Cycle systems in the complete bipartite graph plus a one-factor

Ling, San; Ma, Jun; Pu, Liqun; Shen, Hao

2008

Pu, L., Shen, H., Ma, J., & Ling, S. (2008). Cycle systems in the complete bipartite graph plus a one-factor. *SIAM Journal of discrete math*, 21(4), 1083–1092.

<https://hdl.handle.net/10356/93927>

<https://doi.org/10.1137/06065461X>

©2008 Society for Industrial and Applied Mathematics This paper was published in *SIAM J Discrete Math* and is made available as an electronic reprint (preprint) with permission of Society for Industrial and Applied Mathematics. The paper can be found at <http://dx.doi.org/10.1137/06065461X>. One print or electronic copy may be made for personal use only. Systematic or multiple reproduction, distribution to multiple locations via electronic or other means, duplication of any material in this paper for a fee or for commercial purposes, or modification of the content of the paper is prohibited and is subject to penalties under law.

Downloaded on 26 Apr 2023 21:28:13 SGT

CYCLE SYSTEMS IN THE COMPLETE BIPARTITE GRAPH PLUS A ONE-FACTOR*

LIQUAN PU[†], HAO SHEN[‡], JUN MA[‡], AND SAN LING[§]

Abstract. Let $K_{n,n}$ denote the complete bipartite graph with n vertices in each partite set and $K_{n,n} + I$ denote $K_{n,n}$ with a one-factor added. It is proved in this paper that there exists an m -cycle system of $K_{n,n} + I$ if and only if $n \equiv 1 \pmod{2}$, $m \equiv 0 \pmod{2}$, $4 \leq m \leq 2n$, and $n(n+1) \equiv 0 \pmod{m}$.

Key words. complete bipartite graph, one-factor, cycle system

AMS subject classification. 05C38

DOI. 10.1137/06065461X

1. Introduction. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An m -cycle system of G is a collection T of m -cycles such that each edge of G is contained in a unique m -cycle of T .

It is easy to get the necessary conditions for the existence of an m -cycle system of G :

$$\begin{cases} 3 \leq m \leq |V(G)|; \\ |E(G)| \equiv 0 \pmod{m}; \\ d(u) \equiv 0 \pmod{2} \text{ for each } u \in V(G), \end{cases}$$

where $d(u)$ denotes the number of edges incident with u in G .

Let K_n denote the complete graph of order n , and let $K_{x,y}$ denote a complete bipartite graph with partite sets of sizes x and y . For $G = K_n$ or $K_{n,n}$, let $G + I$ denote G with a one-factor added and $G - I$ denote G with a one-factor removed. The existence of m -cycle systems has been studied extensively, and the following results are known.

THEOREM 1.1 (see [1, 6]). *Let m and n be positive integers. Then there exists an m -cycle system of K_n if and only if $n \equiv 1 \pmod{2}$, $3 \leq m \leq n$, and $n(n-1) \equiv 0 \pmod{2m}$.*

THEOREM 1.2 (see [7]). *Let $m \equiv 0 \pmod{2}$ and $m \geq 4$. Then there exists an m -cycle system of $K_{x,y}$ if and only if $x, y \geq \frac{1}{2}m$, $x \equiv y \equiv 0 \pmod{2}$, and $xy \equiv 0 \pmod{m}$.*

THEOREM 1.3 (see [5]). *Let n be an even integer and m be an integer in the range $3 \leq m \leq n$. Then there exists an m -cycle system of $K_n + I$ if and only if m divides $\frac{n^2}{2}$.*

*Received by the editors March 19, 2006; accepted for publication (in revised form) July 29, 2007; published electronically January 25, 2008. The research of Liquan Pu and San Ling was supported by NTU Research grant M58110003, Singapore.

<http://www.siam.org/journals/sidma/21-4/65461.html>

[†]Department of Mathematics, Zhengzhou University, Zhengzhou 450001, China, and School of Physical and Mathematical Sciences, Nanyang Technology University, Singapore 637616 (liqunpu@yahoo.com.cn).

[‡]Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China (haoshen@sjtu.edu.cn, mj904@sjtu.edu.cn).

[§]School of Physical and Mathematical Sciences, Nanyang Technology University, Singapore 637616 (lingsan@ntu.edu.sg).

THEOREM 1.4 (see [1, 6]). *Let m and n be positive integers. Then there exists an m -cycle system of $K_n - I$ if and only if $n \equiv 0 \pmod{2}$, $3 \leq m \leq n$, and $n(n-2) \equiv 0 \pmod{2m}$.*

THEOREM 1.5 (see [2, 4]). *Let m and n be positive integers. Then there exists an m -cycle system of $K_{n,n} - I$ if and only if $n \equiv 1 \pmod{2}$, $m \equiv 0 \pmod{2}$, $4 \leq m \leq 2n$, and $n(n-1) \equiv 0 \pmod{m}$.*

In this paper, we study the existence and construction of m -cycle systems for the bipartite graph $K_{n,n} + I$. Since in $K_{n,n} + I$ there are $2n$ vertices, $n^2 + n$ edges, $d(u) = n + 1$ for each vertex u , and m must be even, we have the following necessary conditions for the existence of an m -cycle system of $K_{n,n} + I$.

LEMMA 1.6. *If there exists an m -cycle system of $K_{n,n} + I$, then*

$$\begin{cases} n \equiv 1 \pmod{2}, \\ m \equiv 0 \pmod{2} \text{ and } 4 \leq m \leq 2n, \\ n(n+1) \equiv 0 \pmod{m}. \end{cases}$$

The purpose of this paper is to prove that these conditions are also sufficient for the existence of an m -cycle system of $K_{n,n} + I$. This is an extension of the result in [4].

2. Construction techniques. A cycle on m vertices is denoted by C_m . A C_n in a graph with n vertices is called a Hamilton cycle. If there exists an m -cycle system of G , then G is C_m -decomposable and is denoted by $C_m|G$.

In this section, we will provide some construction techniques for m -cycle systems of $K_{n,n} + I$. For our first construction, we need the following result.

LEMMA 2.1 (see [3]). *Let n be an integer, $n \geq 3$. Then there exists an n -cycle system of K_n if and only if $n \equiv 1 \pmod{2}$.*

When m is even, we can construct m -cycle systems of $K_{\frac{1}{2}m, \frac{1}{2}m} + I$ by applying $\frac{1}{2}m$ -cycle systems of $K_{\frac{1}{2}m}$.

THEOREM 2.2. *Let m be a positive integer such that $m \equiv 2 \pmod{4}$ and $m \geq 6$. Then $C_m|K_{\frac{1}{2}m, \frac{1}{2}m} + I$.*

Proof. Let $V(K_{\frac{1}{2}m, \frac{1}{2}m}) = \{u_0, u_1, \dots, u_{\frac{1}{2}m-1}\} \cup \{v_0, v_1, \dots, v_{\frac{1}{2}m-1}\}$. Since $m \equiv 2 \pmod{4}$ and $m \geq 6$, we have $\frac{1}{2}m \equiv 1 \pmod{2}$ and $\frac{1}{2}m \geq 3$. Hence by Lemma 2.1, $K_{\frac{1}{2}m}$ has a $\frac{1}{2}m$ -cycle system, denoted by T . Let $V(K_{\frac{1}{2}m}) = \{w_0, w_1, \dots, w_{\frac{1}{2}m-1}\}$.

For

$$C' = (w_{j_0}, w_{j_1}, w_{j_2}, w_{j_3}, \dots, w_{j_{\frac{1}{2}m-1}}) \in T,$$

let

$$C'^{1*} = (u_{j_0}, v_{j_0}, u_{j_1}, v_{j_1}, u_{j_2}, v_{j_2}, u_{j_3}, v_{j_3}, \dots, u_{j_{\frac{1}{2}m-1}}, v_{j_{\frac{1}{2}m-1}})$$

and

$$C'^{2*} = (v_{j_0}, u_{j_0}, v_{j_1}, u_{j_1}, v_{j_2}, u_{j_2}, v_{j_3}, u_{j_3}, \dots, v_{j_{\frac{1}{2}m-1}}, u_{j_{\frac{1}{2}m-1}}).$$

For each

$$C = (w_{i_0}, w_{i_1}, w_{i_2}, w_{i_3}, \dots, w_{i_{\frac{1}{2}m-1}}) \in T \setminus \{C'\},$$

let

$$C^* = (u_{i_0}, v_{i_1}, u_{i_2}, v_{i_3}, \dots, u_{i_{\frac{1}{2}m-1}}, v_{i_0}, u_{i_1}, v_{i_2}, u_{i_3}, \dots, v_{i_{\frac{1}{2}m-1}}).$$

Let $T^* = \{C^* | C \in T \setminus \{C'\}\} \cup \{C'^{1*}, C'^{2*}\}$ and $I = \{u_i v_i | 0 \leq i \leq \frac{1}{2}m - 1\}$. Then T^* is an m -cycle system of $K_{\frac{1}{2}m, \frac{1}{2}m} + I$. $\square$

Now for a positive integer n , let $D \subseteq Z_n$ and let $X(n; D)$ be a graph with vertex set $V(X(n; D)) = \{i_j | i \in Z_n, j \in Z_2\}$ and edge set $E(X(n; D)) = \{\{i_0, (i+d)_1\} | d \in D, i \in Z_n\}$. Clearly, $K_{n,n} = X(n; Z_n)$. The elements of D are called $(0, 1)$ -mixed differences. We say that $\{i_0, (i+d)_1\}$ is an edge of difference d .

Suppose that $C = ((i_1)_0, (i_2)_1, \dots, (i_{m-1})_0, (i_m)_1)$ is a C_m in $X(n; D)$. For $x \in Z_n$, let $C + x = ((i_1 + x)_0, (i_2 + x)_1, \dots, (i_{m-1} + x)_0, (i_m + x)_1)$. Obviously, $C + x$ is still a C_m . Let $(C) = \{C + x | x \in Z_n\}$. Here, (C) is called the orbit generated by C , and C is called a base cycle of (C) .

In our proof, we denote the union of multisets by $\uplus$, for example, $\{1, 1, 2\} \uplus \{2, 3\} = \{1, 1, 2, 2, 3\}$.

We use the difference method to give constructions of m -cycle systems of $X(n; D)$ which we need in this paper.

LEMMA 2.3. *For an even integer m , $m \geq 4$, $C_m | K_{m-1, m-1} + I$, where I is a one-factor of $K_{m-1, m-1}$.*

Proof. We view $K_{m-1, m-1}$ as $X(m-1; Z_{m-1})$ and $I = \{\{i_0, i_1\} | i \in Z_{m-1}\}$. Let $d_r \in Z_{m-1} \uplus \{0\}$ and

$$d_{r+1} = \begin{cases} r & \text{if } 0 \leq r \leq \frac{1}{2}m - 1, \\ 0 & \text{if } r = \frac{1}{2}m, \\ r - 1 & \text{if } \frac{1}{2}m + 1 \leq r \leq m - 1. \end{cases}$$

Let $e_r = \sum_{i=1}^r (-1)^{i+1} d_i$ for $1 \leq r \leq m$. Then

$$e_r = e_{r-1} + (-1)^{r+1} d_r.$$

When $m \equiv 0 \pmod{4}$,

$$e_i = \begin{cases} -\frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, 1 \leq i \leq \frac{1}{2}m; \\ \frac{i-1}{2} & \text{if } i \equiv 1 \pmod{2}, 1 \leq i \leq \frac{1}{2}m; \\ -\frac{m+1-i}{2} & \text{if } i \equiv 1 \pmod{2}, \frac{1}{2}m + 1 \leq i \leq m; \\ -(m-1 - \frac{m-i}{2}) & \text{if } i \equiv 0 \pmod{2}, \frac{1}{2}m + 1 \leq i \leq m. \end{cases}$$

When $m \equiv 2 \pmod{4}$,

$$e_i = \begin{cases} -\frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, 1 \leq i \leq \frac{1}{2}m; \\ \frac{i-1}{2} & \text{if } i \equiv 1 \pmod{2}, 1 \leq i \leq \frac{1}{2}m; \\ \frac{m-i}{2} & \text{if } i \equiv 0 \pmod{2}, \frac{1}{2}m + 1 \leq i \leq m; \\ m-1 - \frac{m+1-i}{2} & \text{if } i \equiv 1 \pmod{2}, \frac{1}{2}m + 1 \leq i \leq m. \end{cases}$$

That is,

$$(1) \quad e_i = e_{m+1-i} \pmod{(m-1)} \quad \text{for } 1 \leq i \leq \frac{1}{2}m.$$

When $m \equiv 0 \pmod{4}$, let C be the following closed trail:

$$((e_1)_0, (e_2)_1, (e_3)_0, \dots, (e_{\frac{1}{2}m-2})_1, (e_{\frac{1}{2}m-1})_0, (e_{\frac{1}{2}m})_1, (e_{\frac{1}{2}m+1})_0, \dots, (e_{m-1})_0, (e_m)_1).$$

By (1), C can also be written as

$$((e_1)_0, (e_2)_1, \dots, (e_{\frac{1}{2}m-1})_0, (e_{\frac{1}{2}m})_1, (e_{\frac{1}{2}m})_0, (e_{\frac{1}{2}m-1})_1, \dots, (e_2)_0, (e_1)_1).$$

The differences used in C are $d_1, d_2, \dots, d_m$.

Since

$$0 = e_1 < e_3 < \dots < e_{\frac{1}{2}m-1} = \frac{1}{4}m - 1$$

and

$$m - 2 = m - 1 + e_2 > m - 1 + e_4 > \dots > m - 1 + e_{\frac{1}{2}m} = \frac{3}{4}m - 1 > 0,$$

it follows that the vertices of C are distinct so that C is an m -cycle.

When $m \equiv 2 \pmod{4}$, let C be the following closed trail:

$$((e_1)_0, (e_2)_1, (e_3)_0, \dots, (e_{\frac{1}{2}m-1})_1, (e_{\frac{1}{2}m})_0, (e_{\frac{1}{2}m+1})_1, (e_{\frac{1}{2}m+2})_0, \dots, (e_{m-1})_0, (e_m)_1).$$

By (1), C can also be written as

$$((e_1)_0, (e_2)_1, (e_3)_0, \dots, (e_{\frac{1}{2}m-1})_1, (e_{\frac{1}{2}m})_0, (e_{\frac{1}{2}m})_1, (e_{\frac{1}{2}m-1})_0, (e_{\frac{1}{2}m-2})_1, \dots, (e_2)_0, (e_1)_1).$$

The differences used in C are $d_1, d_2, \dots, d_m$.

As before, it is easy to check that the vertices of C are distinct so that C is an m -cycle. Let $T = (C)$. Then T is an m -cycle system of $K_{m-1, m-1} + I$ and $C_m | K_{m-1, m-1} + I$. $\square$

3. Cycle decomposition of $K_{n,n} + I$ with $\frac{1}{2}m < n < \frac{3}{2}m$. The main purpose of this section is to prove Theorem 3.4, which considers cycle decomposition of $K_{n,n} + I$ with $\frac{1}{2}m < n < \frac{3}{2}m$. Lemmas 3.1, 3.2, and 3.3 will be needed in the proof of Theorem 3.4. The following notation will appear in the three lemmas.

For any integer x , let

$$\varepsilon(x) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod{2}, \\ 1 & \text{if } x \equiv 1 \pmod{2}. \end{cases}$$

LEMMA 3.1. *Let m and n be positive integers with $m \equiv 0 \pmod{2}$, $n \equiv 1 \pmod{2}$, and $\frac{1}{2}m < n < \frac{3}{2}m$. Let $g = \gcd(m, n) > 1$ and $n = s\frac{m}{g} - 1$. Let $D = \{2, 3, \dots, \frac{m}{g}, \frac{n}{g} + \frac{m}{2g} + 1\}$. Then $C_m | X(n; D)$.*

Proof. Let $V(X(n; D)) = \{i_j | i \in Z_n, j \in Z_2\}$. Let

$$d_i = \begin{cases} 0 & \text{if } i = 0, \\ i + 1 & \text{if } 1 \leq i \leq \frac{m}{g} - 1, \\ \frac{n}{g} + \frac{m}{2g} + 1 & \text{if } i = \frac{m}{g}. \end{cases}$$

For $1 \leq i \leq \frac{m}{g}$, let

$$\begin{cases} e_0 = 0, \\ e_i = e_{i-1} + (-1)^i d_i. \end{cases}$$

Then

$$e_i = \begin{cases} \frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, 0 \leq i \leq \frac{m}{g} - 2, \\ -\frac{i+3}{2} & \text{if } i \equiv 1 \pmod{2}, 1 \leq i \leq \frac{m}{g} - 1, \\ \frac{n}{g} & \text{if } i = \frac{m}{g}. \end{cases}$$

Let P be the trail of length $\frac{m}{g}$ given by

$$P = (e_0)_0, (e_1)_1, (e_2)_0, (e_3)_1, \dots, (e_{\frac{m}{g}-2})_0, (e_{\frac{m}{g}-1})_1, (e_{\frac{m}{g}})_0.$$

The differences used in P are $d_1, d_2, d_3, \dots, d_{\frac{m}{g}}$.

Since

$$0 = e_0 < e_2 < e_4 < \dots < e_{\frac{m}{g}} = \frac{n}{g}$$

and

$$s\frac{m}{g} - 3 = n + e_1 > n + e_3 > n + e_5 > \dots > n + e_{\frac{m}{g}-1} = s\frac{m}{g} - \frac{m}{2g} - 2,$$

the vertices of P are distinct so that P is a path. Moreover, the first and last vertices of P are the only ones which are congruent modulo $\frac{n}{g}$. It follows that

$$C = P \cup \left(P + \frac{n}{g}\right) \cup \left(P + \frac{2n}{g}\right) \cup \dots \cup \left(P + \frac{(g-1)n}{g}\right)$$

is a C_m .

In C , each difference in D occurs exactly g times, and $\{i_0, (i+d)_1\}$ incident with edges of difference d are all congruent modulo $\frac{n}{g}$. Let $T = (C)$. It follows that T is an m -cycle system of $X(n; D)$ and $C_m | X(n; D)$. $\square$

LEMMA 3.2. *Let m and n be positive integers with $m \equiv 0 \pmod{4}$, $n \equiv 1 \pmod{2}$, and $\frac{1}{2}m < n < \frac{3}{2}m$. Let $g = \gcd(m, n) > 1$ and $n = s\frac{m}{g} - 1$. Let $D_l = \{(l-1)\frac{m}{g} + 1, (l-1)\frac{m}{g} + 2, \dots, l\frac{m}{g} - 1, l\frac{m}{g}, (l-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + \varepsilon(l)\} \setminus \{(l-2)\frac{m}{g} + \frac{m}{2g} + \frac{n}{g} + \varepsilon(l+1)\}$ for $2 \leq l \leq s$. Then $C_m | X(n; D_l)$.*

Proof. Let $V(X(n; D_l)) = \{i_j | i \in Z_n, j \in Z_2\}$. For $l \equiv 0 \pmod{2}$, let

$$d_i = \begin{cases} 0 & \text{if } i = 0, \\ (l-1)\frac{m}{g} + i & \text{if } 1 \leq i < \frac{n}{g} - \frac{m}{2g} + 1, \\ (l-1)\frac{m}{g} + i + 1 & \text{if } \frac{n}{g} - \frac{m}{2g} + 1 \leq i \leq \frac{m}{g} - 1, \\ (l-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} & \text{if } i = \frac{m}{g}. \end{cases}$$

For $1 \leq i \leq \frac{m}{g}$, let

$$\begin{cases} e_0 = 0, \\ e_i = e_{i-1} + (-1)^i d_i. \end{cases}$$

Then

$$e_i = \begin{cases} \frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, 0 \leq i \leq \frac{n}{g} - \frac{m}{2g}, \\ -(l-1)\frac{m}{g} - \frac{i+1}{2} & \text{if } i \equiv 1 \pmod{2}, 0 \leq i \leq \frac{n}{g} - \frac{m}{2g}, \\ \frac{i}{2} + 1 & \text{if } i \equiv 0 \pmod{2}, \frac{n}{g} - \frac{m}{2g} + 1 \leq i \leq \frac{m}{g} - 1, \\ -(l-1)\frac{m}{g} - \frac{i+1}{2} & \text{if } i \equiv 1 \pmod{2}, \frac{n}{g} - \frac{m}{2g} + 1 \leq i \leq \frac{m}{g} - 1, \\ \frac{n}{g} & \text{if } i = \frac{m}{g}. \end{cases}$$

Let P be the trail of length $\frac{m}{g}$ given by

$$P = (e_0)_0, (e_1)_1, (e_2)_0, (e_3)_1, \dots, (e_{\frac{m}{g}-2})_0, (e_{\frac{m}{g}-1})_1, (e_{\frac{m}{g}})_0.$$

The differences used in P are $d_1, d_2, d_3, \dots, d_{\frac{m}{g}}$.

Since

$$0 = e_0 < e_2 < e_4 < \dots < e_{\frac{m}{g}} = \frac{n}{g}$$

and

$$(s-l+1)\frac{m}{g} - 2 = n + e_1 > n + e_3 > n + e_5 > \dots > n + e_{\frac{m}{g}-1} = (s-l+1)\frac{m}{g} - \frac{m}{2g} - 1,$$

the vertices of P are distinct so that P is a path. Moreover, the first and last vertices are the only ones which are congruent modulo $\frac{n}{g}$. It follows that

$$C = P \cup \left(P + \frac{n}{g}\right) \cup \left(P + \frac{2n}{g}\right) \cup \dots \cup \left(P + \frac{(g-1)n}{g}\right)$$

is a C_m .

In C , each difference in D occurs exactly g times, and $\{i_0, (i+d)_1\}$ incident with edges of difference d are congruent modulo $\frac{n}{g}$. Let $T = (C)$. It follows that T is an m -cycle system of $X(n; D_l)$ and $C_m|X(n; D_l)$ for l even.

For $l \equiv 1 \pmod{2}$, let

$$d_i = \begin{cases} 0 & \text{if } i = 0, \\ (l-1)\frac{m}{g} + i & \text{if } 1 \leq i < \frac{n}{g} - \frac{m}{2g}, \\ (l-1)\frac{m}{g} + i + 1 & \text{if } \frac{n}{g} - \frac{m}{2g} \leq i \leq \frac{m}{g} - 1, \\ (l-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1 & \text{if } i = \frac{m}{g}. \end{cases}$$

Then

$$e_i = \begin{cases} \frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, 0 \leq i \leq \frac{n}{g} - \frac{m}{2g} - 1, \\ -(l-1)\frac{m}{g} - \frac{i+1}{2} & \text{if } i \equiv 1 \pmod{2}, 0 \leq i \leq \frac{n}{g} - \frac{m}{2g} - 1, \\ \frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, \frac{n}{g} - \frac{m}{2g} \leq i \leq \frac{m}{g} - 1, \\ -(l-1)\frac{m}{g} - \frac{i+3}{2} & \text{if } i \equiv 1 \pmod{2}, \frac{n}{g} - \frac{m}{2g} \leq i \leq \frac{m}{g} - 1, \\ \frac{n}{g} & \text{if } i = \frac{m}{g}. \end{cases}$$

Let P be the trail of length $\frac{m}{g}$ given by

$$P = (e_0)_0, (e_1)_1, (e_2)_0, (e_3)_1, \dots, (e_{\frac{m}{g}-2})_0, (e_{\frac{m}{g}-1})_1, (e_{\frac{m}{g}})_0.$$

The differences used in P are $d_1, d_2, d_3, \dots, d_{\frac{m}{g}}$.

Since

$$0 = e_0 < e_2 < e_4 < \dots < e_{\frac{m}{g}} = \frac{n}{g}$$

and

$$(s-l+1)\frac{m}{g} - 2 = n + e_1 > n + e_3 > n + e_5 > \dots > n + e_{\frac{m}{g}-1} = (s-l+1)\frac{m}{g} - \frac{m}{2g} - 2,$$

the vertices of P are distinct so that P is a path. Moreover, the first and last vertices are the only ones which are congruent modulo $\frac{n}{g}$. It follows that

$$C = P \cup \left(P + \frac{n}{g}\right) \cup \left(P + \frac{2n}{g}\right) \cup \dots \cup \left(P + \frac{(g-1)n}{g}\right)$$

is a C_m .

In C , each difference in D occurs exactly g times, and $\{i_0, (i+d)_1\}$ incident with edges of difference d are congruent modulo $\frac{n}{g}$. Let $T = (C)$. It follows that T is an m -cycle system of $X(n; D_l)$ and $C_m|X(n; D_l)$ for l odd. $\square$

LEMMA 3.3. *Let m and n be positive integers with $m \equiv 2 \pmod{4}$, $n \equiv 1 \pmod{2}$, and $\frac{1}{2}m < n < \frac{3}{2}m$. Let $g = \gcd(m, n) > 1$ and $n = s\frac{m}{g} - 1$. Let $D_l = \{(l-1)\frac{m}{g} + 1, (l-1)\frac{m}{g} + 2, \dots, l\frac{m}{g} - 1, l\frac{m}{g}, (l-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1\} \setminus \{(l-2)\frac{m}{g} + \frac{m}{2g} + \frac{n}{g} + 1\}$ for $2 \leq l \leq s$. Then $C_m|X(n; D_l)$.*

Proof. Let $V(X(n; D)) = \{i_j | i \in Z_n, j \in Z_2\}$ and let

$$d_i = \begin{cases} 0 & \text{if } i = 0, \\ (l-1)\frac{m}{g} + i & \text{if } 1 \leq i < \frac{n}{g} - \frac{m}{2g} + 1, \\ (l-1)\frac{m}{g} + i + 1 & \text{if } \frac{n}{g} - \frac{m}{2g} + 1 \leq i \leq \frac{m}{g} - 1, \\ (l-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1 & \text{if } i = \frac{m}{g}. \end{cases}$$

For $1 \leq i \leq \frac{m}{g}$, let

$$\begin{cases} e_0 = 0, \\ e_i = e_{i-1} + (-1)^i d_i. \end{cases}$$

Then

$$e_i = \begin{cases} \frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, 0 \leq i \leq \frac{n}{g} - \frac{m}{2g}, \\ -(l-1)\frac{m}{g} - \frac{i+1}{2} & \text{if } i \equiv 1 \pmod{2}, 0 \leq i \leq \frac{n}{g} - \frac{m}{2g}, \\ \frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, \frac{n}{g} - \frac{m}{2g} + 1 \leq i \leq \frac{m}{g} - 1, \\ -(l-1)\frac{m}{g} - \frac{i+3}{2} & \text{if } i \equiv 1 \pmod{2}, \frac{n}{g} - \frac{m}{2g} + 1 \leq i \leq \frac{m}{g} - 1, \\ \frac{n}{g} & \text{if } i = \frac{m}{g}. \end{cases}$$

Let P be the trail of length $\frac{m}{g}$ given by

$$P = (e_0)_0, (e_1)_1, (e_2)_0, (e_3)_1, \dots, (e_{\frac{m}{g}-2})_0, (e_{\frac{m}{g}-1})_1, (e_{\frac{m}{g}})_0.$$

The differences used in P are $d_1, d_2, d_3, \dots, d_{\frac{m}{g}}$.

Since

$$0 = e_0 < e_2 < e_4 < \dots < e_{\frac{m}{g}} = \frac{n}{g}$$

and

$$(s-l+1)\frac{m}{g} - 2 = n + e_1 > n + e_3 > n + e_5 > \dots > n + e_{\frac{m}{g}-1} = (s-l+1)\frac{m}{g} - \frac{m}{2g} - 2,$$

the vertices of P are distinct so that P is a path. Moreover, the first and last vertices are the only ones which are congruent modulo $\frac{n}{g}$. It follows that

$$C = P \cup \left(P + \frac{n}{g}\right) \cup \left(P + \frac{2n}{g}\right) \cup \dots \cup \left(P + \frac{(g-1)n}{g}\right)$$

is a C_m . In C , each difference in D occurs exactly g times, and $\{i_0, (i+d)_1\}$ incident with edges of difference d are congruent modulo $\frac{n}{g}$. Let $T = (C)$. It follows that T is an m -cycle system of $X(n; D_l)$ and $C_m|X(n; D_l)$. $\square$

With the above preparations, we now prove the following theorem.

THEOREM 3.4. *Let m be an even integer and n be an odd integer with $\frac{1}{2}m < n < \frac{3}{2}m$. Then there exists an m -cycle system of $K_{n,n} + I$ if and only if m divides $n^2 + n$.*

Proof. The necessity is similar to that in Lemma 1.6; here we consider only the sufficiency. Let $g = \gcd(m, n)$. If $g = 1$, then since $n(n+1) \equiv 0 \pmod{m}$ and $n+1 < 2m$, we have $n = m-1$. By Lemma 2.3, there exists an m -cycle system of $K_{m-1, m-1} + I$.

If $n \neq m-1$, then $g > 1$. Since $n(n+1) \equiv 0 \pmod{m}$, we have $n+1 = s\frac{m}{g}$. When $m \equiv 0 \pmod{4}$, let

$$I = \left\{ \left\{ i_0, \left(i + (s-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + \varepsilon(s) \right)_1 \right\} \mid i \in Z_n \right\}.$$

We can put an additional difference $(s-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + \varepsilon(s)$ on Z_n . Then

$$Z_n \uplus \left\{ (s-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + \varepsilon(s) \right\} = \bigcup_{l=1}^{s-1} D_l \uplus D_s$$

where

$$D_1 = \left\{ 2, \dots, \frac{m}{g}, \frac{n}{g} + \frac{m}{2g} + 1 \right\}$$

and

$$D_l = \left\{ (l-1)\frac{m}{g} + 1, (l-1)\frac{m}{g} + 2, \dots, l\frac{m}{g} - 1, l\frac{m}{g}, (l-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + \varepsilon(l) \right\} \\ \setminus \left\{ (l-2)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + \varepsilon(l+1) \right\}$$

for $2 \leq l \leq s$.

By Lemmas 3.1 and 3.2, there exists an m -cycle system of $K_{n,n} + I$. This completes this case.

When $m \equiv 2 \pmod{4}$, let

$$I = \left\{ \left\{ i_0, \left((s-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1 \right)_1 \right\} \mid i \in Z_n \right\}.$$

We can put an additional difference $(s-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1$ on Z_n . Then

$$Z_n \uplus \left\{ (s-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1 \right\} = \bigcup_{l=1}^{s-1} D_l \uplus D_s,$$

where

$$D_1 = \left\{ 2, \dots, \frac{m}{g}, \frac{n}{g} + \frac{m}{2g} + 1 \right\}$$

and

$$D_l = \left\{ (l-1)\frac{m}{g} + 1, (l-1)\frac{m}{g} + 2, \dots, l\frac{m}{g} - 1, l\frac{m}{g}, (l-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1 \right\}$$

$$\setminus \left\{ (l-2)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1 \right\}$$

for $2 \leq l \leq s$.

By Lemmas 3.1 and 3.3, there exists an m -cycle system of $K_{n,n} + I$. This completes the proof. $\square$

4. Main result. Now we are in position to prove the main theorem of this paper.

THEOREM 4.1. *Let m be an even integer and n be an odd integer with $4 \leq m \leq 2n$. Then $K_{n,n} + I$ can be decomposed into cycles of length m if and only if m divides $n^2 + n$.*

Proof. The necessity can be found in Lemma 1.6; we need only prove the sufficiency. For $\frac{1}{2}m \leq n < \frac{3}{2}m$, we have proved the result in Theorems 2.2 and 3.4.

Now we need to consider the case $n \geq \frac{3}{2}m$. Let $n = qm + r$, where q is a positive integer and $\frac{1}{2}m \leq r < \frac{3}{2}m$. Let $V(K_{n,n}) = \{v_1, v_2, \dots, v_{qm+r}\} \cup \{u_1, u_2, \dots, u_{qm+r}\}$, $V_i = \{v_{(i-1)m+j} | 1 \leq j \leq m\}$, and $U_i = \{u_{(i-1)m+j} | 1 \leq j \leq m\}$ for $1 \leq i \leq q$. Let $V_{q+1} = \{v_{qm+j} | 1 \leq j \leq r\}$, $U_{q+1} = \{u_{qm+j} | 1 \leq j \leq r\}$, and $I = \{u_i v_i | 1 \leq i \leq n\}$.

Let $H_{i,i}$ be a subgraph of $K_{m-1,m-1} + I$ induced by $(V_i \setminus \{v_{(i-1)m+1}\}) \cup (U_i \setminus \{u_{(i-1)m+1}\})$ for $1 \leq i \leq q$. Then $H_{i,i} = K_{m-1,m-1} + I_{i,i}$, where

$$I_{i,i} = \{u_{(i-1)m+r} v_{(i-1)m+r} | 2 \leq r \leq m\}.$$

By Lemma 2.3, $C_m | H_{i,i}$ for $1 \leq i \leq q$. Let $T_{i,i}$ be the m -cycle system of $H_{i,i}$.

Let $H_{i,j}$ be a subgraph of $K_{m,m}$ induced by $V_i \cup U_j$, where $1 \leq i, j \leq q$ and $i \neq j$. Then $H_{i,j} = K_{m,m}$. By Theorem 1.2, $C_m | H_{i,j}$. Let $T_{i,j}$ be the m -cycle system of $H_{i,j}$.

Let $H_{r+1,m}^i$ be a subgraph of $K_{r+1,m}$ induced by $(V_{q+1} \cup \{v_{(i-1)m+1}\}) \cup U_i$ for $1 \leq i \leq q$. Then $H_{r+1,m}^i = K_{r+1,m}$. By Theorem 1.2, $C_m | H_{r+1,m}^i$ for $1 \leq i \leq q$. Let $T_{r+1,m}^i$ be the m -cycle system of $H_{r+1,m}^i$.

Let $H_{m,r+1}^i$ be a subgraph of $K_{m,r+1}$ induced by $V_i \cup (U_{q+1} \cup \{u_{(i-1)m+1}\})$, where $1 \leq i \leq q$. Then $H_{m,r+1}^i = K_{m,r+1}$. By Theorem 1.2, $C_m | H_{m,r+1}^i$ for $1 \leq i \leq q$. Let $T_{m,r+1}^i$ be the m -cycle system of $H_{m,r+1}^i$.

Let $H_{r,r}$ be a subgraph of $K_{r,r} + I$ induced by $V_{q+1} \cup U_{q+1}$. Then $H_{r,r} = K_{r,r} + I_{r,r}$, where $I_{r,r} = \{u_{qm+j} v_{qm+j} | 1 \leq j \leq r\}$. By Theorem 1.2, $C_m | H_{r,r}$. Let $T_{r,r}$ be the m -cycle system of $H_{r,r}$.

Let

$$T = \bigcup_{1 \leq i, j \leq q} T_{i,j} \bigcup_{1 \leq i \leq q} \left(T_{m,r+1}^i \bigcup T_{r+1,m}^i \right) \bigcup T_{r,r}.$$

Then T is an m -cycle system of $K_{n,n} + I$. This concludes the proof. $\square$

Acknowledgment. The authors would like to thank the referees for their helpful comments to improve the paper.

REFERENCES

- [1] B. ALSPACH AND H. GAVLAS, *Cycle decompositions of K_n and $K_n - I$* , J. Combin. Theory Ser. B, 81 (2001), pp. 77–99.
- [2] D. ARCHDEACON, M. DEBOWSKY, J. DINITZ, AND H. GAVLAS, *Cycle systems in the complete bipartite graph minus a one factor*, Discrete Math., 284 (2004), pp. 37–43.
- [3] B. BOLLOBAS, *Modern Graph Theory*, Springer-Verlag, New York, 1998.

- [4] J. MA, L. PU, AND H. SHEN, *Cycle decompositions of $K_{n,n} - I$* , SIAM J. Discrete Math., 20 (2006), pp. 603–609.
- [5] M. ŠAJNA, *Decomposition of the complete graph plus a 1-factor into cycles of equal length*, J. Combin. Des., 11 (2003), pp. 170–207.
- [6] M. ŠAJNA, *Cycle decompositions III: Complete graphs and fixed length cycles*, J. Combin. Des., 10 (2002), pp. 27–78.
- [7] D. SOTTEAU, *Decompositions of $K_{m,n}(K_{m,n}^*)$ into cycles (circuits) of length $2k$* , J. Combin. Theory Ser. B, 29 (1981), pp. 75–81.