fading Wiretap Channel.

I. INTRODUCTION

A. Related work

The wiretap channel was introduced by Wyner [9] as a discrete memoryless broadcast channel where the sender, Alice, transmits confidential messages to a legal receiver Bob, in the presence of an eavesdropper Eve. Wyner defined the perfect secrecy capacity as the maximum amount of information that Alice can send to Bob while insuring that Eve gets a negligible amount of information. He also described a generic coset coding strategy, where both data and random bits are encoded, in order to confuse the eavesdropper (see also [7]). The question of determining the secrecy capacity of many classes of channels has been addressed extensively recently, yielding a plethora of information theoretical results on secrecy capacity (see [6] for a survey of many such results).

There is a sharp contrast with the situation of wiretap code designs, where very little is known. The most exploited approach to get practical codes so far has been to use LDPC codes (see [8] for binary erasure and symmetric channels, [4] for Gaussian channels with binary inputs). Finally, lattice codes for Gaussian channels have been considered from an information theoretical point of view in [3].

A design criterion for constructing explicit lattice codes on the Gaussian Wiretap channel has been proposed in [1], based on the analysis of Eve's correct decision probability. This design criterion relies on a new lattice invariant called "secrecy gain" which is based on the lattice theta series. The secrecy gain of unimodular lattice was further studied in [2].

B. Contribution and organization

We propose here to find the appropriate design criterion for both the wiretap fast fading and block fading channels and to give some intuition on lattice codes which are optimal for this criterion.

This paper is organized as follows. Section II presents the system model and recalls the design criterion for the Gaussian wiretap channel. Sections III and IV are the main contributions where we give the code design criterion for, respectively, the fast fading and the block fading channel. The particular case of algebraic lattices is discussed in both cases.

II. SYSTEM MODEL AND THE GAUSSIAN CASE

A. Fast fading channels

Alice wants to send data to Bob on a wiretap fading channel, where an eavesdropper Eve is trying to intercept the data through another fading channel. Perfect channel state information (CSI) is assumed at both receivers. Thus it is possible to remove the phase of the complex fading coefficients to obtain a real fading which is Rayleigh distributed, with the aid of an in-phase/quadrature component interleaver to guarantee that the fading coefficients are independent from one real symbol to the next [5, sec 2.1]. This is modeled by

$$\begin{aligned} \mathbf{y} &= \text{diag}(\mathbf{h}_b)\mathbf{x} + \mathbf{v}_b \\ \mathbf{z} &= \text{diag}(\mathbf{h}_e)\mathbf{x} + \mathbf{v}_e, \end{aligned} \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the transmitted signal, $\mathbf{v}_b$ and $\mathbf{v}_e$ denote the Gaussian noise at Bob, respectively Eve's side, both with zero mean, and respective variance σ_b^2 and σ_e^2 , and

$$\begin{aligned} \text{diag}(\mathbf{h}_b) &= \begin{pmatrix} |h_{b,1}| & & \\ & \ddots & \\ & & |h_{b,n}| \end{pmatrix}, \\ \text{diag}(\mathbf{h}_e) &= \begin{pmatrix} |h_{e,1}| & & \\ & \ddots & \\ & & |h_{e,n}| \end{pmatrix} \end{aligned} \quad (2)$$

are the channel matrices containing the fading coefficients where $h_{b,i}, h_{e,i}$ are complex Gaussian random variables with variance $\sigma_{h,b}^2$, resp. $\sigma_{h,e}^2$, so that $|h_{b,i}|, |h_{e,i}|$ are Rayleigh distributed, $i = 1, \dots, N$, with parameter $\sigma_{h,b}^2$, resp. $\sigma_{h,e}^2$. We assume that Bob has a good SNR, but that $\sigma_b^2 = N_0 \ll N_1 = \sigma_e^2$, so that Eve has a poor SNR with respect to Bob.

The transmitted codeword $\mathbf{x} \in \mathbb{R}^n$ comes from a lattice Λ_b intended to Bob, that is

$$\mathbf{x} = M_b \mathbf{u}, \quad \mathbf{u} \in \mathbb{Z}^n$$

where M_b is the generator matrix of the lattice Λ_b . We can rewrite the channel accordingly:

$$\begin{aligned} \mathbf{y} &= \text{diag}(\mathbf{h}_b)M_b\mathbf{u} + \mathbf{v}_b \\ \mathbf{z} &= \text{diag}(\mathbf{h}_e)M_b\mathbf{u} + \mathbf{v}_e, \end{aligned} \quad (3)$$

and we set

$$M_{b,\mathbf{h}_b} = \text{diag}(\mathbf{h}_b)M_b, \quad M_{b,\mathbf{h}_e} = \text{diag}(\mathbf{h}_e)M_b$$

which can be interpreted as the generator matrix of the lattice $\Lambda_{b,\mathbf{h}_b}$, resp. $\Lambda_{b,\mathbf{h}_e}$. In words, these are the lattice intended to Bob seen through Bob's, resp. Eve's channel.

Coset encoding is used, namely the lattice Λ_b is partitioned into a union of disjoint cosets of the form

$$\Lambda_e + \mathbf{c},$$

with Λ_e a sublattice of Λ_b and $\mathbf{c}$ an n -dimensional vector. To send k bits $\mathbf{s}$ of data, we need 2^k cosets

$$\Lambda_b = \cup_{j=1}^{2^k} (\Lambda_e + \mathbf{c}_j)$$

to be labelled by

$$\mathbf{s} \mapsto \Lambda_e + \mathbf{c}_{j(\mathbf{s})}.$$

Alice then randomly chooses a point $\mathbf{x} \in \Lambda_e + \mathbf{c}_{j(\mathbf{s})}$ and sends it over the wiretap channel. This is equivalent to choose a random vector $\mathbf{r} \in \Lambda_e$. The transmitted lattice point $\mathbf{x} \in \Lambda_b$ is finally of the form

$$\mathbf{x} = \mathbf{r} + \mathbf{c} \in \Lambda_e + \mathbf{c}. \quad (4)$$

We can as above set

$$M_{e,\mathbf{h}_b} = \text{diag}(\mathbf{h}_b)M_e, \quad M_{e,\mathbf{h}_e} = \text{diag}(\mathbf{h}_e)M_e$$

which are the generator matrix of the lattice $\Lambda_{e,\mathbf{h}_b}$, resp. $\Lambda_{e,\mathbf{h}_e}$ corresponding to the lattice Λ_e twisted by the channel of Bob, resp Eve. Bits are transmitted by Alice at a rate equal to $R = R_s + R_r$ where R_s is the secrecy rate of this transmission and R_r is the rate of random bits.

The parameters involved are:

- Λ_b is the lattice intended for Bob,
- Λ_e is a sublattice of Λ_b that encodes the random bits intended for Eve,
- n is the dimension of both lattices,
- $\mathcal{V}(\Lambda_b)$ (resp. $\mathcal{V}(\Lambda_e)$) is the fundamental parallelotope of Λ_b (resp. Λ_e),
- $\text{Vol}(\Lambda_b)$ (resp. $\text{Vol}(\Lambda_e)$) is the volume of Λ_b (resp. Λ_e) where by definition

$$\text{Vol}(\Lambda_b) = \int_{\mathcal{V}(\Lambda_b)} d\mathbf{x} = \det(M_b M_b^T)^{1/2},$$

- the unnormalized second moment $\mathcal{U}(\Lambda_b)$ is

$$\mathcal{U}(\Lambda_b) = \int_{\mathcal{V}(\Lambda_b)} \|\mathbf{x}\|^2 d\mathbf{x}.$$

B. Gaussian channels

Recall from [1] that the probability $P_{c,b}$ of Bob's (resp. $P_{c,e}$ of Eve's) correct decision in doing coset decoding when Λ_b is sent over a Gaussian channel is:

$$\begin{aligned} P_{c,b} &= \frac{1}{(\sqrt{2\pi}\sigma_b)^n} \sum_{\mathbf{r} \in \Lambda_e} \int_{\mathcal{V}(\Lambda_b + \mathbf{r})} e^{-\|\mathbf{u}\|^2/2\sigma_b^2} d\mathbf{u} \\ P_{c,e} &= \frac{1}{(\sqrt{2\pi}\sigma_e)^n} \sum_{\mathbf{r} \in \Lambda_e} \int_{\mathcal{V}(\Lambda_b + \mathbf{r})} e^{-\|\mathbf{u}\|^2/2\sigma_e^2} d\mathbf{u}. \end{aligned}$$

Since Λ_b is designed for Bob to correctly decode, the received point is most likely to be in the coset with $\mathbf{r} = \mathbf{0}$, so that

$$P_{c,b} \simeq \frac{1}{(\sqrt{2\pi}\sigma_b)^n} \int_{\mathcal{V}(\Lambda_b)} e^{-\|\mathbf{u}\|^2/2\sigma_b^2} d\mathbf{u}. \quad (5)$$

As for Eve, σ_e is assumed larger than σ_b , so we need to take into account the cosets where $\mathbf{r} \neq \mathbf{0}$. By writing $\mathbf{u} = \mathbf{w} + \mathbf{r}$, $\mathbf{w} \in \Lambda_b$, a Taylor expansion of $e^{-\|\mathbf{w} + \mathbf{r}\|^2/2\sigma_e^2}$ at order 2 gives

$$\left(1 + \frac{-1}{\sigma_e^2} \langle \mathbf{r}, \mathbf{w} \rangle + \frac{-1}{2\sigma_e^2} \|\mathbf{w}\|^2 + \frac{1}{2\sigma_e^4} \langle \mathbf{r}, \mathbf{w} \rangle^2 \right) + O\left(\frac{1}{\sigma_e^4}\right)$$

and we get, by neglecting $O\left(\frac{1}{\sigma_e^4}\right)$, that

$$\begin{aligned} &\sum_{\mathbf{r} \in \Lambda_e} \int_{\mathcal{V}(\Lambda_b + \mathbf{r})} e^{-\|\mathbf{u}\|^2/2\sigma_e^2} d\mathbf{u} \\ &\simeq \sum_{\mathbf{r} \in \Lambda_e} e^{-\|\mathbf{r}\|^2/2\sigma_e^2} \left(\text{Vol}(\Lambda_b) - \frac{\mathcal{U}(\Lambda_b)}{2\sigma_e^2} \right) \end{aligned}$$

noticing that $\sum_{\mathbf{r} \in \Lambda_e} \int_{\mathcal{V}(\Lambda_b)} \langle \mathbf{r}, \mathbf{w} \rangle d\mathbf{w} = 0$ since the sum is over $\mathbf{r} \in \Lambda_e$, and for each $\mathbf{r}$ in Λ_e , $-\mathbf{r}$ is also in Λ_e .

The probability of making a correct decision for Eve is then

$$P_{c,e} \simeq \frac{1}{(2\pi\sigma_e^2)^{n/2}} \sum_{\mathbf{r} \in \Lambda_e} e^{-\|\mathbf{r}\|^2/2\sigma_e^2} \left(\text{Vol}(\Lambda_b) - \frac{\mathcal{U}(\Lambda_b)}{2\sigma_e^2} \right)$$

and the goal is then to minimize

$$\sum_{\mathbf{r} \in \Lambda_e} e^{-\|\mathbf{r}\|^2/2\sigma_e^2}.$$

By further neglecting the terms in $O\left(\frac{1}{\sigma_e^2}\right)$, we further simplify Eve's probability of correct decision to

$$P_{c,e} \simeq \frac{\text{Vol}(\Lambda_b)}{(2\pi\sigma_e^2)^{n/2}} \sum_{\mathbf{r} \in \Lambda_e} e^{-\|\mathbf{r}\|^2/2\sigma_e^2}. \quad (6)$$

III. CODE DESIGN CRITERION: FAST FADING CHANNELS

For a given realization of the fading $\mathbf{h}$, the channel (3) can be seen as the Gaussian wiretap channel

$$\begin{aligned} \mathbf{y} &= M_{b,\mathbf{h}_b}\mathbf{u} + \mathbf{v}_b \\ \mathbf{z} &= M_{b,\mathbf{h}_e}\mathbf{u} + \mathbf{v}_e, \end{aligned} \quad (7)$$

and we note that for $\mathbf{r} \in \Lambda_{e,\mathbf{h}_e}$

$$\|\mathbf{r}\|^2 = \|\text{diag}(\mathbf{h}_e)M_e\mathbf{u}\|^2 = \sum_{i=1}^n |h_{e,i}x_i|^2, \quad (8)$$

with $\mathbf{u} \in \mathbb{Z}^n$ and $\mathbf{x} \in \Lambda_e$.

Since probability computations for Bob, which is the classical problem of transmitting over a fast Rayleigh fading channel, have been extensively studied in the literature (e.g. [5, sec 2.3]), we focus on Eve.

A. Eve's probability of correct decision

The probability of Eve correctly decoding on channel (7) is from (6), for a given fading realization

$$P_{c,e,h_e} \simeq \left(\frac{1}{2\pi\sigma_e^2} \right)^{n/2} \text{Vol}(\Lambda_{b,h_e}) \sum_{\mathbf{r} \in \Lambda_{e,h_e}} e^{-\frac{\|\mathbf{r}\|^2}{2\sigma_e^2}}. \quad (9)$$

As

$$\text{Vol}(\Lambda_{b,h_e}) = \prod_{i=1}^n |h_{e,i}| \text{Vol}(\Lambda_b)$$

and using (8), we get

$$\sum_{\mathbf{r} \in \Lambda_{e,h_e}} e^{-\frac{\|\mathbf{r}\|^2}{2\sigma_e^2}} = \sum_{\mathbf{x} \in \Lambda_e} e^{-\frac{\sum_{i=1}^n |h_{e,i} x_i|^2}{2\sigma_e^2}}, \quad (10)$$

yielding the following approximate expression for P_{c,e,h_e}

$$\begin{aligned} & \left(\frac{1}{2\pi\sigma_e^2} \right)^{n/2} \text{Vol}(\Lambda_b) \prod_{i=1}^n |h_{e,i}| \sum_{\mathbf{x} \in \Lambda_e} e^{-\frac{\sum_{i=1}^n |h_{e,i} x_i|^2}{2\sigma_e^2}} \\ &= \left(\frac{1}{2\pi\sigma_e^2} \right)^{n/2} \text{Vol}(\Lambda_b) \sum_{\mathbf{x} \in \Lambda_e} \prod_{i=1}^n \left(|h_{e,i}| e^{-\frac{|h_{e,i} x_i|^2}{2\sigma_e^2}} \right). \end{aligned} \quad (11)$$

The average probability $\bar{P}_{c,e}$ of correct decision is now:

$$\begin{aligned} & \mathbb{E}_{h_e} [P_{c,e,h_e}] \\ & \simeq \left(\frac{1}{2\pi\sigma_e^2} \right)^{n/2} \mathbb{E}_{h_e} \left[\text{Vol}(\Lambda_b) \sum_{\mathbf{x} \in \Lambda_e} \prod_{i=1}^n \left(|h_{e,i}| e^{-\frac{|h_{e,i} x_i|^2}{2\sigma_e^2}} \right) \right] \\ &= \left(\frac{1}{2\pi\sigma_e^2} \right)^{n/2} \text{Vol}(\Lambda_b) \sum_{\mathbf{x} \in \Lambda_e} \mathbb{E}_{h_e} \left[\prod_{i=1}^n \left(|h_{e,i}| e^{-\frac{|h_{e,i} x_i|^2}{2\sigma_e^2}} \right) \right] \\ &= \left(\frac{1}{2\pi\sigma_e^2} \right)^{n/2} \text{Vol}(\Lambda_b) \sum_{\mathbf{x} \in \Lambda_e} \prod_{i=1}^n \underbrace{\mathbb{E}_{h_e} \left(|h_{e,i}| e^{-\frac{|h_{e,i} x_i|^2}{2\sigma_e^2}} \right)}_{\mathcal{F}} \end{aligned} \quad (12)$$

since the $|h_{e,i}|$ are independently distributed, $i = 1, \dots, n$. Set $\rho_i = |h_{e,i}|$ which is Rayleigh distributed with parameter $\sigma_{h,e}^2$ and pdf

$$f(\rho_i, \sigma_{h,e}^2) = \frac{\rho_i}{\sigma_{h,e}^2} e^{-\frac{\rho_i^2}{2\sigma_{h,e}^2}}.$$

Thus

$$\begin{aligned} \mathcal{F} &= \frac{1}{\sigma_{h,e}^2} \int_0^\infty \rho_i e^{-\frac{\rho_i^2 |x_i|^2}{2\sigma_e^2}} \rho_i e^{-\frac{\rho_i^2}{2\sigma_{h,e}^2}} d\rho_i \\ &= \frac{1}{\sigma_{h,e}^2} \int_0^\infty \rho_i^2 e^{-\rho_i^2 \left(\frac{|x_i|^2}{2\sigma_e^2} + \frac{1}{2\sigma_{h,e}^2} \right)} d\rho_i \\ &= \frac{1}{\sigma_{h,e}^2} \frac{\sqrt{\pi}}{4 \left(\frac{|x_i|^2}{2\sigma_e^2} + \frac{1}{2\sigma_{h,e}^2} \right)^{3/2}} \end{aligned}$$

since for $a > 0$, we have

$$\int_0^{+\infty} x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{4a^{3/2}}.$$

Thus $\sum_{\mathbf{x} \in \Lambda_e} \prod_{i=1}^n \mathcal{F}$ in (12) becomes

$$\begin{aligned} & \sum_{\mathbf{x} \in \Lambda_e} \left(\frac{\sqrt{\pi}}{4\sigma_{h,e}^2} \right)^n \prod_{i=1}^n \frac{1}{\left(\frac{1}{2\sigma_{h,e}^2} + \frac{|x_i|^2}{2\sigma_e^2} \right)^{3/2}} \\ &= \sum_{\mathbf{x} \in \Lambda_e} \left(\frac{\sqrt{\pi}}{4\sigma_{h,e}^2} \right)^n (2\sigma_{h,e}^2)^{3n/2} \prod_{i=1}^n \frac{1}{\left(1 + |x_i|^2 \frac{\sigma_{h,e}^2}{\sigma_e^2} \right)^{3/2}} \\ &= \sum_{\mathbf{x} \in \Lambda_e} \left(\frac{\sqrt{\pi}\sigma_{h,e}}{\sqrt{2}} \right)^n \prod_{i=1}^n \frac{1}{\left(1 + |x_i|^2 \frac{\sigma_{h,e}^2}{\sigma_e^2} \right)^{3/2}} \end{aligned}$$

and (12) can be rewritten as

$$\bar{P}_{c,e} \simeq \left(\frac{\sigma_{h,e}}{2\sigma_e} \right)^n \text{Vol}(\Lambda_b) \sum_{\mathbf{x} \in \Lambda_e} \prod_{i=1}^n \frac{1}{\left(1 + |x_i|^2 \frac{\sigma_{h,e}^2}{\sigma_e^2} \right)^{3/2}}.$$

Now, let γ_e denote Eve's average SNR defined as

$$\gamma_e = \frac{\sigma_{h,e}^2}{\sigma_e^2}. \quad (13)$$

We finally get

$$\bar{P}_{c,e} \simeq \left(\frac{\gamma_e}{4} \right)^{n/2} \text{Vol}(\Lambda_b) \sum_{\mathbf{x} \in \Lambda_e} \prod_{i=1}^n \frac{1}{\left(1 + |x_i|^2 \gamma_e \right)^{3/2}} \quad (14)$$

As $\bar{P}_{c,e}$ is the average probability of correct decision for Eve, it has to be minimized. We remark that the terms inside the summation in (14) are very similar to the terms we have when we express the error probability on a Rayleigh fast fading channel [5]. We further have

$$\begin{aligned} \prod_{i=1}^n \frac{1}{(1 + \gamma_e |x_i|^2)^{3/2}} &= \prod_{i=1}^n \frac{1}{\gamma_e^{3/2} \left(\frac{1}{\gamma_e} + |x_i|^2 \right)^{3/2}} \\ &\simeq \frac{1}{\gamma_e^{3/2} d_{\mathbf{x}}} \prod_{i \in \mathcal{J}_{\mathbf{x}}} \frac{1}{|x_i|^3} \end{aligned} \quad (15)$$

where γ_e is big enough to consider $1/\gamma_e$ as negligible¹ and $\mathcal{J}_{\mathbf{x}}$ is the set of indices i such that $x_i \neq 0$ and $d_{\mathbf{x}} = |\mathcal{J}_{\mathbf{x}}|$ is called the diversity of $\mathbf{x}$. We have that $d_{\mathbf{x}}$ is at most n , and if it is n for all $\mathbf{x} \in \Lambda_e$, then we have a full diversity lattice Λ_e

$$d_{\mathbf{x}} = n, \forall \mathbf{x} \in \Lambda_e.$$

¹This assumption is realistic since Λ_e is a lattice which should be "perfectly" decoded by Eve.

In this case, using (14) and (15), we derive

$$\begin{aligned}\bar{P}_{c,e} &\simeq \left(\frac{\gamma_e}{4}\right)^{\frac{n}{2}} \frac{1}{\gamma_e^{3n/2}} \text{Vol}(\Lambda_b) \sum_{x \in \Lambda_e} \prod_{i=1}^n \frac{1}{|x_i|^3} \\ &= \left(\frac{1}{4\gamma_e^2}\right)^{\frac{n}{2}} \text{Vol}(\Lambda_b) \sum_{x \in \Lambda_e} \prod_{i=1}^n \frac{1}{|x_i|^3}.\end{aligned}$$

B. Full-diversity algebraic lattices

Full-diversity lattices can be obtained using algebraic lattices [5], that is lattices obtained by embedding the ring of integers of a number field. Let $K/\mathbb{Q}$ be a number field of degree n with embeddings $\sigma_1, \dots, \sigma_n$ into $\mathbb{C}$, and denote by $\mathcal{O}_K$ its ring of integers. We assume that the lattice Λ_e is obtained via the canonical embedding of either $\mathcal{O}_K$ or an integral ideal $\mathcal{I}$ of $\mathcal{O}_K$. In that case, $x_i = \sigma_i(x)$ for $x \in \mathcal{O}_K$. Then

$$\bar{P}_{c,e} \simeq \left(\frac{1}{4\gamma_e^2}\right)^{\frac{n}{2}} \text{Vol}(\Lambda_b) \sum_{x \in \mathcal{O}_K} \frac{1}{|N_{K/\mathbb{Q}}(x)|^3}.$$

IV. CODE DESIGN CRITERION: BLOCK FADING CHANNELS

We now consider the case when the channel between Alice and Bob, resp. Eve, is block fading with coherence time L , instead of being fast fading, that is:

$$\begin{aligned}Y &= \text{diag}(\mathbf{h}_b)X + V_b \\ Z &= \text{diag}(\mathbf{h}_e)X + V_e,\end{aligned}\quad (16)$$

where the transmitted signal X is a $n \times L$ matrix, V_b and V_e are $n \times L$ matrices denoting the Gaussian noise at Bob, respectively Eve's side, both with coefficients zero mean, and respective variance σ_b^2 and σ_e^2 . The fading matrices are given explicitly in (2). When $L = 1$, we are back to the fast fading case.

In the setting of (16), we assume that the fading is constant over L time slots and that the channel coefficients $h_{b,1}, \dots, h_{b,n}$, resp. $h_{e,1}, \dots, h_{e,n}$, on the n parallel paths from Alice to Bob, resp. Eve, are supposed independent. In order to focus on the Ln -dimensional lattice structure of the transmitted signal, we vectorize the received signal (16) and obtain

$$\begin{aligned}\text{vec}(Y) &= \text{vec}(\text{diag}(\mathbf{h}_b)X) + \text{vec}(V_b) \\ &= \begin{pmatrix} \text{diag}(\mathbf{h}_b) & & \\ & \ddots & \\ & & \text{diag}(\mathbf{h}_b) \end{pmatrix} \text{vec}(X) + \text{vec}(V_b) \\ \text{vec}(Z) &= \text{vec}(\text{diag}(\mathbf{h}_e)X) + \text{vec}(V_e) \\ &= \begin{pmatrix} \text{diag}(\mathbf{h}_e) & & \\ & \ddots & \\ & & \text{diag}(\mathbf{h}_e) \end{pmatrix} \text{vec}(X) + \text{vec}(V_e).\end{aligned}$$

We now interpret the $n \times L$ codeword X as coming from a lattice by writing

$$\text{vec}(X) = M_b \mathbf{u}, \text{ resp. } \text{vec}(X) = M_e \mathbf{u} \quad (17)$$

where $\mathbf{u} \in \mathbb{Z}^{Ln}$ and M_b (resp. M_e) denotes the $Ln \times Ln$ generator matrix of the lattice intended to Bob (resp. Eve).

Thus in what follows, by a lattice point $\mathbf{x} \in \Lambda_b$ (resp. Λ_e), we mean that

$$\mathbf{x} = \text{vec}(X)$$

with $\text{vec}(X)$ as (17).

By setting as for the fast fading case

$$\begin{aligned}M_{b,\mathbf{h}_b} &= \text{diag}(\text{diag}(\mathbf{h}_b), \dots, \text{diag}(\mathbf{h}_b))M_b, \\ M_{b,\mathbf{h}_e} &= \text{diag}(\text{diag}(\mathbf{h}_e), \dots, \text{diag}(\mathbf{h}_e))M_b\end{aligned}$$

we can rewrite (16) as

$$\begin{aligned}\text{vec}(Y) &= M_{b,\mathbf{h}_b} \mathbf{u} + \text{vec}(V_b) \\ \text{vec}(Z) &= M_{b,\mathbf{h}_e} \mathbf{u} + \text{vec}(V_e),\end{aligned}$$

where $M_{b,\mathbf{h}_b}$, resp. $M_{b,\mathbf{h}_e}$ can be interpreted as the lattice generators of the lattices $\Lambda_{b,\mathbf{h}_b}$, resp. $\Lambda_{b,\mathbf{h}_e}$ and thus we get in particular for Eve

$$\text{Vol}(\Lambda_{b,\mathbf{h}_e}) = \left(\prod_{i=1}^n |h_{e,i}| \right)^L \text{Vol}(\Lambda_b).$$

A. Eve's probability of correct decision

First, we have from (6) that

$$\begin{aligned}P_{c,e} &\simeq \left(\frac{1}{2\pi\sigma_e^2}\right)^{\frac{Ln}{2}} \text{Vol}(\Lambda_{b,\mathbf{h}_e}) \sum_{\mathbf{r} \in \Lambda_{e,\mathbf{h}_e}} e^{-\|\mathbf{r}\|^2/2\sigma_e^2} \\ &= \frac{\text{Vol}(\Lambda_b)}{(2\pi\sigma_e^2)^{\frac{Ln}{2}}} \left(\prod_{i=1}^n |h_{e,i}| \right)^L \sum_{\mathbf{x} \in \Lambda_e} e^{-\sum_{j=1}^L |h_{e,j}x_j|^2/2\sigma_e^2}\end{aligned}$$

where $\Lambda_{e,\mathbf{h}_e}$ is the lattice with generator matrix $M_{e,\mathbf{h}_e} = \text{diag}(\text{diag}(\mathbf{h}_e), \dots, \text{diag}(\mathbf{h}_e))M_e$, $\mathbf{x} = \text{vec}(X)$ as explained in (17) and $\|\mathbf{r}\|^2$ is computed as in (8). Since $M_{e,\mathbf{h}_e}$ contains L copies of $\text{diag}(\mathbf{h}_e)$, we can further adopt a double indexing for coefficients of $\mathbf{x}$ and write

$$\sum_{j=1}^{Ln} |h_{e,j}x_j|^2 = \sum_{j=1}^L \sum_{i=1}^n |h_{e,i}x_{ij}|^2 = \sum_{i=1}^n |h_{e,i}|^2 \sum_{j=1}^L |x_{ij}|^2.$$

Note for further usage that since $\mathbf{x} = \text{vec}(X)$, x_{ij} actually corresponds to the (i, j) coefficient of X , and $\sum_{j=1}^L |x_{ij}|^2$ is a summation over the L components of the i th row of X , that we denote by $\mathbf{x}_i = (x_{i1}, \dots, x_{iL})$.

The average probability of correct decision for Eve is then

$$\bar{P}_{c,e} = \frac{\text{Vol}(\Lambda_b)}{(2\pi\sigma_e^2)^{\frac{Ln}{2}}} \sum_{\mathbf{x} \in \Lambda_e} \prod_{i=1}^n \underbrace{\mathbb{E}_{\mathbf{h}} \left(|h_{e,i}|^L e^{-\frac{|h_{e,i}|^2 \sum_{j=1}^L |x_{ij}|^2}{2\sigma_e^2}} \right)}_{\mathcal{F}}$$

where

$$\begin{aligned}\mathcal{F} &= \frac{1}{\sigma_{h,e}^2} \int_0^\infty \rho_i^L e^{-\frac{\rho_i^2 \sum_{j=1}^L |x_{ij}|^2}{2\sigma_e^2}} \rho_i e^{-\frac{\rho_i^2}{2\sigma_{h,e}^2}} d\rho_i \\ &= \frac{1}{\sigma_{h,e}^2} \int_0^\infty \rho_i^{L+1} e^{-\rho_i^2 \left(\frac{\sum_{j=1}^L |x_{ij}|^2}{2\sigma_e^2} + \frac{1}{2\sigma_{h,e}^2} \right)} d\rho_i \\ &= \frac{1}{\sigma_{h,e}^2} \frac{\Gamma\left(\frac{L}{2} + 1\right)}{\left(\frac{\|\mathbf{x}_i\|^2}{2\sigma_e^2} + \frac{1}{2\sigma_{h,e}^2} \right)^{\frac{L}{2} + 1}}\end{aligned}$$

since for $a > 0$, we have

$$\int_0^{+\infty} x^{L+1} e^{-ax^2} dx = \frac{\Gamma(\frac{L}{2} + 1)}{2a^{\frac{L}{2} + 1}}.$$

Now $\sum_{\mathbf{x} \in \Lambda_e} \prod_{i=1}^n \mathcal{F}$ is given, as done earlier, by

$$\begin{aligned} & \sum_{\mathbf{x} \in \Lambda_e} \left(\frac{\Gamma(\frac{L}{2} + 1)}{2\sigma_{h,e}^2} \right)^n \prod_{i=1}^n \frac{1}{\left(\frac{1}{2\sigma_{h,e}^2} + \frac{\|\mathbf{x}_i\|^2}{2\sigma_e^2} \right)^{\frac{L}{2} + 1}} \\ &= \sum_{\mathbf{x} \in \Lambda_e} \left(\Gamma\left(\frac{L}{2} + 1\right) (2\sigma_{h,e}^2)^{\frac{L}{2}} \right)^n \prod_{i=1}^n \frac{1}{\left(1 + \|\mathbf{x}_i\|^2 \frac{\sigma_{h,e}^2}{\sigma_e^2} \right)^{\frac{L}{2} + 1}} \end{aligned}$$

Recall from (13) that Eve's average SNR is

$$\gamma_e = \frac{\sigma_{h,e}^2}{\sigma_e^2}.$$

We finally conclude that

$$\bar{P}_{c,e} \simeq \frac{\gamma_e^{\frac{Ln}{2}} \Gamma(\frac{L}{2} + 1)^n}{(2\pi)^{\frac{Ln}{2}}} \text{Vol}(\Lambda_b) \sum_{\mathbf{x} \in \Lambda_e} \prod_{i=1}^n \frac{1}{\left(1 + \|\mathbf{x}_i\|^2 \gamma_e \right)^{\frac{L}{2} + 1}}.$$

In the same way as in (15), we can express the term inside the summation by assuming that Eve's SNR γ_e is high compared to the minimum distance of Λ_e and get

$$\prod_{i=1}^n \frac{1}{\left(1 + \|\mathbf{x}_i\|^2 \gamma_e \right)^{\frac{L}{2} + 1}} \simeq \frac{1}{\gamma_e^{n(\frac{L}{2} + 1)}} \prod_{i=1}^n \frac{1}{\|\mathbf{x}_i\|^{L+2}}$$

if we assume that none of the $\|\mathbf{x}_i\|$ are equal to zero. This corresponds to the case where the Ln -dimensional lattice Λ_e has diversity order at least $L(n-1) + 1$. Indeed, a diversity of $L(n-1)$ or less means that at most $L(n-1)$ coefficients of a non-zero lattice vector are non-zero, thus there could be L zero coefficients, which, if all located on the same row i , would make $\|\mathbf{x}_i\| = 0$. This cannot happen if the diversity is at least $L(n-1) + 1$.

In this case, we derive that

$$\bar{P}_{c,e} \simeq \left(\frac{\Gamma(\frac{L}{2} + 1)}{(2\pi)^{\frac{L}{2}} \gamma_e} \right)^n \text{Vol}(\Lambda_b) \sum_{\mathbf{x} \in \Lambda_e} \prod_{i=1}^n \frac{1}{\|\mathbf{x}_i\|^{L+2}}. \quad (18)$$

B. Full-diversity algebraic lattices

Again, to make sure that full diversity is achieved, we propose to use algebraic lattices. But this time, we need to control the terms in (18), that is essentially the sum

$$\sum_{\mathbf{x} \in \Lambda_e} \prod_{i=1}^n \frac{1}{\|\mathbf{x}_i\|^{L+2}}. \quad (19)$$

Let $K/\mathbb{Q}$ be a number field of degree n , with n embeddings $(\sigma_1, \sigma_2, \dots, \sigma_n)$ into $\mathbb{C}$, and ring of integers $\mathcal{O}_K$. Recall that a vector point $\mathbf{x} \in \Lambda_e$ is obtained from $\mathbf{x} = \text{vec}(X)$, and X is the codeword sent. Let $\mathbf{x}_1$ be the first row of X , and take $\mathbf{x}_i = \sigma_i(\mathbf{x}_1)$, so that each row of X is obtained by conjugating its first row. Alternatively, each column can be seen as a lattice point from the algebraic lattice build over $\mathcal{O}_K$. In this case, it

is enough for this lattice to be of diversity n to guarantee that $\|\mathbf{x}_i\| \neq 0$ for all i . Indeed, for every non-zero coefficient of the first row $\mathbf{x}_1$, all the corresponding columns will have non-zero coefficients. Conversely, each zero coefficient on the first row gives a column of zeros, and to have $\|\mathbf{x}_i\| = 0$ for one i means to have $\|\mathbf{x}_i\| = 0$ for all i , that is sending X containing only zeros. Now

$$\|\mathbf{x}_i\|^2 = \|\sigma_i(\mathbf{x}_1)\|^2 = \sum_{j=1}^L \sigma_i(x_{1j})^2 = \sigma_i\left(\sum_{j=1}^L x_{1j}^2\right) = \sigma_i(\|\mathbf{x}_1\|^2)$$

and

$$\prod_{i=1}^n \|\mathbf{x}_i\|^2 = \prod_{i=1}^n \|\sigma_i(\mathbf{x}_1)\|^2 = \prod_{i=1}^n \sigma_i(\|\mathbf{x}_1\|^2) = N_{K/\mathbb{Q}}(\|\mathbf{x}_1\|^2).$$

The sum in (19) finally becomes

$$\sum_{\mathbf{x} \in \Lambda_e} \prod_{i=1}^n \frac{1}{\|\sigma_i(\mathbf{x}_1)\|^{L+2}} = \sum_{\mathbf{x} \in \Lambda_e} \frac{1}{N_{K/\mathbb{Q}}(\|\mathbf{x}_1\|^2)^{\frac{L}{2} + 1}}.$$

V. FUTURE WORK

Current and future work naturally involves (i) the analysis of the wiretap MIMO Channel so as to determine the corresponding code design criterion, and (ii) the construction of lattices optimized for fast fading wiretap channel, block fading wiretap channel, and finally MIMO wiretap channel.

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