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APPLICATION OF ERGUN EQUATION TO COMPUTATION OF CRITICAL SHEAR VELOCITY SUBJECT TO SEEPAGE

Nian-Sheng Cheng¹

Abstract

The Ergun equation that is widely used in the chemical engineering is generalised for applying to description of seepage through the sediment boundaries in this study. Comparisons with the measurements of the hydraulic gradient for flows through packed beds show that the Ergun equation may not be suitable for the transitional zone where both viscous and inertial effects are of the same order of magnitude. The generalised Ergun equation is then used to evaluate the critical shear velocity for incipient sediment motion subject to seepage. The computed results are finally compared to the experimental data and those predicted using the previously proposed approach.

Introduction

Seepage occurs across porous boundaries encountered in natural rivers and artificial irrigation canals. Depending on whether it increases or decreases the flow rate in the open channel, seepage can be identified as injection or suction. In the presence of seepage, the structural features of the open channel flow can be modified considerably. This includes variations in the average velocity profiles, turbulence intensities and boundary shear stresses, as reported by Cheng and Chiew (1998a), who conducted an experimental observation of the injection effects on open channel flows.

When sediment transport occurs on a porous bed, the bed particles experience an additional hydrodynamic force due to seepage. To determine the seepage force exerting on interfacial bed particles, Martin and Aral (1971) experimentally observed effects of injection and suction on the angle of repose of sediment particles. Their results were inconclusive in resolving whether the seepage force exerted on the top

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layer of particles was smaller than that on the particles several layers below the bed surface. Watters and Rao (1971) measured the drag and lift forces on a sphere which was placed at four typical positions on the bed. They found that whether the forces increase or decrease is generally dependent on the relative position of the sphere to its neighbours. However, in particular, the total drag force in the presence of injection always reduced regardless of the position of the sphere. Their measurements were affected appreciably by the Reynolds number because of the use of highly viscous fluid in their study.

Recently, Cheng and Chiew (1999) presented an analytical result for the threshold condition of sediment transport by including the additional hydrodynamic effect due to injection. Their derivation yields that the relative critical shear velocity can be associated with the relative seepage velocity in the following form:

$$\left(\frac{u_{*c}}{u_{*oc}}\right)^2 = 1 - \left(\frac{v_s}{v_{sc}}\right)^m \quad (1)$$

where u_{*c} = critical shear velocity for the condition of the bed injection, v_s = seepage velocity, u_{*oc} = critical shear velocity for $v_s = 0$, v_{sc} = critical seepage velocity for the condition of fluidisation, and m = exponent. With the dimensionless void scale L_* defined as

$$L_* = \left(\frac{\rho^2 g}{\mu^2}\right)^{\frac{1}{3}} \frac{\varepsilon}{1 - \varepsilon} d \quad (2)$$

where ρ = density of fluid, g = gravitational acceleration, μ = dynamic viscosity of fluid, ε = porosity, and d = diameter of particle, the exponent m can be evaluated using the following empirical equation:

$$m = \frac{1 + 0.00015L_*^2}{1 + 0.000075L_*^2} \quad (3)$$

It is noted that in the derivation made by Cheng and Chiew (1999), an exponential function was used to represent the relationship between the hydraulic gradient and seepage velocity. This approach was subsequently discussed by Niven (2000), who reported that the exponential relationship could be indirectly derived by performing a series of numerical computations based on the following binomial function, i.e., the Ergun equation:

$$i = 150 \frac{\mu(1-\varepsilon)^2}{\rho g d_v^2 \varepsilon^3} v_s + 1.75 \frac{1-\varepsilon}{g d_v \varepsilon^3} v_s^2 \quad (4)$$

where i = hydraulic gradient of seepage, $d_v = \phi d$ = diameter of the equivalent-volume sphere, and ϕ = sphericity of particle. For natural sediments, $\phi \approx 0.8$, and for angular particles, such as crushed sands, $\phi \approx$ (or $<$) 0.6 .

Eq. (4) was originally given by Ergun (1952) as an extension of Darcy law and has widely been used in the chemical engineering (Churchill 1988). This equation implies that the energy loss can be computed simply by summing up the two components, one caused by the viscous effect and the other due to the inertial effect (Niven 2002). Similar summations can also be found in other studies. For example, some studies on the settling velocity of particles suggest that the total drag could be obtained by adding the Stokes' term to that caused by the inertial flow (Chien and Wan 1999). Such approaches often yield relatively simple formulations of the physical phenomena. However, the relevant predictions may not agree well with measurements for the transitional zone where both viscous and inertial effects are of the same order of magnitude (Cheng 1997).

In this study, an alternative relationship between the hydraulic gradient and seepage velocity is first derived. This leads to a generalised Ergun equation, which can be used for different flow regimes including the transitional zone. This equation is then applied to evaluation of the critical shear velocity for the incipient motion of sediment particles subject to injection. The computed results of the critical shear velocity are finally compared with the previous analysis and experimental data.

Derivation

It is well known that if the flow through the porous medium is very slow, its equation of motion can be given by Darcy's law, which indicates that the hydraulic gradient, i , is linearly proportional to the seepage velocity, v_s :

$$i = \frac{I}{K} v_s \quad (5)$$

where K = coefficient of permeability. Since not including the effect of the kinetic energy of the flow, Darcy's law may be invalid, for example, for a porous medium comprised of coarse granular materials where the kinetic energy of the flow may be

significant. Therefore, extensive attempts have been made to develop non-linear relationships between the seepage velocity and hydraulic gradient for the transitional and inertial regions. Typically, these relationships can be expressed either in the binomial form ($i_s = av_s + bv_s^2$, where a and b = coefficients) or exponential form ($i_s = cv_s^m$, where $m = 1 \sim 2$ and c = coefficient), as mentioned previously.

Even being different from each other, both the binomial and exponential relationships have been proposed on the same ground that if the viscous effect is significant, Darcy's law applies and if the inertial force is dominant, the hydraulic gradient is proportional to the square of the velocity. Given the same properties of fluid and particles, the above argument can also be formulated as follows:

$$\frac{d \ln i}{d \ln v_s} = 1 \quad \text{for small } v_s \quad (6)$$

$$\frac{d \ln i}{d \ln v_s} = 2 \quad \text{for large } v_s \quad (7)$$

Furthermore, (6) and (7) can be combined in the form:

$$\frac{d \ln i}{d \ln v_s} = \frac{1 + 2\alpha v_s^n}{1 + \alpha v_s^n} \quad (8)$$

where α = coefficient and n = exponent. The interpolation between the two extreme conditions given by (8) is not unique. However, it is preferred here because it is mathematically simple. Integration of (8) with respect to v_s yields

$$i = \beta v_s \left(1 + \alpha v_s^n\right)^{1/n} \quad (9)$$

where β = coefficient. It is noted that for $n = 1$, (9) reduces to the binominal function. Evaluation of the two coefficients α and β can be made by comparing (9) and (4) for two limiting conditions, i.e., one dominated by the viscous effect and the other by the inertial effect. For these two conditions, two different forms of (9) can be obtained, respectively:

$$i = \beta v_s \quad \text{for small } v_s \quad (10)$$

and

$$i = \beta \alpha^{1/n} v_s^2 \quad \text{for large } v_s \quad (11)$$

Correspondingly, (4) can also be simplified, respectively, to

$$i = 150 \frac{\mu(1-\varepsilon)^2}{\rho g d_v^2 \varepsilon^3} v_s \quad \text{for small } v_s \quad (12)$$

$$i = 1.75 \frac{1-\varepsilon}{g d_v \varepsilon^3} v_s^2 \quad \text{for large } v_s \quad (13)$$

By comparing (10) with (12) and (11) with (13), one gets

$$\beta = 150 \frac{\mu(1-\varepsilon)^2}{\rho g d_v^2 \varepsilon^3} \quad (14)$$

$$\beta \alpha^{1/n} = 1.75 \frac{1-\varepsilon}{g d_v \varepsilon^3} \quad (15)$$

Substituting (14) into (15) leads to

$$\alpha = \left(\frac{1.75}{150} \frac{\rho d_v}{\mu(1-\varepsilon)} \right)^n \quad (16)$$

With (14) and (16), (9) can be rewritten as

$$i = 150 \frac{\mu(1-\varepsilon)^2}{\rho g d_v^2 \varepsilon^3} v_s \left[1 + \left(\frac{1.75}{150} \frac{\rho d_v}{\mu(1-\varepsilon)} v_s \right)^n \right]^{\frac{1}{n}} \quad (17)$$

Eq. (17) is identical to (4) for $n = 1$. Therefore, it can be referred to as a generalized Ergun equation. In addition, using the dimensionless parameters L^* given by (2) where d is replaced with the equivalent diameter d_v , and the seepage Reynolds number defined as

$$R_s = \frac{\rho v_s d_v}{\mu(1-\varepsilon)} \quad (18)$$

where $v_s/(1-\varepsilon)$ represents the average interstitial velocity, (17) can further be expressed as

$$i = \frac{150}{L_*^3} R_s \left[1 + \left(\frac{1.75}{150} R_s \right)^n \right]^{\frac{1}{n}} \quad (19)$$

Eq. (19) is plotted in Fig. 1 as iL_*^3 against R_s for several n -values. The figure shows that for the intermediate R_s -values, for example, $R_s = 10 \sim 1000$, the relationship of iL_*^3 and R_s changes obviously with the n -value.

To determine possible variations in the n -value, (19) is further plotted in Fig. 2

against laboratory measurements. Seven sets of experimental data are therefore used, which were collected previously by Mintz and Shubert (1957) for flows through various packed beds. The granular materials selected for the experiment included steel bearings ($d = 0.1646$ cm, $\rho_s/\rho = 7.43$, where ρ_s = density of particles and ρ = density of fluid), coal particles ($d = 0.0937\sim 0.779$ cm, $\rho_s/\rho = 1.661$) and quartz gravels ($d = 0.3665$ cm, $\rho_s/\rho = 2.64$). In computing L^* and R_s , the sphericity ϕ is taken as 1.0 for the steel bearings, 0.8 for the quartz gravels and 0.6 for the coal particles (Niven 2000 and references cited therein). Fig. 2 shows that each series of data can be represented well with (19) by choosing suitable n -value. For all the seven cases, the n -values vary from 0.63 to 0.80. It is interesting to note that they are all less than 1.0, rather than equal to 1.0 as implied by the Ergun equation. Such differences may not be easily observed if the experimental results are presented on the logarithmic scale, for example, in terms of the Reynolds number as found in many previous studies.

Evaluation of Critical Shear Velocity for Incipient Sediment Motion

As an alternative to (1), the relative critical shear velocity, u_{*c}/u_{*oc} , can also be expressed in terms of the relative hydraulic gradient, i/i_c , as follows (Cheng and Chiew 1999):

$$\left(\frac{u_{*c}}{u_{*oc}}\right)^2 = 1 - \frac{i}{i_c} \quad (20)$$

where i_c = critical hydraulic gradient for quick sand or fluidisation, which depends on the porosity and relative density of particles, i.e.

$$i_c = \left(\frac{\rho_s}{\rho} - 1\right)(1 - \varepsilon) \quad (21)$$

By applying (19) for the case of $i = i_c$, i_c can be related to the critical seepage velocity, v_{sc} , as

$$i_c = \frac{150}{L^3} R_{sc} \left[1 + \left(\frac{1.75}{150} R_{sc} \right)^n \right]^{\frac{1}{n}} \quad (22)$$

where $R_{sc} = \rho v_{sc} d_v [\mu(1-\varepsilon)]$ = critical seepage Reynolds number. Substituting (19) and (22) into (20) gives

$$\left(\frac{u_{*c}}{u_{*oc}}\right)^2 = I - \frac{R_s}{R_{sc}} \left[\frac{I + \left(\frac{1.75}{150} R_s\right)^n}{I + \left(\frac{1.75}{150} R_{sc}\right)^n} \right]^{\frac{1}{n}} \quad (23)$$

Eq. (23) indicates that the shear velocity ratio reduces with increasing seepage Reynolds number, for $R_s \leq R_{sc}$.

Comparison of Eq. (23) with Eq. (1)

For the comparison purpose, (1) can be changed to

$$\left(\frac{u_{*c}}{u_{*oc}}\right)^2 = I - \left(\frac{R_s}{R_{sc}}\right)^m \quad (24)$$

and (23) can be re-written as

$$\left(\frac{u_{*c}}{u_{*oc}}\right)^2 = I - \frac{R_s}{R_{sc}} \left[\frac{\left(\frac{R_s}{R_{sc}}\right)^n + \left(\frac{150}{1.75 R_{sc}}\right)^n}{I + \left(\frac{150}{1.75 R_{sc}}\right)^n} \right]^{\frac{1}{n}} \quad (25)$$

Eq. (24) shows that u_{*c}/u_{*oc} depends on R_s/R_{sc} and m that is a function of L^* , while (25) suggests that the relationship of u_{*c}/u_{*oc} and R_s/R_{sc} varies with R_{sc} if n is known. In the computations made subsequently, for each R_{sc} , L^* is first evaluated using (22) and then m is computed using (3).

It can be seen from (22) that L^* can be computed only if R_{sc} , i_c and n are known. As discussed in the previous section, the range of the n -values derived from the measurements is limited and for simplification, n can be taken as 0.7 as an average. On the other hand, as shown in (21), i_c varies depending on the porosity, ε , and relative density of the particles, ρ_s/ρ . For example, for the seven sets of data used in Fig. 2, the computations with (21) give that $i_c = 0.3 \sim 0.33$ for the coal particles, 0.86 for the quartz gravels, and 3.56 for the steel spheres.

Fig. 3 shows different relationships between the relative shear velocity and seepage Reynolds number, which are computed using (24) and (25), respectively, by setting $i_c = 0.3$ and 1.0, and $R_{sc} = 50, 300, 600$, and 1000. The results suggest that (25) agree with (24) for small i_c -values, and they differ from each other if i_c is increased.

Generally, the critical shear velocity predicted with (25) is larger than that with (24).

Comparison with Measured Critical Shear Velocities

The critical shear velocity for the incipient sediment motion subject to injection was experimentally observed with a glass-sided horizontal flume, which is 7.6 m long, 0.21 m wide and 0.4 m deep. The relevant information was detailed previously by Cheng and Chiew (1999), so only a summary is provided here. As sketched in Fig. 4, a sediment recess in the flume was prepared as a seepage zone and the injection flow introduced from its bottom. Three uniform sediments with median grain diameters of 0.63, 1.02 and 1.95 mm, respectively, were used as the bed materials. The particular sediment particles, which were used in the seepage zone, were also glued to the impermeable bed to furnish a consistent surface roughness throughout the channel for each test. The incipient motion of bed particles was observed according to the criteria of “weak movement” as described in Vanoni (1975). The velocity profiles, which were measured at the middle section of the seepage zone under the threshold condition, were then applied to the computation of the critical shear velocities. This can be done conveniently using the modified logarithmic law, which was proposed by Cheng and Chiew (1998b) by including the seepage effect on the velocity profile in open-channel flows.

Fig. 5 is a plot of the computed critical shear velocities in the presence of injection against the measured results. The agreement of the computations with the measurements is reasonably good considering that the laboratory observations associated with the incipient sediment motion are largely subjective. However, it is found that for the same conditions used in the experiments, in particular, for the test with the sand of 1.95 mm diameter, the predictions using the present method [Eq. (25)] are slightly larger than those obtained using the approach proposed by Cheng and Chiew (1999), i.e., (1) or (24). These differences are highlighted in Fig. 6. The deviations from the best fit line seem systematic for the different size particles. Such a result would be expected if the ‘weak movement’ definition has an inherent scale bias. Additional computations by varying the n -value, for example, from 0.7 to 1.3 indicate that the so-induced decrease in the predicted shear velocity is insignificant, when compared the marked scatter of the measured results.

Conclusions

The relationship of the hydraulic gradient and seepage velocity can be expressed in the form of binominal functions, for example, the Ergun equation that is widely used in the chemical engineering. This equation is generalised in this study, in particular, for including possible variations in the characteristics of flows in the transitional zone, where viscous and inertial effects are in the same order of magnitude. The generalised Ergun equation is then used to derive the formula for computing the critical shear velocity for the threshold condition of bed sediment subject to seepage. The computed results are found to be in good agreement with the measurements conducted in open channel flows with the bed injection, and the predictions obtained with the approach that was proposed previously based on an exponential relationship of the hydraulic gradient and seepage velocity.

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Appendix II. Notation

The following symbols are used in this paper:

- a = coefficient;
- b = coefficient;
- c = coefficient;
- d = diameter of particle;
- d_v = diameter of the equivalent-volume sphere;
- g = gravitational acceleration;
- i = hydraulic gradient of seepage;
- i_c = critical hydraulic gradient for fluidisation;
- K = coefficient of permeability;
- m = exponent;
- n = exponent;
- L^* = dimensionless void dimension;
- R_s = seepage Reynolds number;
- R_{sc} = critical seepage Reynolds number;
- u^*_c = critical shear velocity for the condition of seepage;
- u^*_{oc} = critical shear velocity for $v_s = 0$;

v_s	= (superficial) seepage velocity;
v_{sc}	= critical seepage velocity for the condition of fluidisation;
α	= coefficient;
β	= coefficient;
ε	= porosity;
μ	= dynamic viscosity of fluid;
ρ	= density of fluid;
ρ_s	= density of particles; and
φ	= sphericity of particle.

CAPTION FOR FIGURES

- FIG. 1.** Generalised Ergun Equation in Terms of iL_*^3 and R_s for Various Exponents.
- Fig. 2.** Comparisons of Generalised Ergun Equation to Experimental Results Given by Mintz and Shubert (1957)
- FIG. 3.** Relationships of Relative Shear Velocity and Seepage Reynolds Number
- Fig. 4.** Experimental Set-up for Observing Incipient Sediment Motion Subject to Injection
- FIG. 5.** Comparison of Computed Critical Shear Velocities with Experimental Data
- FIG. 6.** Comparison of Critical Shear Velocities Computed Using (24) and (25), Respectively

Fig.1.

Fig. 2.

Fig. 2.

Fig. 2.

Fig. 2.

Fig. 3.

Fig. 4.

Fig. 5.

Fig. 6.