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MINIMAL REPRESENTATIONS OF LOCALLY PROJECTIVE AMALGAMS

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ABSTRACT

A locally projective amalgam is formed by the stabilizer $G(x)$ of a vertex x and the global stabilizer $G\{x, y\}$ of an edge containing x in a group G , acting faithfully and locally finitely on a connected graph Γ of valency $2^n - 1$ so that (i) the action is 2-arc-transitive, (ii) the sub-constituent $G(x)^{\Gamma(x)}$ is the linear group $\mathrm{SL}_n(2) \cong \mathrm{L}_n(2)$ in its natural doubly transitive action, and (iii) $[t, G\{x, y\}] \leq O_2(G(x) \cap G\{x, y\})$ for some $t \in G\{x, y\} \setminus G(x)$. Djoković and Miller used the classical Tutte theorem to show that there are seven locally projective amalgams for $n = 2$. Trofimov's theorem was used by the first author and Shpectorov to extend the classification to the case $n \geq 3$. It turned out that for $n \geq 3$, besides two infinite series of locally projective amalgams (embedded into the groups $\mathrm{AGL}_n(2)$ and $O_{2n}^+(2)$), there are exactly twelve exceptional ones. Some of the exceptional amalgams are embedded into sporadic simple groups M_{22} , M_{23} , Co_2 , J_4 and BM . For a locally projective amalgam $\mathcal{A}$, the minimal degree $m = m(\mathcal{A})$ of its complex representation (which is a faithful completion into $\mathrm{GL}_m(\mathbb{C})$) is calculated. The minimal representations are analysed and three open questions on exceptional locally projective amalgams are answered. It is shown that

- (a) $\mathcal{A}_4^{(1)}$ possesses $\mathrm{SL}_{20}(13)$ as a faithful completion in which the third geometric subgroup is improper;
- (b) $\mathcal{A}_4^{(2)}$ possesses the alternating group Alt_{64} as a completion constrained at levels 2 and 3;
- (c) $\mathcal{A}_4^{(5)}$ possesses Alt_{256} as a completion which is constrained at level 2 but not at level 3.

1. Locally projective amalgams

An amalgam of rank m is a collection

$$\mathcal{A} = \{(G^{[i]}, *_i) \mid 0 \leq i \leq m-1\}$$

of m groups $(G^{[i]}, *_i)$, $0 \leq i \leq m-1$, such that for all $0 \leq i < j \leq m-1$ the intersection $G^{[ij]} := G^{[i]} \cap G^{[j]}$ of the element sets is non-empty and the group operations $*_i$ and $*_j$ coincide, when restricted to $G^{[ij]}$. If $(G, *)$ is a group and $G^{[0]}, G^{[1]}, \dots, G^{[m-1]}$ are subgroups in G , then $\{(G^{[i]}, *|_{G^{[i]}}) \mid 0 \leq i \leq m-1\}$ is an amalgam.

Let $\mathcal{A} = \{(G^{[i]}, *_i) \mid 0 \leq i \leq m-1\}$ be an amalgam, $(G, *)$ be a group, and φ be a mapping of the union of the element sets of the groups constituting $\mathcal{A}$ into G such that for every $0 \leq i \leq m-1$ and all $g, h \in G^{[i]}$ the equality

$$\varphi(g *_i h) = \varphi(g) * \varphi(h)$$

holds (that is, the restriction of φ to each $G^{[i]}$ is a homomorphism). Then the pair (G, φ) is called a *completion* of $\mathcal{A}$ (here G is the *completion group* and φ is the *completion map*). The completion is said to be *faithful* if φ is injective and *generating* if G is generated by the image of φ . A completion $(\tilde{G}, \tilde{\varphi})$ is said to be *universal* if for every completion (G, φ) there is a homomorphism ψ of $\tilde{G}$ into G , such

that φ is the composition of $\tilde{\varphi}$ and ψ . A universal completion is always generating and the universal completion group is unique up to isomorphism. Furthermore, an amalgam possesses a faithful completion (which is not always the case) if and only if its universal completion is faithful. It is natural to define two representations $(G, \varphi^{(1)})$ and $(G, \varphi^{(2)})$ to be *equivalent* whenever there is $l \in G$, such that

$$\varphi^{(1)}(a) = l^{-1} \varphi^{(2)}(a) l$$

for every $a \in \mathcal{A}$. Clearly equivalent representations generate conjugate subgroups in G .

Whenever the group operations are clear from the context or irrelevant, we simply write

$$\mathcal{A} = \{G^{[i]} \mid 0 \leq i \leq m-1\}$$

for an amalgam of rank m and also for the union of the element sets of the groups constituting the amalgam. We will also drop the explicit reference to the completion maps whenever it does not cause confusion (in this case by ‘completion’ we mean the completion group). Recall that for an amalgam $\mathcal{A} = \{G^{[0]}, G^{[1]}\}$ of rank 2 the universal completion is faithful and the universal completion group is isomorphic to the free product of $G^{[0]}$ and $G^{[1]}$ amalgamated over the common subgroup $G^{[01]}$. The free amalgamated product is infinite whenever $G^{[01]}$ is proper in both $G^{[0]}$ and $G^{[1]}$.

Let Γ be an undirected connected graph whose vertex set is denoted by $V(\Gamma)$ and whose edge set is denoted by $E(\Gamma)$ (we treat the latter as a set of 2-element subsets of $V(\Gamma)$). Let G be a group of automorphisms of Γ , and suppose that the action of G on Γ is 2-arc-transitive, that is, transitive on the set $\{(y, x, z) \mid y, x, z \in V(\Gamma), \{y, x\}, \{x, z\} \in E(\Gamma), y \neq z\}$ of 2-arcs in Γ . For $x \in V(\Gamma)$ let

$$\Gamma(x) = \{y \mid y \in V(\Gamma), \{x, y\} \in E(\Gamma)\}$$

be the set of neighbours of x in Γ and

$$G(x) = \{g \mid g \in G, g(x) = x\}$$

be the stabilizer (subgroup) of x in G . We always assume that the action is *locally finite*, that is, that $G(x)$ is of finite order. Then, because of the 2-arc-transitivity, the permutation group $G(x)^{\Gamma(x)}$ (known as the subconstituent) is doubly transitive. We further assume that the subconstituent $G(x)^{\Gamma(x)}$ is isomorphic to the (projective) linear group $L_n(2) \cong \text{SL}_n(2)$ in dimension n over the field $\text{GF}(2)$ of two elements, acting on the set of 1-dimensional subspaces in the natural module. In particular, the valency $|\Gamma(x)|$ of Γ is $2^n - 1$. It is well known (cf. [4, Proposition 9.4.1]) that a 2-arc-transitive action with projective linear subconstituent is one of the following:

- (a) 3-arc-transitive transitive;
- (b) strictly 2-arc-transitive of collineation type;
- (c) strictly 2-arc-transitive of correlation type (here $n \geq 4$).

In the $\text{GF}(2)$ case when $n \geq 3$, the collineation case (b) can be singled out by the condition

$$G\{x, y\}/O_2(G(x, y)) \cong L_{n-1}(2) \times 2, \tag{1}$$

where $G\{x, y\}$ and $G(x, y) = G(x) \cap G(y)$ are the global and the vertex-wise stabilizers in G of the edge $\{x, y\}$, while $O_2(G(x, y))$ is the largest normal 2-subgroup

of $G(x, y)$. Condition (1) is equivalent to

$$[t, G\{x, y\}] \leq O_2(G(x, y)) \quad (2)$$

for some $t \in G\{x, y\} \setminus G(x, y)$ (modulo $O_2(G(x, y))$, the element t generates the direct factor of order 2 in (1)). Notice that in the $\text{GF}(2)$ case, (1) and (2) always hold when $n = 2$.

DEFINITION 1.1. Let G be a group acting faithfully, locally finitely and 2-arc-transitively on a connected graph Γ of valency $2^n - 1$, $n \geq 2$, so that $G(x)^{\Gamma(x)} \cong L_n(2)$ and the equivalent conditions (1) and (2) hold. Let $x \in V(\Gamma)$, $y \in \Gamma(x)$. Then the amalgam

$$\mathcal{A} = \{G^{[0]} = G(x), G^{[1]} = G\{x, y\}\}$$

is called a *locally projective amalgam* (we denote this amalgam by $\mathcal{A}(G, \Gamma)$).

The following easy lemma provides an abstract definition of locally projective amalgams.

LEMMA 1.2. A rank 2 amalgam $\mathcal{A} = \{G^{[0]}, G^{[1]}\}$ formed by finite groups is a locally projective amalgam if and only if it satisfies the following conditions.

- (A1) $G^{[0]}/O_2(G^{[0]}) \cong L_n(2)$ for some $n \geq 2$.
- (A2) $G^{[01]} := G^{[0]} \cap G^{[1]}$ is the stabilizer of a 1-space V_1 in the natural module V of $G^{[0]}/O_2(G^{[0]})$.
- (A3) $|G^{[1]} : G^{[01]}| = 2$.
- (A4) $G^{[1]}/O_2(G^{[01]}) \cong L_{n-1}(2) \times 2$, or, equivalently, $[t, G^{[1]}] \leq O_2(G^{[01]})$ for some $t \in G^{[1]} \setminus G^{[0]}$.
- (A5) No non-identity subgroup in $G^{[01]}$ is normal in both $G^{[0]}$ and $G^{[1]}$.

Proof. It is easy to check that a locally projective amalgam satisfies (A1)–(A5). On the other hand, let $\mathcal{A} = \{G^{[0]}, G^{[1]}\}$ be an amalgam satisfying (A1)–(A5) and let G be the universal completion of $\mathcal{A}$ (recall that this completion is faithful and generating). We identify $G^{[0]}$ and $G^{[1]}$ with their images under the (implicit) completion map. Define $\Gamma = \Gamma(\mathcal{A}, G)$ to be the graph, such that $V(\Gamma)$ is the set of right cosets of $G^{[0]}$ in G and two distinct cosets from $V(\Gamma)$ are adjacent if there is a coset of $G^{[1]}$ in G which intersects them both. By (A3), every $G^{[1]}$ -coset intersects exactly two $G^{[0]}$ -cosets, so that $E(\Gamma)$ corresponds to the set of right cosets of $G^{[1]}$ in G (notice that Γ is a tree, since the completion G is universal). Consider the natural action of G on $V(\Gamma)$ by right translations. Clearly the action preserves the adjacency (that is, G acts as a group of automorphisms of Γ). Furthermore, in this action $G^{[0]}$ is the stabilizer of the vertex $x := G^{[0]}$, while $G^{[1]}$ is the global stabilizer of the edge $\{x, y\}$, where $y = G^{[0]}t$ with t as in (A4). By (A1) and (A2), the valency of Γ is $2^n - 1$; the action of G on Γ is 2-arc-transitive and the subconstituent $G(x)^{\Gamma(x)}$ is isomorphic to $L_n(2)$. Clearly (1) and (2) hold because of (A4). Since G is generated by $G^{[0]}$ and $G^{[1]}$, Γ is connected. We claim that the action of G on Γ is faithful. By a general result from the permutation group theory, the kernel N of the action of G on Γ is the largest subgroup in $G^{[0]}$ normal in G . Since $G^{[01]}$ is maximal in $G^{[0]}$, either (i) $NG^{[01]} = G^{[0]}$, or (ii) $N \leq G^{[01]}$. In case (i), since $G^{[1]}$ normalizes both $G^{[01]}$ and N , it normalizes the whole $G^{[0]}$, and therefore $G^{[0]}$ has index 2 in G , which is impossible since G is the (infinite) free product of $G^{[0]}$ and

TABLE 1. *Exceptional amalgams.*

n	$\mathcal{A}$	$\frac{G^{[0]}/O_2(G^{[0]})}{O_2(G^{[0]})}$	$\frac{\hat{G}^{[2]}}{K^{[2]}}$	$\frac{\hat{G}^{[3]}}{K^{[3]}}$	$\frac{\hat{G}^{[4]}}{K^{[4]}}$
3	$\mathcal{A}_3^{(1)}$	$\frac{L_3(2)}{2^3}$	$\frac{\text{Sym}_3 \wr \text{Sym}_2}{2^4}$		
	$\mathcal{A}_3^{(2)}$	$\frac{L_3(2)}{2^3}$	$\frac{\text{Sym}_5}{2^4}$		
	$\mathcal{A}_3^{(3)}$	$\frac{L_3(2)}{2^3}$	$\frac{\text{Sym}_5}{2^4}$		
	$\mathcal{A}_3^{(4)}$	$\frac{L_3(2)}{2^3 \times 2}$	$\frac{2^4 : (\text{Sym}_3 \wr \text{Sym}_2)}{2^5}$		
	$\mathcal{A}_3^{(5)}$	$\frac{L_3(2)}{2^3 \times 2}$	$\frac{\text{Sym}_5}{2^5}$		
4	$\mathcal{A}_4^{(1)}$	$\frac{L_4(2)}{1}$	$\frac{\text{Sym}_5}{2^4 : 3}$	$\frac{1}{1}$	
	$\mathcal{A}_4^{(2)}$	$\frac{L_4(2)}{2^6}$	$\frac{\text{Sym}_5 \times 2}{2^{1+8} : \text{Sym}_3}$	$\frac{L_6(2)}{2^6}$	
	$\mathcal{A}_4^{(3)}$	$\frac{L_4(2)}{2^{1+4+6}}$	$\frac{\text{Sym}_5}{2^4+10 : \text{Sym}_3}$	$\frac{\text{Aut } M_{22}}{2^{10}}$	
	$\mathcal{A}_4^{(4)}$	$\frac{L_4(2)}{2^4+4+6}$	$\frac{\text{Sym}_5}{2^3+12+2 : \text{Sym}_3}$	$\frac{3 \cdot \text{Aut } M_{22}}{2^{1+12}}$	
	$\mathcal{A}_4^{(5)}$	$\frac{L_4(2)}{2^4+4+6}$	$\frac{\text{Sym}_5 \times 2}{2^3+12+2 : \text{Sym}_3}$	$\frac{L_6(2):2}{2^{1+12}}$	
5	$\mathcal{A}_5^{(1)}$	$\frac{L_5(2)}{2^{10}}$	$\frac{\text{Sym}_5}{2^3+12 : L_3(2)}$	$\frac{\text{Aut } M_{22}}{2^{1+12} : 3}$	$\frac{1}{1}$
	$\mathcal{A}_5^{(2)}$	$\frac{L_5(2)}{2^5+5+10+10}$	$\frac{\text{Sym}_5}{2^3 : [2^{32}] : L_3(2)}$	$\frac{\text{Aut } M_{22}}{2^2+10+20 : \text{Sym}_3}$	$\frac{Co_2}{2^{1+22}}$

$G^{[1]}$ amalgamated over $G^{[0]}$. In case (ii), N is trivial by (A5). Thus $\mathcal{A}$ is indeed a locally projective amalgam (isomorphic to $\mathcal{A}(G, \Gamma)$). $\square$

The classification of the locally projective amalgams became possible after the results were proved by Trofimov (cf. [11] and references therein) on vertex stabilizers in groups acting on graphs with projective linear subconstituents. By [6, Theorem 1], a locally projective amalgam is either a member of two infinite series or one of twelve explicitly described exceptional examples. The infinite series correspond to the action of $\text{AGL}(n, 2)$ on the (complete graph K_{2^n} on the) set of vectors of an n -dimensional $\text{GF}(2)$ -space and the action of the orthogonal group $O_{2n}^+(2)$ on its dual polar graph $D^+(2n, 2)$. The exceptional amalgams are given in Table 1.

In order to explain the data in rows 3–5 of Table 1, we need some definitions. In the theory of locally projective amalgams (with $n \geq 3$), an important role is played by the geometric subgraphs and subgroups (cf. [4, Chapter 9.5]). By (A2), the subgroup $G^{[01]}$ is the stabilizer in $G^{[0]}$ of a 1-subspace V_1 in the natural module V of $G^{[0]}/O_2(G^{[0]}) \cong \text{L}_n(2)$. Let

$$0 < V_1 < V_2 < \dots < V_{n-1} < V$$

be a maximal flag in V containing V_1 . Let $G^{[0i]}$ be the stabilizer of V_i in $G^{[0]}$, put $G^{[01i]} = G^{[0i]} \cap G^{[1]}$, and let $G^{[1i]}$ be the normalizer of $G^{[01i]}$ in $G^{[1]}$ (it is easy to see that $G^{[1i]} = \langle G^{[01i]}, t \rangle$, where t is as in (A4)). By (A3), $|G^{[1i]} : G^{[01i]}| = 2$ for $2 \leq i \leq n-1$. Let $K^{[i]}$ be the largest subgroup in $G^{[01i]}$ normal in both $G^{[0i]}$ and $G^{[1i]}$. Then

$$\mathcal{A}^{[i]} = \{G^{[0i]}/K^{[i]}, G^{[1i]}/K^{[i]}\}$$

is again a locally projective amalgam (this observation is fundamental in the inductive approach to the classification of the locally projective amalgams). Let G be a faithful completion of $\mathcal{A}$ (we identify $\mathcal{A}$ with its image in G) and let $G^{[i]}$ be the subgroup in G generated by $G^{[0i]}$ and $G^{[1i]}$. Then $G^{[i]}$ is called the *geometric subgroup* at level i . The structure of $G^{[i]}$ depends on the completion. On the other hand, $K^{[i]}$ is contained in $G^{[0i]}$ and therefore it does not depend on G . Similarly, the image $\widehat{G}^{[i]}$ of $G^{[i]}$ in the outer automorphism group of $K^{[i]}$ is generated by the images of $G^{[0i]}$ and $G^{[1i]}$, and hence this is also a characteristic of the amalgam $\mathcal{A}$. The structure of the groups $K^{[i]}$ and $\widehat{G}^{[i]}$ for $i = 3, 4$ and 5 is given in rows 3–5 in Table 1.

It is clear that

$$\widehat{G}^{[i]} \cong G^{[i]} / (K^{[i]} C_{G^{[i]}}(K^{[i]})).$$

It might happen that $G^{[i]}$ is the whole of G , although for this $K^{[i]}$ must be trivial. This never happens in the amalgams which form an infinite sequence, and from Table 1 we observe that the geometric subgroup at level i is proper unless

$$\mathcal{A} = \mathcal{A}_4^{(1)} \quad \text{or} \quad \mathcal{A}_5^{(1)} \quad \text{and } i = 3 \text{ or } 4,$$

respectively. The fourth Janko group J_4 is a completion of $\mathcal{A}_5^{(1)}$ in which the fourth geometric subgroup is improper. Until now it was not known whether $\mathcal{A}_4^{(1)}$ possessed a faithful completion in which the third geometric subgroup was improper.

A remarkable feature of the exceptional locally projective amalgams is that many of them possess sporadic groups as faithful generating completions. These and other ‘nice’ completions can be selected from the vast number of all completions using the following concept.

A completion G is called *constrained at level i* if

$$C_{G^{[i]}}(K^{[i]}) \leq K^{[i]}.$$

Clearly $G^{[i]}$ always contains $K^{[i]}$ as a normal subgroup and $\widehat{G}^{[i]}$ as a factor-group over the normal subgroup $K^{[i]} C_{G^{[i]}}(K^{[i]})$ containing $K^{[i]}$. Thus a completion is constrained at level i if the geometric subgroup at level i attains the lower bound on its order implied by the previous sentence. On the other hand, if G is the universal completion of $\mathcal{A}$, then $G^{[i]}$ is the free amalgamated product of $G^{[0i]}$, $G^{[1i]}$ over $G^{[01i]}$, so in this case the geometric subgroup at level i is the largest possible.

For the amalgams

$$\mathcal{A}_3^{(2)}, \mathcal{A}_3^{(4)}, \mathcal{A}_3^{(5)}, \mathcal{A}_4^{(1)}, \mathcal{A}_4^{(3)}, \mathcal{A}_4^{(4)}, \mathcal{A}_5^{(1)}, \mathcal{A}_5^{(2)},$$

the groups M_{22} , $(A_8 \times A_8).2^2$, $\text{Aut } M_{22}$, M_{23} , Co_2 , J_4 , J_4 and BM respectively are completions which are constrained at level 2. It was shown in [6] that

$$\mathcal{A}_3^{(1)} \quad \text{and} \quad \mathcal{A}_3^{(3)}$$

do not possess completions which are constrained at level 2. The existence question of completions which are constrained at level 2 for the amalgams

$$\mathcal{A}_4^{(2)} \quad \text{and} \quad \mathcal{A}_4^{(5)}$$

was left open in [6].

In this paper we turn to *representations* of locally projective amalgams which are completions into the general linear groups $\text{GL}_m(\mathbb{C})$. Considering representations in the smallest possible dimension m , we were lucky to construct a few new nice completions and to answer three questions left open in [6]. We take a minimal

TABLE 2. *Dimensions of minimal representations.*

$\mathcal{A}$	$\mathcal{A}_3^{(1)}$	$\mathcal{A}_3^{(2)}$	$\mathcal{A}_3^{(3)}$	$\mathcal{A}_3^{(4)}$	$\mathcal{A}_3^{(5)}$	$\mathcal{A}_4^{(1)}$	$\mathcal{A}_4^{(2)}$	$\mathcal{A}_4^{(3)}$	$\mathcal{A}_4^{(4)}$	$\mathcal{A}_4^{(5)}$	$\mathcal{A}_5^{(1)}$	$\mathcal{A}_5^{(2)}$
$m(\mathcal{A})$	7	20	20	14	20	20	63	23	1333	255	1333	4371

representation of $\mathcal{A}_4^{(1)}$ in dimension 20, reduce it modulo 13, and obtain a faithful generating completion in $\mathrm{SL}_{20}(13)$ in which the third geometric subgroup is improper. Adding the trivial 1-dimensional representation to minimal representations of

$$\mathcal{A}_4^{(2)} \quad \text{and} \quad \mathcal{A}_4^{(5)}$$

in dimensions 63 and 255, respectively, we obtain permutation representations leading to completions constrained at level 2 for these amalgams into the alternating groups Alt_{64} and Alt_{256} , respectively.

For the sake of completeness, we calculate the dimensions of minimal representations of all locally projective amalgams for $n \geq 3$.

THEOREM 1.3. *Let $\mathcal{A}$ be a locally projective amalgam and let $m = m(\mathcal{A})$ be the smallest positive integer such that $\mathcal{A}$ possesses $\mathrm{GL}_m(\mathbb{C})$ as a faithful completion group (which may or may not be generating). Then the following hold.*

(i) *If $\mathcal{A} = \mathcal{A}(\mathrm{AGL}_n(2), K_{2^n})$, then $m(\mathcal{A}) = 3, 7$ and $2^n - 2$ for $n = 3, n = 4$ and $n \geq 5$, respectively.*

(ii) *If $\mathcal{A} = \mathcal{A}(O_{2n}^+(2), D^+(2n, 2))$, then $m(\mathcal{A}) = 7, 28$ and $(2^n - 1)(2^{n-1} - 1)/3$ for $n = 3, n = 4$ and $n \geq 5$, respectively.*

(iii) *If $\mathcal{A}$ is an exceptional amalgam, then $m(\mathcal{A})$ is given in Table 2.*

A 1333-dimensional representation of $\mathcal{A}_5^{(1)}$ was used in [9] to establish the uniqueness of the Janko group J_4 . By Theorem 1.3, this representation is minimal.

2. Representations of locally projective amalgams

Let (G, φ) be a faithful completion of a locally projective amalgam $\mathcal{A} = \{G^{[0]}, G^{[1]}\}$, where $G = \mathrm{GL}_m(\mathbb{C})$. Such a completion will be called a faithful representation of $\mathcal{A}$ in dimension m over the complex numbers or simply a representation of $\mathcal{A}$. Let $\varphi^{[0]}$, $\varphi^{[1]}$ and $\varphi^{[01]}$ be the restrictions of φ to $G^{[0]}$, $G^{[1]}$ and $G^{[01]}$, respectively. Then, for $x = 0, 1$ and 01 , the restrictions $\varphi^{[x]}$ are group representations and both $\varphi^{[0]}$ and $\varphi^{[1]}$, restricted to $G^{[01]}$, are equal to $\varphi^{[01]}$. On the other hand, having a pair of group representations $\varphi_i : G^{[i]} \rightarrow \mathrm{GL}_m(\mathbb{C})$ whose restrictions to $G^{[01]}$ coincide, we can reconstruct the whole completion map φ .

LEMMA 2.1. *Let $\chi^{[0]}$ and $\chi^{[1]}$ be characters of $G^{[0]}$ and $G^{[1]}$, respectively. Then for the existence of a representation $(\mathrm{GL}_m(\mathbb{C}), \varphi)$ such that the restrictions of φ to $G^{[0]}$ and $G^{[1]}$ afford $\chi^{[0]}$ and $\chi^{[1]}$, respectively, it is necessary and sufficient for the restrictions to $G^{[01]}$ of $\chi^{[0]}$ and $\chi^{[1]}$ to coincide.*

Proof. We can see that the necessity is clear by considering the characters of the representations $\varphi^{[0]}$ and $\varphi^{[1]}$ in the paragraph before Lemma 2.1. To establish

the sufficiency, suppose that $\chi^{[0]}$ and $\chi^{[1]}$ satisfy the hypothesis. Let $\psi^{[0]}$ and $\psi^{[1]}$ be representations affording these characters. Since the restrictions of the characters to $G^{[01]}$ coincide, there is $l \in \mathrm{GL}_m(\mathbb{C})$ such that $\psi^{[0]}(a) = l^{-1}\psi^{[1]}(a)l$ for every $a \in G^{[01]}$. Define $\varphi : \mathcal{A} \longrightarrow \mathrm{GL}_m(\mathbb{C})$ by putting $\varphi(a) = \psi^{[0]}(a)$ for $a \in G^{[0]}$ and $\varphi(a) = l^{-1}\psi^{[1]}(a)l$ for $a \in G^{[1]}$. Then it only remains to check that φ is injective. Since both $\chi^{[0]}$ and $\chi^{[1]}$ are faithful, the only obstacle is that the whole of $l^{-1}\psi^{[1]}(G^{[1]})l$ might be in $\psi^{[0]}(G^{[0]})$. However, this is impossible, since $G^{[01]}$ is normal in $G^{[1]}$ and self-normalized in $G^{[0]}$. $\square$

The next lemma, which is the main result of [9], supplies the number $k(\chi^{[0]}, \chi^{[1]})$ of equivalence classes of representations corresponding to a pair of characters satisfying the hypothesis of Lemma 2.1.

LEMMA 2.2. *Let $(\mathrm{GL}_m(\mathbb{C}), \varphi)$ be a representation of $\mathcal{A} = \{G^{[0]}, G^{[1]}\}$ such that the restrictions of φ to $G^{[0]}$ and $G^{[1]}$ afford $\chi^{[0]}$ and $\chi^{[1]}$, respectively. Then the equivalence classes of representations corresponding to the pair $(\chi^{[0]}, \chi^{[1]})$ are in one-to-one correspondence with the set of $(C^{[0]}, C^{[1]})$ double cosets in $C^{[01]}$, where*

$$C^{[x]} = C_{\mathrm{GL}_m(\mathbb{C})}(\varphi(G^{[x]})) \quad \text{for } x = 0, 1, 01.$$

In particular, there is precisely one equivalence class if and only if $C^{[01]} = C^{[0]}C^{[1]}$.

By Schur's lemma, the isomorphism type of $C^{[x]}$ is determined by the multiplicities of the irreducible constituents of $\chi^{[x]}$. Suppose that $\chi^{[01]}$ is multiplicity-free. Then $C^{[01]}$ is the set of ordered sequences of elements from $\mathbb{C}^*$, whose components are indexed by the irreducible constituents of $\chi^{[01]}$, and the multiplication is componentwise. Then for $i = 0$ and 1 , the subgroup $C^{[i]}$ consists of the sequences from $C^{[01]}$ subject to the condition that components, corresponding to $\chi^{[01]}$ -constituents fused in $\chi^{[i]}$, are equal. In this case the number $k(\chi^{[0]}, \chi^{[1]})$ is uniquely determined by the characters and easy to calculate.

Next we aim to deduce an existence condition for representations in terms of the characters of $G^{[0]}$ only. If K and χ are a conjugacy class and a character of $G^{[0]}$, then $\chi(K)$ clearly means $\chi(k)$ for some (and hence any) $k \in K$. Since $G^{[1]}$ contains $G^{[01]}$ as a normal subgroup (of index 2), the former acts on the conjugacy classes and on the irreducible characters of the latter. Let $t \in G^{[1]} \setminus G^{[01]}$ (as in (A4), say), let I be a conjugacy class of $G^{[01]}$, and let λ be a character of $G^{[01]}$. Put

$$I^t = \{t^{-1}it \mid i \in I\}, \quad \lambda^t : g \longmapsto \lambda(t^{-1}gt) \quad \text{for } g \in G^{[01]}.$$

Then $\{I, I^t\}$ and $\{\lambda, \lambda^t\}$ are $G^{[1]}$ -orbits on classes and characters of $G^{[01]}$ of lengths 1 or 2. Clearly λ^t is irreducible if and only if λ is such. The following result is rather standard (cf. [1, Section 7.15]).

LEMMA 2.3. *Let μ be a character of $G^{[01]}$, let ν be a character of $G^{[1]}$, and suppose that μ is equal to ν restricted to $G^{[01]}$.*

(i) *If μ is the sum of two irreducibles interchanged by $G^{[1]}$, then ν is uniquely determined.*

(ii) *If μ is irreducible and stable under $G^{[1]}$, then there are two choices for ν , whose values on the elements from $G^{[1]} \setminus G^{[01]}$ are negatives of each other.*

Proof. Let λ be an irreducible constituent of μ and let θ be the character of $G^{[1]}$ induced from λ . By the Frobenius reciprocity, ν is a constituent of θ . Evaluating the inner product of θ with itself, we deduce that in (i), θ is irreducible and hence equal to ν . In (ii), θ is the sum of two irreducibles and the restriction of each to $G^{[01]}$ is μ . Since $\theta(a) = 0$ for every $a \in G^{[1]} \setminus G^{[01]}$, the result follows. $\square$

LEMMA 2.4. *Let $\chi^{[0]}$ be a character of $G^{[0]}$. Then, for the existence of a representation $(\mathrm{GL}_m(\mathbb{C}), \varphi)$, such that the restriction of φ to $G^{[0]}$ affords $\chi^{[0]}$, it is necessary and sufficient that for every conjugacy class I of $G^{[01]}$, the equality $\chi^{[0]}(K) = \chi^{[0]}(L)$ holds, where K and L are the conjugacy classes of $G^{[0]}$ containing I and I^t , respectively.*

Proof. Since the classes I and I^t are fused in $G^{[1]}$, the necessity is obvious. If $\chi^{[0]}$ satisfies the hypothesis, then the restriction of $\chi^{[0]}$ to $G^{[01]}$ is stable under the action of $G^{[1]}$. Therefore we can write this restriction as a sum of characters μ as in Lemma 2.3. By that lemma, the restriction of $\chi^{[0]}$ extends to a character of $G^{[1]}$ and the result follows. $\square$

In practice, the action of t on the classes of $G^{[01]}$ can be determined by calculating the automorphism group of the character table of $G^{[01]}$ and by identifying the image of t in this automorphism group.

The last lemma in this section gives a sufficient condition for the existence of a specific completion into the symmetric group.

LEMMA 2.5. *Let $\mathcal{A} = \{G^{[0]}, G^{[1]}\}$ be a rank 2 amalgam. For $x = 0, 1$ and 01 , let $S^{[x]}$ be a subgroup in $G^{[x]}$ such that the following hold.*

- (i) $S^{[0]} \cap G^{[01]} = S^{[1]} \cap G^{[01]} = S^{[01]}$.
- (ii) $[G^{[0]} : S^{[0]}] = [G^{[1]} : S^{[1]}] = [G^{[01]} : S^{[01]}] = n$.

Then $\mathcal{A}$ possesses a completion $(\mathrm{Sym}_n, \varphi)$, such that $\varphi(G^{[x]})$ is transitive for $x = 0, 1$ and 01 .

Proof. Consider the permutation action of $G^{[y]}$ on the set $\Omega^{[y]}$ of right cosets of $S^{[y]}$ for $y = 0, 1$. Then, by the hypothesis, the restrictions to $G^{[01]}$ are permutation equivalent to the action of $G^{[01]}$ on the cosets of $S^{[01]}$. Therefore there is a bijection

$$\omega : \Omega^{[1]} \longrightarrow \Omega^{[0]}$$

which commutes with the action of $G^{[01]}$. The bijection ω enables us to define the action of $G^{[1]}$ on $\Omega^{[0]}$ which gives the required completion. $\square$

The following beautiful lemma was suggested to us by the referee of this paper.

LEMMA 2.6. *Suppose that $\mathcal{A} = \{G^{[0]}, G^{[1]}\}$ is a locally projective amalgam. Assume that H is a generating completion of $\mathcal{A}$ and that H has no subgroup of index 2. If C is a cyclic group of order 2, then $H \times C$ is a generating completion of $\mathcal{A}$.*

Proof. Set $D = H \times C$. Let $C = \langle c \rangle$ and let G_0 be the image of $G^{[0]}$ in H and G_1 be the image of $G^{[1]}$ in H (where H is considered as a subgroup of D). Then $G_0 = \langle t \rangle (G_0 \cap G_1)$ with t as in Lemma 1.2(A4). Set $t_1 = (t, c) \in D$ and

let $G_1^* = (G_0 \cap G_1)\langle t_1 \rangle$. Then $G_1^* \cong G_1$ and so $\langle G_0, G_1^* \rangle$ is a completion of $\mathcal{A}$. Let $H^* = \langle G_0, G_1^* \rangle$. Obviously $H^*C/C = HC/C = D/C$. Therefore, $H \cap H^*$ has index at most 2 in H . Since H has no subgroup of index 2, we get $H \leq H^*$ and consequently $H^* = D$. $\square$

3. An example

In this section we apply Lemma 2.4 to the locally projective amalgams $\mathcal{A} = \{G^{[0]}, G^{[1]}\}$ with $G^{[0]} \cong 2^3 : L_3(2)$. There are four such amalgams, which are described as follows (cf. [6, Section 6]). The group $G^{[0]}$ is the semi-direct product (with respect to the natural action) of $L \cong L_3(2)$ and its natural module $Q = O_2(G^{[0]})$, while $G^{[01]}$ is the centralizer in $G^{[0]}$ of a subgroup Z of order 2 from Q . Let z denote the generator of Z , let P be a subgroup of order 4 in Q , trivially intersecting Z , and let $S \cong \text{Sym}_3$ be the stabilizer in L of the direct sum decomposition $Q = Z \oplus P$. Let R be the subgroup of order 4 in L generated by the transvections with axis z , and let x and y be elements of order 3 and 2 in S . Then

$$G^{[01]} = ZPRS,$$

where every subgroup in the factorization is normalized by S . The structure of $\mathcal{A}$ is determined by the automorphism of $G^{[01]}$ induced by the element t as in (A4) and by the square of this element t . There are four choices for t , say t_0, t_1, t_2 and t_3 . Every t_i can be chosen to commute with z and x and to interchange every $p \in P$ with the only non-identity element in R which commutes with p . Furthermore, t_0 and t_1 commute with y , while t_2 and t_3 interchange y and zy . Finally, $t_0^2 = t_2^2 = 1$, while $t_1^2 = t_3^2 = z$. Then

$$\mathcal{A}(i) := \mathcal{A}\{G^{[0]}, \langle G^{[01]}, t_i \rangle\}$$

is isomorphic to $\mathcal{A}(O_6^+(2), D^+(6, 2))$ for $i = 0$ and to $\mathcal{A}_3^{(i)}$ for $i = 1, 2$ and 3 .

We are interested in the characters of $G^{[0]}$ which correspond to representations of the $\mathcal{A}(i)$. The character table of $G^{[0]}$ (as computed by the GAP package [3]) is shown in Table 3. The first three rows show the orders of the centralizers in the prime power decomposition form; the fourth row gives the classes and the fifth one shows the classes of the squares.

The condition in Lemma 2.4 only depends on the fusion of the $G^{[01]}$ -classes under the action of t (and not on the square of t). Hence, for $i = 0$ and 2 , a character χ of $G^{[0]}$ corresponds to a representation of $\mathcal{A}(i)$ if and only if it corresponds to a representation of $\mathcal{A}(i + 1)$. Next, t can only fuse elements of the same order. Therefore, the condition amounts to some equalities between the values $\chi(2a)$, $\chi(2b)$ and $\chi(2c)$, and/or between the values $\chi(4a)$, $\chi(4b)$, $\chi(4c)$, although, since every class of elements of order 4 is uniquely determined by the class of the squares of its elements, everything is determined by the fusion pattern of the involutions.

Making use of the factorization $G^{[01]} = ZPRS$, we can easily see that there are six classes of involutions in $G^{[01]}$ with representatives

$$z, p, r, pr, y, yz,$$

where $p \in P$, $r \in R$ and $[p, r] = 1$. The group $G^{[0]} \cong 2^3 : L_3(2)$ possesses an outer automorphism which interchanges the classes of $L_3(2)$ -complements and also the classes $2b$ and $2c$ of involutions. Taking into account this symmetry, we assume that z, p are $2a$ -involutions, r, y are $2b$ -involutions, and pr, yz are $2c$ -involutions. Then

TABLE 3. *The irreducible characters of $2^3 : L_3(2)$.*

2	6	6	5	5	1	1	4	3	3	.	.
3	1	1	.	.	1	1	.	.	.	.	.
7	1	.	.	.	.	.	.	.	.	1	1
g	$1a$	$2a$	$2b$	$2c$	$3a$	$6a$	$4a$	$4b$	$4c$	$7a$	$7b$
g^2	$1a$	$1a$	$1a$	$1a$	$3a$	$3a$	$2a$	$2b$	$2c$	$7a$	$7b$
λ_1	1	1	1	1	1	1	1	1	1	1	1
$^1\lambda_3$	3	3	-1	-1	.	.	-1	1	1	**	$b7$
$^2\lambda_3$	3	3	-1	-1	.	.	-1	1	1	$b7$	**
λ_6	6	6	2	2	.	.	2	.	.	-1	-1
$^1\lambda_7$	7	-1	-1	3	1	-1	-1	-1	1	.	.
$^2\lambda_7$	7	-1	3	-1	1	-1	-1	1	-1	.	.
$^3\lambda_7$	7	7	-1	-1	1	1	-1	-1	-1	.	.
λ_8	8	8	.	.	-1	-1	.	.	.	1	1
λ_{14}	14	-2	2	2	-1	1	-2	.	.	.	.
$^1\lambda_{21}$	21	-3	1	-3	.	.	1	-1	1	.	.
$^2\lambda_{21}$	21	-3	-3	1	.	.	1	1	-1	.	.

the fusions of involutions performed by the t_i are quite clear, and Lemma 2.4 gives the following.

LEMMA 3.1. *Let χ be a character of $G^{[0]}$. A representation of $\mathcal{A}(i)$ whose restriction to $G^{[0]}$ affords χ exists if and only if the following hold.*

- (i) $\chi(2a) = \chi(2b)$ and $\chi(4a) = \chi(4b)$ for $i = 0$ or 1 .
- (ii) $\chi(2a) = \chi(2b) = \chi(2c)$ and $\chi(4a) = \chi(4b)$ for $i = 2$ or 3 .

Now an easy inspection of the character table of $G^{[0]}$ shows that the minimal degree of a faithful character which satisfies the equalities in Lemma 3.1(i) is 7; the unique such character is the irreducible one, $^1\lambda_7$. The minimal faithful degree necessary for the equalities in Lemma 3.1(ii) to hold is 20, and the bound is attained by two characters, $\lambda_6 + ^1\lambda_7 + ^2\lambda_7$ and $\lambda_6 + \lambda_{14}$. This gives the following.

LEMMA 3.2. *The following equalities hold.*

$$m(\mathcal{A}(O_6^+(2), D^+(6, 2))) = m(\mathcal{A}_3^{(1)}) = 7,$$

$$m(\mathcal{A}_3^{(2)}) = m(\mathcal{A}_3^{(3)}) = 20.$$

The group $O_6^+(2) \cong \text{Sym}_8$ is a faithful completion of $\mathcal{A}(O_6^+(2), D^+(6, 2))$; its minimal faithful representation is 7-dimensional (the faithful constituent of the natural permutation representation of degree 8). Thus the restriction of this representation to the amalgam is the minimal one. The Mathieu group M_{22} is a faithful completion of $\mathcal{A}_3^{(2)}$. The minimal faithful representation of M_{22} is 21-dimensional (the faithful constituent of the permutation representation of degree 22). By Lemma 3.1, the restriction of the 21-dimensional representation to the amalgam $\mathcal{A}_3^{(2)}$ is not minimal.

4. The minimal representations

In this section we prove Theorem 1.3. For a group G by $m(G)$ we denote the minimal degree of a faithful complex character of G . The following result is rather obvious.

LEMMA 4.1. *Let $\mathcal{A} = \{G^{[0]}, G^{[1]}\}$ be a locally projective amalgam and let G be a faithful completion of $\mathcal{A}$. Then*

$$\max\{m(G^{[0]}), m(G^{[1]})\} \leq m(\mathcal{A}) \leq m(G).$$

The following easy lemma is one of the main methods for obtaining lower bounds on $m(G)$ for groups G of Lie type pioneered by Landázuri and Seitz in [7].

LEMMA 4.2. *Let F be a group which contains an elementary abelian 2-group Q as a normal subgroup, and let l be the length of the shortest F -orbit on the set of hyperplanes in Q . Then $l \leq m(F)$.*

We proceed to the proof of Theorem 1.3 by dealing with cases (i) and (ii), and then with the individual amalgams in (iii) separately.

4.1. $\mathcal{A}(\mathrm{AGL}_n(2), K_{2^n})$

In this case, $G^{[0]} \cong \mathrm{L}_n(2)$ and $G^{[1]}$ is the direct product of $G^{[0]}$ and a group of order 2. Therefore the element t in Lemma 1.2(A4) can be chosen to be an involution commuting with $G^{[0]}$. Then the conditions in Lemma 2.4 are trivial (satisfied by any character of $G^{[0]}$) and therefore $m(\mathcal{A}) = m(G^{[0]}) = m(\mathrm{L}_n(2))$. The latter number (as given in [10, Table II]) is 3, 7 and $2^n - 2$ for $n = 3$, $n = 4$ and $n \geq 5$, respectively. This takes care of case (i). A minimal representation of $\mathcal{A}$ can be constructed by taking $G^{[0]} \cong \mathrm{L}_n(2)$ and adjoining the element t acting as the (-1) -scalar matrix. This representation generates the direct product $G = \mathrm{L}_n(2) \times 2$, which is a faithful completion of $\mathcal{A}$, although somewhat degenerate, since the coset graph $\Gamma(\mathcal{A}, G)$ defined as in Lemma 1.2 has just two vertices.

4.2. $\mathcal{A}(O_{2n}^+(2), D^+(2n, 2))$

Here $G^{[0]}$ is the semi-direct product of $L \cong \mathrm{L}_n(2)$ and the exterior square Q of a natural module of L . Let $l(n)$ be the length of the shortest orbit of L on the hyperplanes in Q . Then it is well known and easy to check that $l(3) = 7$, $l(4) = 28$ and $l(n) = (2^n - 1)(2^{n-1} - 1)/3$ for $n \geq 5$. On the other hand, we use [10, Table II] for $n \geq 5$ and [1] for $n = 3, 4$ to conclude that $l(n) = m(O_{2n}^+(2))$ for all $n \geq 3$. Since $O_{2n}^+(2)$ is a faithful completion of the considered amalgam, Lemmas 4.1 and 4.2 show that $l(n)$ is both an upper and a lower bound for $m(\mathcal{A})$; hence the result.

4.3. $\mathcal{A}_3^{(i)}$, $i = 1, 2, 3$

These cases are covered by Lemma 3.2.

4.4. $\mathcal{A}_3^{(4)}$ and $\mathcal{A}_3^{(5)}$

In this case, $G^{[0]}$ is the direct product of $H \cong 2^3 : L_3(2)$ and the group of order 2 (with generator c , say). The case can be handled by the methods presented in Section 3. By a general principle, the character table of $G^{[0]}$ can easily be described in terms of the character table of H (shown in Table 3) in the following way.

$$G^{[0]} = \{(\alpha, \beta) \mid \alpha \in H \cong 2^3 : L_3(2), \beta \in \{1, c\}\}.$$

The conjugacy class of (α, β) is denoted by the conjugacy class K of α if $\beta = 1$, and by K^- if $\beta = c$. In these terms $G^{[0]}$ contains exactly seven classes of involutions:

$$1a^-, 2a, 2a^-, 2b, 2b^-, 2c, 2c^-.$$

Similarly, let ψ be an irreducible character of $G^{[0]}$, and suppose that the restriction of ψ to H is χ . Then ψ will be denoted by χ^+ if c is in the kernel of ψ and by χ^- otherwise. Notice that in a representation which affords χ^ε , the element c acts as the ε -scalar matrix.

Taking into account the symmetry induced by the outer automorphism group of H , we have the following.

LEMMA 4.3. *Let ψ be a character of $G^{[0]} \cong 2^3 : L_3(2) \times 2$. A representation of $\mathcal{A}_3^{(i)}$ whose restriction to $G^{[0]}$ affords ψ exists if and only if the following hold.*

- (i) $\psi(2a) = \psi(2b)$, $\psi(1a^-) = \psi(2a^-) = \psi(2b^-) = \psi(2c^-)$, $\psi(4a) = \psi(4b)$, $\psi(4a^-) = \psi(4b^-)$, $\psi(3a^-) = \psi(6a^-)$ for $i = 4$.
- (ii) $\psi(2a) = \psi(2b) = \psi(2c)$, $\psi(1a^-) = \psi(2a^-) = \psi(2b^-)$, $\psi(4a) = \psi(4b)$, $\psi(4a^-) = \psi(4b^-)$, $\psi(3a^-) = \psi(6a^-)$ for $i = 5$. $\square$

Since $\mathcal{A}_3^{(4)}$ contains $\mathcal{A}_3^{(1)}$, while $\mathcal{A}_3^{(5)}$ contains $\mathcal{A}_3^{(2)}$, we have

$$\begin{aligned} m(\mathcal{A}_3^{(4)}) &\geq m(\mathcal{A}_3^{(1)}) = 7, \\ m(\mathcal{A}_3^{(5)}) &\geq m(\mathcal{A}_3^{(2)}) = 20. \end{aligned}$$

It is easy to see that the 7-dimensional representation of $\mathcal{A}_3^{(1)}$ cannot be extended to a representation of $\mathcal{A}_3^{(4)}$. On the other hand, ${}^1\lambda_7^+ + {}^1\lambda_7^-$ satisfies the conditions in Lemma 4.3(i). We know (cf. [6]) that $\mathcal{A}_3^{(4)}$ possesses a completion in a subgroup of index 2 in the wreath product $\text{Sym}_8 \wr \text{Sym}_2$ and the latter clearly has a faithful 14-dimensional representation.

The 20-dimensional representation of $\mathcal{A}_3^{(2)}$ corresponding to the character $\lambda_6 + \lambda_{14}$ cannot be extended to a representation of $\mathcal{A}_3^{(5)}$. On the other hand, the representation corresponding to $\lambda_6 + {}^1\lambda_7 + {}^2\lambda_7$ can be lifted, since $\lambda_6^+ + {}^1\lambda_7^- + {}^2\lambda_7^+$ satisfies the conditions in Lemma 4.3(ii).

4.5. $\mathcal{A}_4^{(1)}$

In this case, $G^{[0]} \cong L_4(2) \cong \text{Alt}_8$ and $G^{[01]} \cong 2^3 : L_3(2)$. The character table of the latter is given in Table 3, and the element t as in (A4) induces an outer automorphism of $G^{[01]}$, exchanging the classes $2b$ and $2c$ of involutions and the classes $4b$ and $4c$ of elements of order 4. Applying Lemma 2.4, we conclude that a character of $G^{[0]}$ leads to representation of $\mathcal{A}_4^{(1)}$ if and only if (a) its values on the classes of elements of cycle type $2^2 1^4$ and 2^4 are equal, and (b) its values on

the classes of elements of cyclic type $4\,1^4$ and $4\,2\,1^2$ are equal (we refer to the cyclic type in the natural representation of degree 8). A brief look at the character table of Alt_8 in [1] (or anywhere else) shows that the minimal degree of a character satisfying these conditions is 20 and the minimum is attained by the 20-dimensional Specht module corresponding to the partition $(6, 2)$. The 20-dimensional representations of $\mathcal{A}_4^{(1)}$ are discussed in Section 5.

4.6. $\mathcal{A}_4^{(2)}$ and $\mathcal{A}_4^{(5)}$

In these two cases, the character tables of $G^{[0]}$ and $G^{[01]}$ along with the fusion pattern were explicitly calculated with the GAP computer package [3], and the action of $G^{[1]}$ on the conjugacy classes of $G^{[01]}$ was reconstructed. This enabled us to apply Lemma 2.4. The smallest degree of a character satisfying the conditions is 63 and 255 for $\mathcal{A}_4^{(2)}$ and $\mathcal{A}_4^{(5)}$, respectively. The minimal representations are discussed in Sections 6 and 7.

4.7. $\mathcal{A}_4^{(3)}$

The group Co_2 is a faithful completion of $\mathcal{A}_4^{(3)}$; therefore $m(\mathcal{A}_4^{(3)})$ is at most $m(Co_2) = 23$. On the other hand, $K^{[3]}$ is elementary abelian of order 2^{10} , and isomorphic to the irreducible Golay code module with respect to the action of $\widehat{G}^{[3]} \cong \text{Aut } M_{22}$. The shortest $(\text{Aut } M_{22})$ -orbit on the hyperplanes in $K^{[3]}$ is of length 22. Thus, by Lemma 4.2, $m(\mathcal{A}_4^{(3)})$ is at least 22, and all we have to show is that it cannot be exactly 22. Suppose to the contrary that W is a 22-dimensional $\mathbb{C}$ -module supporting a faithful representation of $\mathcal{A}_4^{(3)}$. Then, by the above, $W = [W, K^{[3]}]$ and $C_W(M)$ is 1-dimensional for every hyperplane M in $K^{[3]}$ from the 22-orbit. We make use of the following properties of the amalgam $\mathcal{A}_4^{(3)}$, which can either be found in [6] or checked in the completion Co_2 of $\mathcal{A}_4^{(3)}$.

- (a) $G^{[0]} \sim 2.2^4.2^6.L_4(2)$, so that the centre $Z^{[0]}$ of $G^{[0]}$ is of order 2.
- (b) $O_2(G^{[0]})/Z^{[0]}$ is an (indecomposable) extension of the natural module of $G^{[0]}/O_2(G^{[0]}) \cong L_4(2)$ by the exterior square of the natural module.
- (c) $Z^{[0]} \leq K^{[3]}$ and the $\widehat{G}^{[3]}$ -orbit of $Z^{[0]}$ is of length 330.
- (d) For t as in (A4), we have $(Z^{[0]})^t \leq O_2(G^{[0]})$.

Calculating in the irreducible Golay code module, we deduce from (c) that $Z^{[0]}$ is contained in exactly 14 hyperplanes from the 22-orbit, so that $C_W(Z^{[0]})$ is 14-dimensional. We claim that $O_2(G^{[0]})$ acts non-trivially on $C_W(Z^{[0]})$. Indeed, otherwise, by (c) and (d),

$$(C_W(Z^{[0]}))^t = C_W((Z^{[0]})^t) \leq C_W(Z^{[0]}),$$

and the latter two centralizers must coincide. Therefore, $C_W(Z^{[0]})$ is normalized by both $G^{[0]}$ and $G^{[1]}$, which readily leads to a contradiction. Therefore, by Lemma 4.2, the dimension of $C_W(Z^{[0]})$ is at least the length of the shortest $G^{[0]}/O_2(G^{[0]})$ -orbit on the hyperplanes in $O_2(G^{[0]})/Z^{[0]}$, which is 15.

4.8. $\mathcal{A}_4^{(4)}$ and $\mathcal{A}_5^{(1)}$

The amalgam $\mathcal{A}_4^{(4)}$ is contained in $\mathcal{A}_5^{(1)}$; therefore

$$m(\mathcal{A}_5^{(1)}) \geq m(\mathcal{A}_4^{(4)}).$$

On the other hand, each of the two amalgams possesses the fourth Janko group J_4 as a faithful completion; hence

$$m(\mathcal{A}_5^{(1)}) \leq m(J_4) = 1333.$$

We have checked the equality $m(\mathcal{A}_4^{(4)}) = 1333$ by calculating the character table of $G^{[0]}$ and applying Lemma 2.4.

4.9. $\mathcal{A}_5^{(2)}$

The Baby Monster group BM is a completion of $\mathcal{A}_5^{(2)}$, and hence $m(\mathcal{A}_5^{(2)}) \leq m(BM) = 4371$. Let W be the module supporting a minimal faithful representation of $\mathcal{A}_5^{(2)}$, so that $\dim W \leq 4371$. The subgroup $K^{[4]}$ is extra-special of type 2_+^{1+22} with centre $Z^{[4]}$ of order 2 and $\widehat{G}^{[4]} \cong Co_2$. Put

$$W_0^{[4]} = [W, Z^{[4]}], \quad W_1^{[4]} = [C_W(Z^{[4]}), K^{[4]}], \quad W_2^{[4]} = C_W(K^{[4]}).$$

Then $W_0^{[4]}$ is non-trivial and supports a faithful representation of $K^{[4]}$. The representation theory of extra-special groups gives

$$\dim W_0^{[4]} = l \cdot 2^{11} = l \cdot 2048 \quad \text{for } l \geq 1.$$

By Lemma 4.2, if $W_1^{[4]}$ is non-trivial, then its dimension is at least the length of the shortest Co_2 -orbit on $K^{[4]}/Z^{[4]}$. The latter module is the only faithful section of the Leech lattice taken modulo 2, and it is well known that the shortest orbit has length 2300 and the second shortest one is of length 46 575. Finally, $W_2^{[4]}$ supports a representation of the amalgam

$$\{G^{[04]}/K^{[4]}, G^{[14]}/K^{[4]}\} \cong \mathcal{A}_4^{(3)},$$

so, if non-trivial, $W_2^{[4]}$ must be at least 23-dimensional by the $\mathcal{A}_4^{(3)}$ -case considered above. Since

$$2048 + 2300 + 23 = 4371,$$

it only remains to show that $W_1^{[4]}$ and $W_2^{[4]}$ are non-zero and that $G^{[014]}$ acts non-trivially on the latter.

In order to accomplish this, we consider $K^{[3]}$, which possesses the lower central series

$$1 < Z^{[3]} < Y^{[3]} < K^{[3]}.$$

The factors of this series are elementary abelian of order 2^2 , 2^{10} and 2^{20} , respectively. The group $\widehat{G}^{[3]} \cong \text{Aut } M_{22} \times \text{Sym}_3$ induces, respectively, Sym_3 and $\text{Aut } M_{22}$ on the first and the second chief factors and acts faithfully on the last factor. Furthermore, the first factor is the natural module of $\text{Sym}_3 \cong L_2(2)$, the second one is the irreducible Todd module, and the third one is the tensor product of the natural Sym_3 -module and the irreducible Golay code module.

Let χ be the character afforded by the representation of $\mathcal{A}_5^{(2)}$ in W , and for an element a , let $\chi_i^{[4]}(a)$ denote the character of a acting on $W_i^{[4]}$ for $i = 0, 1, 2$. Let $\langle z \rangle = Z^{[4]}$ and $\langle z, y \rangle = Z^{[3]}$. Then z and y are conjugate in $G^{[3]}$; therefore $\chi(z) = \chi(y)$. On the other hand,

$$\begin{aligned} \chi_2^{[4]}(z) &= \chi_2^{[4]}(y) = \dim W_2^{[4]}, \\ \chi_0^{[4]}(z) &= -\dim W_0^{[4]}, \quad \chi_0^{[4]}(y) = 0. \end{aligned}$$

Thus, in order for $\chi(z)$ and $\chi(y)$ to be equal, $W_1^{[4]}$ must be non-zero, and by the above, $\dim W_0^{[4]} = 2048$ and $\dim W_1^{[4]} = 2300$.

Put

$$W_0^{[3]} = [W, Z^{[3]}], \quad W_1^{[3]} = [C_W(Z^{[3]}), Y^{[3]}], \quad W_2^{[3]} = [C_W(Y^{[3]}), K^{[3]}].$$

Since $\chi(z) = 2048$, we have $\dim W_0^{[3]} = 3 \cdot 1024$. It follows from the structure of $\mathcal{A}_5^{(2)}$ that

$$Y^{[3]} = C_{K^{[4]}}(Z^{[3]}),$$

and hence $W_1^{[3]} \leq W_1^{[4]}$. Direct calculations in the Leech lattice taken modulo 2 (cf. [4, Section 4.11]) show that $Y^{[3]}/Z^{[4]}$ is contained in exactly 44 hyperplanes from the 2300-orbit of Co_2 on $K^{[4]}/Z^{[4]}$. Hence

$$\dim(W_2^{[3]} \cap (W_0^{[4]} \oplus W_1^{[4]})) = 44.$$

On the other hand, by Lemma 4.2, the dimension of $W_2^{[3]}$ must be at least the length of the shortest $\widehat{G}^{[3]}$ -orbit on the hyperplanes in $K^{[3]}/Y^{[3]}$. Since the latter quotient is the tensor product of the irreducible Golay code module for $\text{Aut } M_{22}$ and the natural module of Sym_3 , the shortest orbit length is 66. Therefore $W_2^{[4]} \cap W_2^{[3]}$ is a non-trivial module for $K^{[3]} \leq G^{[014]}$ of dimension 22 or more; hence the result. $\square$

5. 20-dimensional representations of $\mathcal{A}_4^{(1)}$

In this section, $\mathcal{A} = \{G^{[0]}, G^{[1]}\}$ stays for the amalgam $\mathcal{A}_4^{(1)}$, so that

$$G^{[0]} \cong L_4(2) \cong \text{Alt}_8, \quad G^{[01]} \cong 2^3 : L_3(2), \quad G^{[1]} \cong 2^4 : L_3(2).$$

Let $R = O_2(G^{[01]})$, $L \cong L_3(2)$ be a complement to R in $G^{[01]}$, $T = O_2(G^{[1]})$, and t be as in Lemma 1.2(A4). Then R is the natural module for L , $T = \langle R, t \rangle$, L is also a complement to T in $G^{[1]}$, and t cannot be chosen to normalize L , so that T (as an L -module) is an indecomposable extension of R by the trivial 1-dimensional module.

Let $\chi^{[0]}$ be the 20-dimensional Specht module of $G^{[0]}$. The values of $\chi^{[0]}$ in terms of the cyclic types of the elements of $G^{[0]}$ are given in Table 4.

We follow the notation as in the character table in Table 3 for the conjugacy classes and the irreducible characters of $G^{[01]}$. In these terms, the restriction $\chi^{[01]}$ of $\chi^{[0]}$ to $G^{[01]}$ equals $\lambda_6 + \lambda_{14}$. Let $\chi^{[1]}$ be a character of $G^{[1]}$ whose restriction to $G^{[01]}$ coincides with $\chi^{[01]}$. By Lemma 2.4, $\chi^{[1]}$ consists of two irreducible components, say $\chi_6^{[1]}$ and $\chi_{14}^{[1]}$. Furthermore, $\chi_6^{[1]}$ restricted to $G^{[01]}$ is λ_6 , while $\chi_{14}^{[1]}$ restricted to $G^{[01]}$ is λ_{14} . Thus altogether we obtain four possibilities for $\chi^{[1]}$.

In order to specify the possibilities for $\chi^{[1]}$, we let W denote the corresponding 20-dimensional $\mathcal{C}$ -module for $G^{[1]}$. Then

$$W_6 := C_W(R) \quad \text{and} \quad W_{14} := [R, W]$$

TABLE 4. The degree 20 character of Alt_8 .

	1^8	2^4	$2^2 1^4$	$3^2 1^2$	$3 1^5$	4^2	$4 2 1$	$6 2$	$3 2^2 1$	$7 1$	$5 1^3$	$5 3$
$\chi^{[0]}$	20	4	4	-1	5	0	0	1	1	-1	0	0

are the proper irreducible submodules. Furthermore, λ_6 is the faithful constituent of a transitive permutation action of

$$G^{[01]}/R \cong L \cong L_3(2)$$

of degree 7. Since t commutes with L modulo R , the action of t on W_6 is either trivial or multiplies every vector by -1 . This gives two possibilities for $\chi_6^{[1]}$.

It is convenient to describe

$$\lambda_{14} \quad \text{and} \quad \chi_{14}^{[1]}$$

in terms of induced characters. The complement L acts transitively on the seven hyperplanes in R . If X is one of these hyperplanes, then $N_L(X) \cong \text{Sym}_4$, by the symmetry reason $C_{W_{14}}(X)$ is 2-dimensional, and by the Frobenius reciprocity λ_{14} is induced from the character of the action of $N_{G^{[01]}}(X)$ on $C_{W_{14}}(X)$. The kernel of the latter action is $\langle X, O_2(N_L(X)) \rangle$ and the acting group $\text{Sym}_3 \times 2$ possesses a unique faithful character of degree 2. This specifies λ_{14} . The same complement L acts transitively on the fourteen hyperplanes in T other than R . If Y is one of the hyperplanes, then $N_L(Y) \cong \text{Alt}_4$, $C_{W_{14}}(Y)$ is 1-dimensional, and $\chi_{14}^{[1]}$ is induced from the linear character θ of the action of $N_{G^{[1]}}(Y)$ on $C_{W_{14}}(Y)$. The kernel of the action is $\langle Y, O_2(N_L(Y)) \rangle$, the acting group is cyclic of order 6, and θ is one of the two faithful linear characters of the latter cyclic group.

By Lemma 2.2, each of the four choices of $\chi^{[1]}$ determines a unique equivalence class of representations. Thus (up to equivalence) there are four minimal representations of $\mathcal{A}_4^{(1)}$. These representations were explicitly constructed using GAP in terms of generating matrices. In calculations of the orders of a few thousand pseudo-random elements and in each of the four cases, both the generating group and the third geometric subgroup showed strong symptoms of infiniteness. On the other hand, the second geometric subgroup appeared to be independent of the choice between the four representations with

$$C_{G^{[2]}}(K^{[2]}) \cong 3^2 : 2^2$$

(exactly as in the modulo 13 case discussed below.) Since the resources of GAP in dealing with linear groups over infinite domains are still rather limited, we decided to switch to calculations modulo 13.

Since the multiplicative group of the field with thirteen elements has order divisible by 6, the above characters $\chi^{[x]}$, $x = 0, 1, 01$, can be realized over that field (the field of seven elements could probably also do the job, but we went for thirteen to be sure that we were still in the semi-simple case). The results are summarised in the next proposition. (Recall that in the amalgam $\mathcal{A}_4^{(1)}$, the kernel $K^{[2]}$ is an elementary abelian group of order 16 extended by a fixed-point-free automorphism of order 3. In particular, its centre is trivial.)

PROPOSITION 5.1. *Let (G, φ) be a faithful completion of the amalgam $\mathcal{A}_4^{(1)}$, where $G = \text{SL}_{20}(13)$. Then the following hold.*

- (i) *The completion is generating.*
- (ii) *$C_{G^{[2]}}(K^{[2]})$ is a semi-direct product with respect to a faithful action of an elementary abelian group of order 3^2 and a group of order 2^2 .*
- (iii) *$G^{[3]} = G$.*

Proof. The subgroup $G^{[2]}$ is small enough for its chief factors to be identified in GAP. The knowledge of $\widehat{G}^{[2]} \cong \text{Sym}_5$ and $K^{[2]} \cong 2^4 : 3$ gives (ii). By generating

pseudo-random elements in $G^{[3]}$ and calculating their orders, we have shown that every prime number dividing the order of $\text{SL}_{20}(13)$ also divides that of $G^{[3]}$. By [8, Corollary 6], this is enough to identify $G^{[3]}$ with the whole of $\text{SL}_{20}(13)$, which gives (i) and (iii). $\square$

6. A completion of $\mathcal{A}_4^{(2)}$ in Alt_{64}

Let $\mathcal{A} = \{G^{[0]}, G^{[1]}\}$ be a locally projective amalgam such that $G^{[0]}$ is the semi-direct product of $L \cong \text{L}_4(2)$ and $Q := O_2(G^{[0]})$, the latter being the exterior square of the natural module V of L . Therefore $G^{[01]}$ is the semi-direct product of Q and $C_L(V_1) \cong 2^3 : \text{L}_3(2)$, where V_1 is a 1-subspace in V . Let $K \cong \text{L}_3(2)$ be a complement to $A := O_2(C_L(V_1))$ in $C_L(V_1)$, which stabilizes a hyperplane in V (K is said to be a Levi complement). Then, as a module for K , Q is a direct sum of $Z = Z(O_2(G^{[01]}))$ and a submodule B . Furthermore, Z is isomorphic to V/V_1 and dual to both A and B .

Let α and β be automorphisms of $G^{[01]}$ defined in the following way: α centralizes ZK , interchanging A and B ; β centralizes $ZAB = O_2(G^{[01]})$ and induces an outer automorphism of $ZK \cong 2^3 : \text{L}_3(2)$. If $\mathcal{A} \cong \mathcal{A}(O_8^+(2), D^+(8, 2))$, then the element t as in Lemma 1.2(A4) can be chosen to be an involution inducing the automorphism α , and if $\mathcal{A} = \mathcal{A}_4^{(2)}$, then t can be chosen to be an involution inducing the automorphism $\alpha\beta$. In this section we work with the latter situation.

We have explicitly constructed using GAP the character tables of $G^{[0]}$ and $G^{[01]}$, and determined the class fusion. After that we determined the action of $\alpha\beta$ on the conjugacy classes and irreducible characters of $G^{[01]}$, which enabled us to apply Lemma 2.4. The smallest degree of a faithful character satisfying the conditions in Lemma 2.4 turned out to be 63. This character is the sum of two irreducibles

$$\chi_{28}^{[0]} \quad \text{and} \quad \chi_{35}^{[0]}$$

of degrees 28 and 35, respectively. When restricted to $G^{[01]}$, the former one is an irreducible ${}^1\chi_{28}$, while the latter is the sum of two irreducibles ${}^2\chi_{28}$ and χ_7 of degrees 28 and 7, respectively. The automorphism $\alpha\beta$ stabilizes χ_7 and interchanges ${}^1\chi_{28}$ and ${}^2\chi_{28}$. By Lemma 2.3, this gives two possibilities for $\chi^{[1]}$. Furthermore, one can see that the element t as in (A4) commutes with $G^{[01]}$ modulo the kernel of χ_7 . Hence the choice between the two possibilities is determined by whether $\chi^{[1]}(t) = 7$ or $\chi^{[1]}(t) = -7$. By Lemma 2.2, $\chi^{[1]}$ determines the equivalence class of the corresponding representations of $\mathcal{A}$.

The minimal representation (G, φ) with $\chi^{[1]}(t) = 7$ was constructed explicitly in terms of generating matrices. Calculations of the orders of pseudo-random elements suggested that a prime p divides the order of the subgroup I generated by the image of φ if and only if $p \leq 61$. This in turn suggested that I could be the alternating group Alt_{64} (notice that I possesses a 63-dimensional complex representation.) This turned out to be true. We start with the following lemma (which chronologically was proved last).

LEMMA 6.1. *There is a faithful completion (Sym_{64}, ψ) of $\mathcal{A}_4^{(2)}$.*

Before proceeding to the proof, we recall some standard properties of $\text{L}_4(2)$.

LEMMA 6.2. *Let $L \cong \text{L}_4(2) \cong \text{Alt}_8$, let V be the natural 4-dimensional $\text{GF}(2)$ -module of L , let Q be the exterior square of V , and let Δ be an 8-set on which L acts as the alternating group.*

- (i) L contains three classes of $L_3(2)$ -subgroups with representatives $K^{(0)}$, $K^{(1)}$ and $K^{(2)}$ such that the following hold.
- (a) $K^{(0)}$ acts transitively on Δ , stabilizes a direct sum decomposition of V into a 1-subspace and a 3-subspace, and stabilizes a direct sum decomposition of Q into two 3-subspaces dual to each other.
 - (b) $K^{(1)}$ stabilizes an element of Δ , acts indecomposably on V stabilizing a 1-subspace, and acts indecomposably on Q .
 - (c) $K^{(2)}$ stabilizes an element of Δ , acts indecomposably on V stabilizing a 3-subspace, and acts indecomposably on Q .
- (ii) The semi-direct product of L and Q contains two classes of complements to Q ; if L' is a complement not in the class of L , then the intersection $L \cap L'$ is either Alt_7 or $(\text{Sym}_5 \times \text{Sym}_3)^+$.

Proof of Lemma 6.1. We apply Lemma 2.5. Let $S^{[0]} \cong L_4(2)$ be a complement to Q in $G^{[0]}$ which is not conjugate to L . Then $S^{[01]} := S^{[0]} \cap G^{[01]}$ is isomorphic to $C_L(V_1)$, and in particular, $S^{[01]} \cong 2^3 : L_3(2)$. In order to apply Lemma 2.5, it is sufficient for us to show that

$$S^{[1]} := N_{G^{[1]}}(S^{[01]}) \neq S^{[01]}. \quad (3)$$

Let $C = O_2(S^{[01]})$. Then $\{AZ, BZ, CZ\}$ is the complete set of maximal elementary abelian normal subgroups in $G^{[01]}$. It is known (cf. [6]) and easy to check that $G^{[01]}$ contains six classes of $L_3(2)$ -complements to $O_2(G^{[01]})$. Representatives of these classes can be obtained as follows. Let K be the complement that we have started with (a Levi complement contained in L). For $X = A$ and B , let K_X be the image of K under an outer automorphism of $XK \cong 2^3 : L_3(2)$, and for a complement M , let M^Z be the image of M under an outer automorphism of $ZM \cong 2^3 : L_3(2)$. Then

$$\mathcal{K} = \{K, K^Z, K_A, K_A^Z, K_B, K_B^Z\}$$

is a complete set of representatives of the classes of complements to $O_2(G^{[01]})$ in $G^{[01]}$. The group

$$O := \text{Out } G^{[01]} \cong \text{Sym}_3 \times 2$$

permutes the classes of complements transitively; the pairs $\{M, M^Z\}$ form an imprimitivity system, and the image of the automorphism β is the only non-identity element of O which stabilizes every block of this system as a whole.

Let F be a Sylow 7-subgroup in $G^{[01]}$. Then without loss of generality we assume that every subgroup from $\mathcal{K}$ contains F (subject to this condition, $\mathcal{K}$ is uniquely determined). The above defined automorphism α centralizes F ; we may assume that the same is true for the automorphism β . We claim that they induce the following permutations of $\mathcal{K}$.

$$\begin{aligned} \alpha &: (K)(K^Z)(K_A, K_B)(K_A^Z, K_B^Z), \\ \beta &: (K, K^Z)(K_A, K_A^Z)(K_B, K_B^Z). \end{aligned}$$

In fact, α centralizes both K and Z and maps AK onto BK , while β centralizes the whole of $O_2(G^{[01]})$; therefore $ZM^\beta = ZM = ZM^Z$ for every $M \in \mathcal{K}$, since Z is the centre of $O_2(G^{[01]})$. Hence the actions are as claimed and $\alpha\beta$ induces the permutation

$$\alpha\beta : (K, K^Z)(K_A, K_B^Z)(K_B, K_A^Z).$$

Again, without loss of generality, we assume that F is contained in $S^{[01]}$. Then the two complements to $C = O_2(S^{[01]})$ in $S^{[01]}$ containing F are in $\mathcal{K}$. Our next aim is to identify these complements.

First, L and $S^{[0]}$ are in different classes of complements to Q in $G^{[01]}$, and by our choice they share a 7-subgroup. By Lemma 6.2(ii), this means that $L \cap S^{[0]} \cong \text{Alt}_7$, and by Lemma 6.2(i), $C_L(V) = AK$ and $S^{[01]}$ share an $L_3(2)$ -subgroup containing F and acting indecomposably on Q . Since K is a Levi complement, the shared subgroup must be K_A , and hence $S^{[01]} = CK_A$. In accordance with the notation introduced above, let Y denote the second complement to C in $S^{[01]}$ containing F . We claim that

$$Y = K_B^Z. \quad (4)$$

Let D be a subgroup of order 3 normalizing F . Then D is contained in every complement $M \in \mathcal{K}$. Let E be a Sylow 2-subgroup of $N_{G^{[01]}}(D)$. Then E is elementary abelian of order 2^4 generated by the non-identity elements z , a , b and k in the intersections of E with Z , A , B and K , respectively (notice that $\langle z, a, b \rangle = C_E(D)$). Then the non-identity elements of E contained in $N_M(D) \cap E$ are k , kz , ka , kaz , kb and kbz for $M = K$, K^Z , K_A , K_A^Z , K_B and K_B^Z , respectively. On the other hand, $N_Y(D) \cap E$ contains kac , where c is the non-identity element in $E \cap C$. Thus (4) is equivalent to

$$c = abz. \quad (5)$$

The centralizer of D in ZC is of order 2^2 generated by the elements z and ab . The orbit of ab under F coincides with its orbit under K , and since the action of K on ZC is known to be indecomposable, this orbit must generate the whole of ZC . Hence ab is not in C and the latter must contain abz . This gives (5) and hence (4) as well. Thus $S^{[01]} = \langle K_A, K_B^Z \rangle$ and hence $S^{[01]}$ is normalized by the automorphism $\alpha\beta$ (which has been adjusted to centralize F). Therefore (3) follows, completing the proof. $\square$

Within the above proof we had a chance to appreciate the remarkable beauty of the group

$$G^{[01]} = ZABK \cong 2^3 : (2^3 \times 2^3) : L_3(2).$$

This group will reappear in the next section, so it needs a special name, and we call it the *trident group* (recall that it contains exactly three maximal elementary abelian normal subgroups of order 2^6 , which we call *dents*). The remarkable feature of the trident group is that a given dent switches from direct sum to indecomposable module while we switch from one class of $L_3(2)$ -complements to another one.

PROPOSITION 6.3. *Let (G, φ) be a faithful completion of $\mathcal{A}_4^{(2)}$, where $G = \text{GL}_{63}(\mathbb{C})$. Let I be the subgroup generated by the image of φ . Then either $\varphi(t)$ has trace 7 and $I \cong \text{Alt}_{64}$ or $\varphi(t)$ has trace -7 and $I \cong 2 \times \text{Alt}_{64}$. Furthermore, in the former case we have the following.*

- (i) $I \cong \text{Alt}_{64}$.
- (ii) $G^{[2]} \cong 2_+^{1+8} \cdot \text{Sym}_3 \cdot (\text{Sym}_5 \times 2)$.
- (iii) $G^{[3]} \cong 2_+^{6+6} : L_6(2)$.

In particular, the former of the completions is constrained at levels 2 and 3.

Proof. The completion (Sym_{64}, ψ) , as in Lemma 6.1, leads to a 63-dimensional completion in the faithful component of the corresponding permutation module over $\mathbb{C}$. This completion must be the former of the completions (G, φ) under consideration and hence $I \leq \text{Sym}_{64}$. Since $G^{[0]}$ has no subgroups of index 2, $\psi(G^{[0]}) \leq \text{Alt}_{64}$. Since $\chi^{[1]}(t) = 7$, the permutation character of t is 8; hence it is realized by an even permutation, and hence $I \leq \text{Alt}_{64}$. On the other hand, among the pseudo-random elements generated, we have found an element of order $183 = 61 \times 3$. The 61st power of this element must be a 3-cycle. Finally, since the image of φ is irreducible, the image of ψ is doubly transitive and (i) follows. The subgroups $G^{[2]}$ and $G^{[3]}$ were constructed explicitly, and the information given in (ii) and (iii) has been checked using the relevant GAP packages. By Lemma 2.6, $2 \times \text{Alt}_{64}$ is also a generating completion of $\mathcal{A}_4^{(2)}$. Of course $2 \times \text{Alt}_{64}$ is a subgroup of $\text{GL}_{63}(\mathbb{C})$, and we obtain the latter of the completions. $\square$

7. A completion of $\mathcal{A}_4^{(5)}$ in Alt_{256}

The amalgam $\mathcal{A}_4^{(5)}$ was defined in [6] in terms of

$$\mathcal{H} = \{H^{[0]}, H^{[1]}\} \cong \mathcal{A}(O_{10}^+(2), D^+(10, 2)).$$

Let $L \cong \text{L}_5(2)$, let W be the natural module of L , and let Q be the exterior square of the dual of W . Let $H^{[0]}$ be the semi-direct product of Q and L with respect to the natural action, and let $H^{[01]}$ be the semi-direct product of Q and $C_L(W_1) \cong 2^4 : \text{L}_4(2)$, where W_1 is a 1-subspace in W . Let $F = O_2(C_L(W_1))$ and $N \cong \text{L}_4(2)$ be a complement to F in $C_L(W_1)$. Then (as a module for N) Q is the direct sum of E and D , where E is the centre of $O_2(H^{[01]})$, isomorphic to the exterior square of F , while D is a submodule isomorphic to F . Let α be an automorphism of $H^{[01]}$, which centralizes EM interchanging F and D . Let β be an automorphism which centralizes $EDF = O_2(H^{[01]})$ and induces an outer automorphism of $EN \cong 2^6 : \text{L}_4(2)$. Then the element $t \in H^{[1]} \setminus H^{[0]}$ as in Lemma 1.2(A4) induces the automorphism α (if it would induce $\alpha\beta$, the amalgam would be $\mathcal{A}_5^{(2)}$). Let W_4 be a hyperplane in W containing W_1 . Let $G^{[0]}$ be the semi-direct product of Q and

$$C_N(U_1) \cap N_N(U_4) \cong 2_+^{1+6} : \text{L}_3(2),$$

and let $G^{[01]} = G^{[0]} \cap H^{[01]}$. Abusing notation, we denote by α and β the restrictions of the above defined automorphisms to $G^{[01]}$. The centre P of $O_2(G^{[01]})$ is elementary abelian of order 2^3 . Let γ be an automorphism of $G^{[01]}$ which centralizes $O_2(G^{[01]})$, and for a complement $K \cong \text{L}_3(2)$ to $O_2(G^{[01]})$ in $G^{[01]}$ it induces an outer automorphism of $PK \cong 2^3 : \text{L}_3(2)$. Let $G^{[1]}$ be a semi-direct product of $G^{[01]}$ and a group of order 2 generated by an element t inducing the automorphism $\alpha\beta\gamma$. Then $\{G^{[0]}, G^{[1]}\}$ is isomorphic to $\mathcal{A}_4^{(5)}$.

Similarly to the $\mathcal{A}_4^{(2)}$ case, we have constructed the character tables of $G^{[0]}$ and $G^{[01]}$ using GAP and determined the action of $\alpha\beta\gamma$ on the classes and irreducibles of $G^{[01]}$. The smallest character $\chi^{[0]}$ satisfying the conditions in Lemma 2.4 has degree 255, and it is the sum of three irreducibles, $\chi_{15}^{[0]}$, ${}^1\chi_{120}^{[0]}$ and ${}^2\chi_{120}^{[0]}$. When restricted to $G^{[01]}$, each irreducible constituent of $\chi^{[0]}$ becomes the sum of two irreducibles of $G^{[01]}$: $\chi_7 + {}^1\chi_8$, ${}^2\chi_8 + {}^1\chi_{112}$ and ${}^3\chi_8 + {}^2\chi_{112}$, respectively. The automorphism $\alpha\beta\gamma$ stabilizes χ_7 and ${}^3\chi_8$, interchanging ${}^1\chi_8$ with ${}^2\chi_8$ and ${}^1\chi_{112}$ with ${}^2\chi_{112}$. Thus there are four possibilities for $\chi^{[1]}$, depending on whether $\chi^{[1]}$ is 15, 1, -1 or -15

(notice that t commutes with $G^{[01]}$ modulo the kernel of χ_7 or ${}^3\chi_8$). By Lemma 2.2, the character $\chi^{[1]}$ determines the completion up to equivalence.

LEMMA 7.1. *There is a faithful completion (Sym_{256}, ψ) of $\mathcal{A}_4^{(5)}$.*

Proof. It is implicit from the first paragraph of this section that if (H, θ) is the faithful completion of the amalgam

$$\mathcal{H} = \{H^{[0]}, H^{[1]}\} \cong \mathcal{A}(O_{10}^+(2), D^+(10, 2)),$$

where $H \cong O_{10}^+(2)$, then $G^{[01]} \cong H^{[014]}$. On the other hand, $H^{[4]}$ is the semi-direct product of $R \cong O_8^+(2)$ and its natural module $X \cong 2^8$; $F^{[04]} \cong G^{[0]}$ is the semi-direct product of $N_R(U) \cong 2^6 : L_4(2)$ and X , where U is a maximal (4-dimensional) totally isotropic subspace in X (with respect to the non-zero quadratic form on X preserved by R). Let $M \cong L_4(2)$ be a complement to $T := O_2(N_R(U))$ in $N_R(U)$ which stabilizes a complement V to U in X . Then

$$F^{[04]} \cong G^{[0]} = UVTM \cong (2^4 \times 2^4) : 2^6 : L_4(2),$$

where V is the natural module of $M \cong L_4(2)$, U is the dual natural module, and T is the exterior square of V . Let λ be the bijections of the set of hyperplanes from V into U which commutes with the action of M , and let μ be the bijection of the set of 2-subspaces from V into T which commutes with the action of M . Then the structure of $G^{[0]}$ is essentially determined by the following relations. Let V_2 be a 2-subspace in W (so that $\mu(V_2) \in T$) and let $v \in V \setminus V_2$. Then

$$[w, \mu(V_2)] = \lambda^{-1}(\langle v, V_2 \rangle_V).$$

Finally, $H^{[014]} \cong G^{[01]}$ is the semi-direct product of UVT and $C_M(V_1)$, where V_1 is a 1-subspace in W . Let K be a complement to $A := O_2(C_M(V_1))$ in $C_M(V_1) \cong 2^3 : L_3(2)$, which stabilizes a direct sum decomposition

$$T = Z \oplus B$$

(where Z is dual to A , and B is isomorphic to A). Then $ZBAK$ is the trident group as defined in the previous section and $G^{[01]} = UVZBAK$. Furthermore,

$$U = V_1^* \oplus J, \quad V = V_1 \oplus I$$

(as K -modules) where V_1^* is isomorphic to A , while I is dual to A . In what follows we refer to I as the natural module for K . In these terms, I and Z are natural modules, while V_1^* , B and A are dual natural modules.

Thus we have the factorization

$$G^{[01]} = V_1^* J V_1 I Z B A K. \tag{6}$$

Our nearest goal is to describe (in terms of the factorization (6)) the action of the automorphisms α , β and γ of $G^{[01]}$ and hence the automorphism $\alpha\beta\gamma$, induced by the element $t \in G^{[1]} \setminus G^{[01]}$ as in (A4).

We summarise below some relations between the terms in (6) which follow by straightforward calculations using the above description.

(i) $V_1^* \cong 2^3$ is the centre if $O_2(G^{[01]}) = UVZBA$.

(ii) $V_1^* J V_1 I Z K \cong V_1^* I Z K \times J V_1$, the former direct factor being the trident group, while the latter one is elementary abelian of order 2^2 .

(iii) $V_1^*JV_1IZ$ is characteristic in $G^{[01]}$ and contains exactly three elementary abelian subgroups of order 2^8 normal in $G^{[01]}$. Two of them are $X = V_1^*JV_1I$ and $V_1^*JV_1Z$, while the third one will be denoted by Y .

(iv) $[B, Z] = 1$, $[B, I] = J$, $[B, V_1] = V_1^*$.

(v) $[A, B] = Z$, $[A, I] = V_1$, $[A, J] = V_1^*$.

(vi) $Q = O_2(H^{[0]}) = V_1^*JZB$, and the only other elementary abelian normal subgroup of order 2^{10} in $G^{[01]}$ is $V_1^*V_1ZA$.

(vii) $E = V_1^*Z$ and $IK = N \cap G^{[01]}$.

The properties (i)–(vii) enable one to re-prove [6, Proposition 9.11], stating that the outer automorphism group of $G^{[01]}$ is elementary abelian of order 8, generated by the images of α , β and γ . It is even easier to describe the action of these automorphisms on the factors in (6).

The automorphism γ is the easiest to describe. It centralizes $O_2(G^{[01]}) = V_1^*JV_1IZBA$ and maps every $L_3(2)$ -complement N onto its image $N^{V_1^*}$ under an outer automorphism of $V_1^*N \cong 2^3 : L_3(2)$. The action of α is also rather transparent. It commutes with the trident group V_1ZIK , and interchanges J with V_1 and A with B (commuting with the action of K).

Finally, let us turn to β . By (vii), β centralizes JV_1AB and induces an automorphism of the trident group $\Phi := V_1^*ZIK$ which is the restriction of an outer automorphism of $EN \cong 2^6 : L_4(2)$ which centralizes E and permutes two classes of $L_4(2)$ -complements. As in the proof of Lemma 6.1, let

$$\mathcal{K} = \{K, K^{V_1^*}, K_Z, K_Z^{V_1^*}, K_I, K_I^{V_1^*}\}$$

be the representatives of classes of $L_3(2)$ -complements in Φ , where K_Z and K_I are the images of K under outer automorphisms of ZK and IK ; for a complement N by $N^{V_1^*}$, we denote the image of N under an outer automorphism of $V_1^*K \cong 2^3 : L_3(2)$. Arguing as in the proof of Lemma 6.1, and in particular using (4), we observe that β normalizes V_1^*Z and induces the following permutation of $\mathcal{K}$:

$$\beta : (K_I)(K_I^{V_1^*})(K, K_Z^{V_1^*})(K_Z, K^{V_1^*}).$$

Therefore $\beta\gamma$ induces the permutation

$$\beta\gamma : (K_I, K_I^{V_1^*})(K, K_Z)(K^{V_1^*}, K_Z^{V_1^*}).$$

Now it is time to apply Lemma 2.5. Put $S^{[0]} = TM \cong 2^6 : L_4(2)$. Then $S^{[01]} := S^{[0]} \cap G^{[01]} = ZBAK$ (the trident group), and we have seen that the latter is normalized by $\alpha\beta\gamma$. Therefore $S^{[1]} := \langle S^{[01]}, t \rangle$ contains $S^{[01]}$ with index 2 and the hypothesis of Lemma 2.5 holds, which completes the proof. $\square$

It only remains to summarise the results of computer calculations using GAP with the completion in Lemma 7.1.

PROPOSITION 7.2. *Let (G, φ) be a faithful completion of $\mathcal{A}_4^{(5)}$, where $G = \text{GL}_{255}(\mathbb{C})$, such that $\text{tr}(\varphi(t)) = 15$ for t as in (A4), and let I be the subgroup generated by the image of φ . Then the following hold.*

(i) $I \cong \text{Alt}_{256}$.

(ii) $G^{[2]} \cong 2^{3+12+2}.\text{Sym}_3.(\text{Sym}_5 \times 2)$.

(iii) $G^{[3]} \cong (2_+^{1+12} * 2^3).L_6(2) : 2$, so that $C_{G^{[3]}}(K^{[3]})/Z(K^{[3]}) \cong 2^2$.

In particular, the completion is constrained at level 2 but not at level 3.

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