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Codes over Matrix Rings for Space-Time Coded Modulations

Frédérique Oggier, Patrick Solé, Jean-Claude Belfiore

Abstract

It is known that, for transmission over quasi-static MIMO fading channels with n transmit antennas, diversity can be obtained by using an inner fully diverse space-time block code while coding gain, derived from the determinant criterion, comes from an appropriate outer code. When the inner code has a cyclic algebra structure over a number field, as for perfect space-time codes, an outer code can be designed via coset coding, more precisely, by taking the quotient of the algebra by a two-sided ideal which leads to matrices over finite alphabets for the outer code. In this paper, we show that the determinant criterion induces various metrics on the outer code, such as the Hamming and Bachoc distances. When $n = 2$, partitioning the 2×2 Golden code by using an ideal above the prime 2 leads to consider codes over either $\mathcal{M}_2(\mathbb{F}_2)$ or $\mathcal{M}_2(\mathbb{F}_2[i])$, both being non-commutative alphabets. By identifying them as algebras over a finite field or a finite ring respectively, we establish an unexpected connection with classical error-correcting codes over $\mathbb{F}_4$ and $\mathbb{F}_4[i]$. Matrix rings of higher dimension, suitable for 3×3 and 4×4 perfect codes, give rise to more complex examples.

Keywords: Space-time codes, codes over rings, cyclic algebras, number fields, finite rings, Golden code.

I. INTRODUCTION

We consider the problem of coding over a quasi-static (slow) fading MIMO channel, for example in a mobile wireless setting, where the channel is assumed to be fixed over the duration of a frame. Compared to standard MIMO channels, slow fading induces a loss in diversity, which can be compensated by using concatenated coding schemes, as for example space-time trellis codes [15]. Finer concatenated schemes enable to distinguish the two main design criteria, namely the rank and determinant criteria: an inner

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code guarantees full diversity, while combining with an outer code brings coding gain. Any fully diverse space-time code can be used as inner code, but in this work, we will focus on codes built over cyclic division algebras [13], [8] whose algebraic structure is easier to analyze.

A. Related work

Most attempts in the literature to obtain coded modulation schemes for algebraic space-time codes focused on having the so-called Golden code [2] as inner code. In the first attempt [3], the Golden code was concatenated with an outer trellis code, whose drawback is its high trellis complexity. Trellis coded modulation using a set partitioning of the Golden code is studied in [5], where a systematic design approach is proposed: partitions of the Golden code with increasing minimum determinant correspond to $\mathbb{Z}^8$ lattice partitions, which are labeled by using a sequence of nested binary codes. The disadvantage of trellis approaches is that the number of trellis states and branches needed grows too fast (starting from 256 states, each having 256 branches). In [7], the Golden code is serially concatenated with a convolution code, and simulations results show that this approach only yields a marginal gain compared to spatial division multiplexing with the same outer code. In [6], the algebraic structure of the Golden code partitions is investigated, and the authors show that they are actually dealing with matrices over the finite field $\mathbb{F}_2$, or over the finite ring $\mathbb{F}_2[i]$. The problem becomes thus the one of designing a suitable outer code over the given ring of matrices, for which only two examples are given: one repetition code of length 2, and one ad hoc construction using Reed Solomon codes. In [9], codes over $\mathcal{M}_2(\mathbb{F}_2)$ and $\mathcal{M}_2(\mathbb{F}_2[i])$ have been proposed, with applications to modulation schemes for the Golden code.

Generalizations to higher dimensional perfect codes are reported in [11] where a partition of a 4×4 perfect code is considered, similar to what had been proposed earlier for the Golden code, that is using construction A at the encoder and constructing the lattices corresponding to perfect subcodes at different levels of the partition chain. In [10], it has been shown that for dimensions 3 and 4, codes to be designed are over respectively $\mathcal{M}_3(\mathbb{F}_4)$ and $\mathcal{M}_4(\mathbb{F}_2)$.

B. Contribution and organization

The original motivation for this paper is the observation that all previous works base their code design on a coarse bound depending on the minimum Hamming distance of the outer code. Instead, the determinant criterion drives us, here, to consider other weights than the Hamming weight. These alternative weights can take more than one nonzero value, a feature that allows us to derive finer lower bounds. New codes

both for the Hamming distance (extending thus the work of [6]) and for the newly introduced distances are proposed.

Furthermore, the present paper deepens previous results in two main ways. First, we explore outer code constructions when the inner code has higher dimension than the Golden code: we propose, for instance, a multilevel code construction over $\mathcal{M}_4(\mathbb{F}_2)$. Second, for $n = 2$, we go one level deeper in the partition of the Golden code, by quotienting with an ideal of higher norm. This enlarges the base ring, thus moving matrix entries from $\mathbb{F}_2$ to $\mathbb{F}_2[i]$. This has for consequence to improve the bound on the minimum determinant, on which depends the determinant criterion.

The material is organized in the following way. Section II gives a general framework for dealing with n -dimensional coset codes. It gives a sequence of isomorphisms yielding four different representations of the outer code alphabet: the quotient of the inner code by a two-sided ideal, an algebra of matrices over a finite field, a cyclic algebra over a finite field, Cartesian products of finite fields. Section III studies weights on the outer code in relation with determinantal lower bounds, for $n = 2, 3, 4$ and presents a multilevel construction for $n = 4$. Section IV is dedicated to the special case $\mathcal{M}_2(\mathbb{F}_2)$, where codes for both the Hamming and Bachoc distances are considered. In Section V, we extend the Bachoc weight to $\mathcal{M}_2(\mathbb{F}_2[i])$ where a bidimensional Lee-like distance is derived. Corresponding codes are proposed. Section VI puts the preceding results into perspective and points out some challenging open problems.

II. COSET CODES

A. Background

For a slow block fading channel, where the fading coefficients are assumed to be constant for L time blocks, the goal is to design a codebook $\tilde{\mathcal{C}}$ of codewords

$$\mathbf{X} = (X_1, \dots, X_L), \quad X_i \in \mathcal{S}$$

for $i = 1, \dots, L$, where $\mathcal{S}$ is a set of codewords from a fully diverse space-time codebook, such that the minimum determinant Δ_{min} of $\tilde{\mathcal{C}}$, given by

$$\Delta_{min} = \min_{\mathbf{X} \neq \mathbf{0}} \det(\mathbf{X}\mathbf{X}^*) \tag{1}$$

$$\begin{aligned} &= \min_{\mathbf{X} \neq \mathbf{0}} \det(X_1 X_1^* + \dots + X_L X_L^*) \\ &\geq \min_{\mathbf{X} \neq \mathbf{0}} \left(\sum_{i=1}^L |\det(X_i)| \right)^2 \end{aligned} \tag{2}$$

is maximized. In this paper, $\mathcal{S}$ will be a set of $n \times n$ perfect space-time codewords [8], [14]. These codes are not only fully diverse, they further offer a good minimum determinant, independently of the size of the signal constellation.

It is known that choosing the blocks X_i independently does not bring coding gain. This is remedied by using outer codes, or more particularly in this setting, coset codes, as proposed in [6]. Consider the projection

$$\begin{aligned} \pi : \mathcal{S} &\rightarrow \mathcal{S}/\mathcal{I} \simeq R \\ X &\mapsto \pi(X) \end{aligned} \quad (3)$$

where $\mathcal{I}$ is a two-sided ideal of $\mathcal{S}$ seen as a ring, so that the quotient $\mathcal{S}/\mathcal{I} \simeq R$ is a ring. We now take a code $\mathcal{C}$ over R . The coset code $\tilde{\mathcal{C}}$ is obtained by considering $\pi^{-1}(\mathcal{C})$.

To evaluate the spectral efficiency of $\tilde{\mathcal{C}}$ independently of the size of the signal constellation in use, we employ the notion of normalized redundancy ρ_{norm} in bits per channel use, defined by

$$\rho_{\text{norm}} = \frac{\text{number of outer code redundancy bits}}{Ln}. \quad (4)$$

Let us now assume that for example a 2^R -QAM is used. We get $nR - \rho_{\text{norm}}$ bits of information per channel use.

We similarly define the notion of normalized redundancy ρ_s in bits per total QAM symbols used:

$$\rho_s = \frac{\text{number of outer code redundancy bits}}{n^2 L} = \frac{\rho_{\text{norm}}}{n}.$$

Let us denote by $\Delta_{\min,c}$ and $\Delta_{\min,u}$ the minimum determinant in the coded, respectively uncoded case.

To compare the gain obtained in the coded case, we compute

$$\frac{\Delta_{\min,c}^{1/n}}{E_c} \frac{E_u}{\Delta_{\min,u}^{1/n}},$$

where $E_u \approx 2^R n^2 L$ refers to the energy needed for sending the $n^2 L$ 2^R -QAM symbols in the uncoded case (where L independent copies of the inner code are sent). Since coding might introduce a constellation expansion, the ratio E_u/E_c has to be taken into account. Now E_c is the energy in the coded case, which must correspond to the same spectral efficiency, that is the use of $n^2 L$ $2^{R+\rho_s}$ -QAM, so that $E_s \approx n^2 L 2^{R+\rho_s}$. This yields

$$\frac{\Delta_{\min,c}^{1/n}}{\Delta_{\min,u}^{1/n}} 2^{-\rho_s} = \left(\frac{\Delta_{\min,c}}{\Delta_{\min,u}} \right)^{1/n} 2^{-\rho_{\text{norm}}/n}.$$

In the following, $\Delta_{\min,c} = \Delta_{\min}$ will be computed as well as ρ_{norm} , from which the normalized minimum determinant can be computed as shown above, since the gain in the uncoded gain is just that of the inner code.

To build coset codes as described above, the first step is to identify the quotient ring $\mathcal{S}/\mathcal{I} \simeq R$. In this section, we show that if we start with $\mathcal{S}$ a code built over a cyclic algebra, then $\mathcal{S}/\mathcal{I}$ also has a cyclic algebra structure however over a finite field.

B. Cyclic algebras

Let us briefly recall the definition of codes built over cyclic algebras, introduced in [13], since perfect space-time codes that are of interest for this work are a subclass.

Definition 1: Let L/K be a cyclic extension of degree n , with Galois group $\text{Gal}(L/K) = \langle \sigma \rangle$, where σ is the generator of the cyclic group. Let $\mathcal{A} = (L/K, \sigma, \gamma)$ be its corresponding *cyclic algebra* of degree n , that is

$$\mathcal{A} = 1L \oplus eL \oplus \dots \oplus e^{n-1}L$$

with $e \in \mathcal{A}$ such that $le = e\sigma(l)$ for all $l \in L$ and $e^n = \gamma \in K$, $\gamma \neq 0$.

Note that L/K is a priori any cyclic field extension. In this paper, we use both cyclic algebras over number fields and cyclic algebras over finite fields.

One can associate a matrix to any element $x \in \mathcal{A}$ using the map λ_x , the multiplication by x of an element $y \in \mathcal{A}$:

$$\begin{aligned} \lambda_x : \mathcal{A} &\rightarrow \mathcal{A} \\ y &\mapsto \lambda_x(y) = x \cdot y. \end{aligned}$$

The matrix of the multiplication by λ_x , with

$$x = x_0 + ex_1 + \dots + e^{n-1}x_{n-1},$$

is given by

$$\begin{pmatrix} x_0 & \gamma\sigma(x_{n-1}) & \gamma\sigma^2(x_{n-2}) & \dots & \gamma\sigma^{n-1}(x_1) \\ x_1 & \sigma(x_0) & \gamma\sigma^2(x_{n-1}) & \dots & \gamma\sigma^{n-1}(x_2) \\ \vdots & & \vdots & & \vdots \\ x_{n-2} & \sigma(x_{n-3}) & \sigma^2(x_{n-4}) & \dots & \gamma\sigma^{n-2}(x_{n-1}) \\ x_{n-1} & \sigma(x_{n-2}) & \sigma^2(x_{n-3}) & \dots & \sigma^{n-1}(x_0) \end{pmatrix}. \quad (5)$$

Perfect codes [8] are codes built over cyclic division algebras, with in particular the property that their minimum determinant is lower bounded by a constant independent of the size of the signal constellation.

n	L/K	$\mathcal{O}_L$
2	$\mathbb{Q}(i, \sqrt{5})/\mathbb{Q}(i)$	$\mathbb{Z}[i, (1 + \sqrt{5})/2]$
3	$\mathbb{Q}(\zeta_3, \zeta_7 + \zeta_7^{-1})/\mathbb{Q}(\zeta_3)$	$\mathbb{Z}[\zeta_3, \zeta_7 + \zeta_7^{-1}]$
4	$\mathbb{Q}(i, \zeta_{15} + \zeta_{15}^{-1})/\mathbb{Q}(i)$	$\mathbb{Z}[i, \zeta_{15} + \zeta_{15}^{-1}]$

TABLE I
ORDERS CORRESPONDING TO SOME PERFECT CODES

This can be achieved by considering the subset of elements $x = x_0 + ex_1 + \dots + e^{n-1}x_{n-1}$, x_i in $\mathcal{O}_L$ instead of L , $k = 1, \dots, n$, where $\mathcal{O}_L$ denote the ring of integers of L . In other words, we consider the subset $\Lambda \subset \mathcal{A}$ given by

$$\Lambda = 1\mathcal{O}_L \oplus e\mathcal{O}_L \oplus \dots \oplus e^{n-1}\mathcal{O}_L,$$

which is actually an order of $\mathcal{A}$, as identified in [4].

For the case of interest to us, K is typically $\mathbb{Q}(i)$ or $\mathbb{Q}(\zeta_3)$, where ζ_3 is a primitive third root of unity, to allow the use of either QAM or HEX symbols. Since their respective rings of integers $\mathbb{Z}[i]$ and $\mathbb{Z}[\zeta_3]$ are principal ideal domains, it makes sense to speak of an $\mathcal{O}_K$ -basis for $\mathcal{O}_L$. We can now be more precise, and recall that for R a Noetherian integral domain with quotient field K , and $\mathcal{A}$ a finite dimensional K -algebra, we have the following definition.

Definition 2: An R -order in the K -algebra $\mathcal{A}$ is a subring Λ of $\mathcal{A}$, having the same identity element as $\mathcal{A}$, and such that Λ is a finitely generated module over R and generates $\mathcal{A}$ as a linear space over K . An order Λ is called *maximal* if it is not properly contained in any other R -order.

In the cyclic algebra $\mathcal{A}$, we can choose the element $0 \neq \gamma \in K$ to be an algebraic integer. We see that the order Λ given above is more precisely an $\mathcal{O}_K$ -order in $\mathcal{A}$. Orders Λ corresponding to the codes from [8], for dimensions 2,3 and 4, are reported in Table I. The table reads that for an $n \times n$ space-time block code, the cyclic field extension used to construct the cyclic algebra $\mathcal{A}$ is L/K , and the order Λ in $\mathcal{A}$ is given by $\Lambda = 1\mathcal{O}_L \oplus e\mathcal{O}_L \oplus \dots \oplus e^{n-1}\mathcal{O}_L$. When $\mathcal{S}$ is a set of codewords coming from division algebras, we can really consider Λ , an order of the algebra as in Definition 2, which has a ring structure.

The $\mathcal{O}_K$ -order Λ of $\mathcal{A}$ is a free module over $\mathcal{O}_K = \mathbb{Z}[i]$ or $\mathbb{Z}[\zeta_3]$, with basis $\{b_i\}$, $i = 1, \dots, n^2$:

$$\Lambda \simeq \bigoplus_{i=1}^{n^2} b_i \mathcal{O}_K,$$

since for us $\mathcal{O}_L$ is a free $\mathcal{O}_K$ -module of rank n (say with basis β_k , $k = 1, \dots, n$):

$$\begin{aligned}\Lambda &\simeq \bigoplus_{j=1}^n e^j \mathcal{O}_L \\ &\simeq \bigoplus_{j=1}^n e^j \bigoplus_{k=1}^n \beta_k \mathcal{O}_K.\end{aligned}$$

The basis vectors $\{b_i\}$ are thus given by

$$\{e^j \beta_k\}, \quad j, k = 1, \dots, n.$$

Let $\mathfrak{a}$ be a two-sided ideal of $\mathcal{O}_K$. Since $\mathcal{O}_K$ is commutative, we have that

$$\Lambda/\mathfrak{a}\Lambda \simeq \bigoplus_{i=1}^{n^2} b_i \mathcal{O}_K / b_i \mathfrak{a}$$

is a free module over the ring $\mathcal{O}_K/\mathfrak{a}\mathcal{O}_K$, with basis $\{\pi(b_i)\}$, $i = 1, \dots, n^2$, where π is the canonical projection

$$\pi : \Lambda \rightarrow \Lambda/\mathfrak{a}\Lambda.$$

The above considerations mean the following for our setting.

Lemma 1: For $n = 2, 3, 4$ respectively, we have:

- 1) If $\Lambda = \mathbb{Z}[i, (1 + \sqrt{5})/2] + e\mathbb{Z}[i, (1 + \sqrt{5})/2]$, $\mathfrak{a} = (1 + i)$, then

$$\mathbb{Z}[i]/\mathfrak{a}\mathbb{Z}[i] \simeq \mathbb{F}_2$$

and $\Lambda/\mathfrak{a}\Lambda$ is a $\mathbb{F}_2$ -module of rank 4. In particular, we have that

$$|\Lambda/\mathfrak{a}\Lambda| = 2^4.$$

- 2) If $\Lambda = \mathbb{Z}[\zeta_3, \zeta_7 + \zeta_7^{-1}] + e\mathbb{Z}[\zeta_3, \zeta_7 + \zeta_7^{-1}] + e^2\mathbb{Z}[\zeta_3, \zeta_7 + \zeta_7^{-1}]$, $\mathfrak{a} = 2$, then

$$\mathbb{Z}[\zeta_3]/\mathfrak{a}\mathbb{Z}[\zeta_3] \simeq \mathbb{F}_2^2$$

and $\Lambda/\mathfrak{a}\Lambda$ is a $\mathbb{F}_4$ -module of rank 9. In particular, we have that

$$|\Lambda/\mathfrak{a}\Lambda| = 4^9.$$

- 3) If $\Lambda = \mathbb{Z}[i, \zeta_{15} + \zeta_{15}^{-1}] + e\mathbb{Z}[i, \zeta_{15} + \zeta_{15}^{-1}] + e^2\mathbb{Z}[i, \zeta_{15} + \zeta_{15}^{-1}] + e^3\mathbb{Z}[i, \zeta_{15} + \zeta_{15}^{-1}]$, $\mathfrak{a} = (1 + i)$, then

$$\mathbb{Z}[i]/\mathfrak{a}\mathbb{Z}[i] \simeq \mathbb{F}_2$$

and $\Lambda/\mathfrak{a}\Lambda$ is a $\mathbb{F}_2$ -module of rank 16. In particular, we have that

$$|\Lambda/\mathfrak{a}\Lambda| = 2^{16}.$$

Note that 2 is prime in $\mathbb{Z}[\zeta_3]$, while 2 ramifies as $2 = (1+i)(1-i) = (1+i)^2$ in $\mathbb{Z}[i]$.

This yields that

Proposition 1: We have

$$\begin{aligned} \Lambda/\mathfrak{a}\Lambda &\simeq \mathcal{M}_n(\mathcal{O}_K/\mathfrak{a}\mathcal{O}_K) \\ &\simeq \begin{cases} \mathcal{M}_2(\mathbb{F}_2) & \text{for } n = 2 \\ \mathcal{M}_3(\mathbb{F}_4) & \text{for } n = 3 \\ \mathcal{M}_4(\mathbb{F}_2) & \text{for } n = 4 \end{cases} \end{aligned}$$

Proof: By the previous lemma, we already know that $\Lambda/\mathfrak{a}\Lambda$ is a $\mathcal{O}_K/\mathfrak{a}\mathcal{O}_K$ -module whose cardinality is the same as $\mathcal{M}_n(\mathcal{O}_K/\mathfrak{a}\mathcal{O}_K)$. It is thus enough to give a ring homomorphism $\psi : \Lambda/\mathfrak{a}\Lambda \rightarrow \mathcal{M}_n(\mathcal{O}_K/\mathfrak{a}\mathcal{O}_K)$ which is one-to-one to conclude, and ψ can be defined by mapping the basis vectors $\pi(e^j \beta^k)$. ■

The particular case for $n = 2$ was proved in [6]. The meaning of this proposition is that when considering the projection (3)

$$\begin{aligned} \pi : \mathcal{S} &\rightarrow \mathcal{S}/\mathcal{I} \simeq R \\ X &\mapsto \pi(X) \end{aligned}$$

to build coset codes with $\mathcal{S}$ coming from perfect codes, we need to build codes over matrices over finite fields. We prove next that an alternative point of view is to ask for codes over cyclic algebras over finite fields.

C. Cyclic algebras over finite fields

Let $\mathbb{F}_2$ be the finite field with 2 elements, and consider the field extension $\mathbb{F}_{2^n}/\mathbb{F}_2$ of degree n , that is $\mathbb{F}_{2^n} \simeq \mathbb{F}_2(w)$ with $p(w) = 0$ and $p \in \mathbb{F}_2[X]$ is an irreducible polynomial of degree n . Its cyclic Galois group is generated by the Frobenius automorphism $\sigma : w \mapsto w^2$. We consider the cyclic algebra $\mathcal{A} = (\mathbb{F}_{2^n}/\mathbb{F}_2, \sigma, 1)$, with

$$\mathcal{A} \simeq \mathbb{F}_{2^n} \oplus \dots e \mathbb{F}_{2^n} \oplus e^{n-1} \mathbb{F}_{2^n}$$

(see Definition 1). We know by Lemma 2.16 in [12] that $\mathcal{A} \simeq \text{End}_{\mathbb{F}_2}(\mathbb{F}_{2^n})$. The isomorphism $j : \mathcal{A} \rightarrow \text{End}_{\mathbb{F}_2}(\mathbb{F}_{2^n})$ is explicitly given by $j(a)$, which is the multiplication by a for all a in $\mathbb{F}_{2^n}$, and $j(e) = \sigma$. Indeed, we have that

$$j(ae)(x) = (j(a)j(e))(x) = j(a)\sigma(x) = a\sigma(x)$$

which in turn can be written

$$j(e)(\sigma(a)x) = j(e)j(\sigma(a))(x) = j(e\sigma(a))(x)$$

thus $j(ae) = j(e\sigma(a))$.

Example 1: We consider the cyclic algebra $\mathcal{A} = (\mathbb{F}_8/\mathbb{F}_2, \sigma, 1)$, where $\mathbb{F}_8 \simeq \mathbb{F}_2(w)$ with $w^3 + w + 1 = 0$ and $\sigma : w \mapsto w^2$. As a vector space, we have $\mathcal{A} \simeq \mathbb{F}_8 \oplus e\mathbb{F}_8 \oplus e^2\mathbb{F}_8$ and multiplication is given by $ae = e\sigma(a)$ for $a \in \mathbb{F}_8$. We have that $\mathcal{A} \simeq \mathcal{M}_3(\mathbb{F}_2)$. The isomorphism is given as follows:

$$\begin{aligned} e &\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ a_0 + a_1w + a_2w^2 &\mapsto \begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_0 + a_2 & a_1 \\ a_1 & a_1 + a_2 & a_0 + a_2 \end{pmatrix}. \end{aligned}$$

It is a straightforward computation to check that

$$\begin{aligned} (a_0 + a_1w + a_2w^2)e &= e\sigma(a_0 + a_1w + a_2w^2) \\ &= e(a_0 + a_1w^2 + a_2(w^2 + w)). \end{aligned}$$

Example 2: Consider now the cyclic algebra $\mathcal{A} = (\mathbb{F}_{16}/\mathbb{F}_2, \sigma, 1)$, where $\mathbb{F}_{16} \simeq \mathbb{F}_2(w)$ with $w^4 + w^2 + 1 = 0$ and $\sigma : w \mapsto w^2$. We have that $\mathcal{A} \simeq \mathcal{M}_4(\mathbb{F}_2)$. The isomorphism is given as follows:

$$e \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad w \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

The above example gives us an explicit isomorphism

$$\mathcal{M}_4(\mathbb{F}_2) \simeq \mathbb{F}_{16} \oplus e\mathbb{F}_{16} \oplus e^2\mathbb{F}_{16} \oplus e^3\mathbb{F}_{16}. \quad (6)$$

We now finish this sequence of isomorphisms, and connect codes on cyclic algebras over finite fields to classical error correcting codes. The isomorphism

$$\mathcal{A} \simeq \mathbb{F}_{2^n} \oplus \dots \mathbb{F}_{2^n}e \oplus \mathbb{F}_{2^n}e^{n-1} \simeq \mathcal{M}_n(\mathbb{F}_2)$$

clearly induces an isomorphism of $\mathbb{F}_2$ -left vector space

$$\phi : \underbrace{\mathbb{F}_{2^n} \times \dots \times \mathbb{F}_{2^n}}_n \rightarrow \mathcal{M}_n(\mathbb{F}_2).$$

Also, ϕ can be extended to L -tuples

$$\phi : (\mathbb{F}_{2^n} \times \dots \times \mathbb{F}_{2^n})^L \rightarrow \mathcal{M}_n(\mathbb{F}_2)^L$$

so that if $\mathcal{C}$ is a code of length L over $\mathcal{M}_n(\mathbb{F}_2)$, then $\phi^{-1}(\mathcal{C})$ is a code of length $2L$ over $\mathbb{F}_{2^n}$.

This connection with classical codes has been introduced first in [1] for the construction of particular lattices.

III. WEIGHTS AND CODES

In this section, we propose a multilevel construction for codes over rings of matrices with coefficients in finite fields. To see which performance the code should reach, we first compute a bound on the minimum determinant.

A. Hamming distance bound

The determinant of a $n \times n$ codeword $X \in \mathcal{S}$ can be bounded depending on its projection $\pi(X) \in \mathcal{S}/\mathcal{I}$, as follows for the case when $\mathcal{I} = a\mathcal{S}$, a a scalar.

Lemma 2: We have that

- 1) $|\det(X)|^2 \geq |a^n|^2 \delta$ if $\pi(X) = \mathbf{0}$, $\mathbf{0} \neq X$,
- 2) $|\det(X)|^2 \geq |a|^2 \delta$ if $\mathbf{0} \neq \pi(X)$ is not a unit,
- 3) $|\det(X)|^2 \geq \delta$ if $\pi(X)$ is a unit,

where $\delta = \min_{X \in \mathcal{S}} |\det(X)|^2$.

Proof:

- 1) If $\pi(X) = \mathbf{0}$, with $\mathbf{0} \neq X$, then $X = a\tilde{X} \in a\mathcal{S}$, and $\det(X) = a^n \det(\tilde{X})$.
- 2) If $\mathbf{0} \neq \pi(X)$ is not a unit, then its determinant has to be zero, that is $\det(\pi(X)) = 0$, implying that $\det(X)$ is a multiple of a .
- 3) If $\pi(X)$ is a unit, then its determinant has to be one too.

■

We have from (2) that

$$\Delta_{min} \geq \min_{\mathbf{X} \neq \mathbf{0}} \left(\sum_{i=1}^L |\det(X_i)| \right)^2.$$

For each of the X_i , we have that $\pi(X_i)$ may or may not be $\mathbf{0}$, and if it is non-zero, it may or not be invertible. Ideally, we would like to be able to distinguish these three cases, since we have by Lemma 2

that

$$\begin{aligned}
|\det(X)| &\geq |a^n|\sqrt{\delta} \text{ if } \pi(X) = \mathbf{0}, X \neq \mathbf{0} \\
|\det(X)| &\geq |a|\sqrt{\delta} \text{ if } \mathbf{0} \neq \pi(X) \text{ is not a unit} \\
|\det(X)| &\geq \sqrt{\delta} \text{ if } \pi(X) \text{ is a unit.}
\end{aligned}$$

To start with, let us give a bound which only takes into account zero and non-zero elements (the all zero elements case is treated afterwards). Let $d_{\min}^H$ be the minimum Hamming distance of the code $\pi(\mathcal{C})$, which is as in the classical case the number of different components between any two pairs of codewords. If not all $\pi(X_j) = \mathbf{0}$, then by definition of $\pi(\mathcal{C})$, there are at least $d_{\min}^H$ terms such that $\pi(X_j) \neq \mathbf{0}$. We give those X_j a weight of $\sqrt{\delta}$ (instead of either $|a|\sqrt{\delta}$ or $\sqrt{\delta}$). Thus

$$\begin{aligned}
\Delta_{\min} &\geq \min_{\mathbf{X} \neq \mathbf{0}} \left(\sum_{i=1}^L |\det(X_i)| \right)^2 \\
&\geq (d_{\min}^H)^2 \delta.
\end{aligned}$$

Note that this is far from a tight bound, since we would like most of the weight to be given to codewords whose projection is either zero or a non-invertible element.

Since the case where all the $\pi(X_j) = 0$ is not included above ($d_{\min}^H$ does not apply), we treat this case separately. Let us thus assume that $\pi(X_j) = 0$ for all $j = 1, \dots, L$. We then have $X_j = a\tilde{X}_j$, and

$$\begin{aligned}
\Delta_{\min} &= \min_{\mathbf{X} \neq \mathbf{0}} \det(X_1 X_1^* + \dots + X_L X_L^*) \\
&= \min_{\mathbf{X} \neq \mathbf{0}} \det(|a|^2 \mathbf{I}_n) \det(\tilde{X}_1 \tilde{X}_1^* + \dots + \tilde{X}_L \tilde{X}_L^*) \\
&= |a|^{2n} \min_{\mathbf{X} \neq \mathbf{0}} \det(\tilde{X}_1 \tilde{X}_1^* + \dots + \tilde{X}_L \tilde{X}_L^*) \\
&\geq |a|^{2n} \delta.
\end{aligned}$$

Note that $\Delta_{\min}$ is actually equal to $|a|^{2n} \delta$ if $X = (a\tilde{X}_1, \mathbf{0}, \dots, \mathbf{0})$. The problem is that the criterion $\pi(X) = \mathbf{0}$ does not allow to distinguish $X \neq \mathbf{0}$ and $\mathbf{0} \neq X \subset a\mathcal{S}$.

Lemma 3: (Hamming distance bound) We have that

$$\Delta_{\min} \geq \min(|a|^{2n} \delta, (d_{\min}^H)^2 \delta).$$

It was proved in [6] for $n = 2$ (that is the Golden code) that

$$\Delta_{\min} \geq \min(4\delta, (d_{\min}^H)^2 \delta), \quad \mathcal{I} = (1 + i),$$

where $d_{\min}^H$ is the minimum Hamming distance of the code over $\mathcal{M}_2(\mathbb{F}_2)$, and $\delta = 1/5$ is the minimum determinant of the Golden code. This above lemma tells us that

$$\Delta_{\min} \geq \min(64\delta', (d_{\min}^H)^2\delta'), \mathcal{I} = (2),$$

for $n = 3$ over $\mathcal{M}_3(\mathbb{F}_4)$, $\delta' = 1/49$, and

$$\Delta_{\min} \geq \min(16\delta'', (d_{\min}^H)^2\delta''), \mathcal{I} = (1+i),$$

for $n = 4$ over $\mathcal{M}_4(\mathbb{F}_2)$ and $\delta'' = 1/1125$.

B. Multilevel coding for $n = 4$

We know from (6) that

$$\mathcal{M}_4(\mathbb{F}_2) \simeq \mathbb{F}_{16} \oplus e\mathbb{F}_{16} \oplus e^2\mathbb{F}_{16} \oplus e^3\mathbb{F}_{16}.$$

Note that $(1+e)^4 = (1+e^2)(1+e^2) = 0$, showing that $1+e$ is nilpotent. We set $f = 1+e$, and do a change of basis to get

$$\mathcal{M}_4(\mathbb{F}_2) \simeq \mathbb{F}_{16} \oplus f\mathbb{F}_{16} \oplus f^2\mathbb{F}_{16} \oplus f^3\mathbb{F}_{16}.$$

Via this isomorphism, we can write a matrix $X \in \mathcal{M}_4(\mathbb{F}_2)$ as an element $x \in \mathbb{F}_{16} \oplus f\mathbb{F}_{16} \oplus f^2\mathbb{F}_{16} \oplus f^3\mathbb{F}_{16}$, given by

$$x = x_0 + fx_1 + f^2x_2 + f^3x_3.$$

Note that if $x_0 = 0$, then x is not invertible, since $x = f(x_1 + fx_2 + f^2x_3)$, with f nilpotent.

Consequently, to a codeword $(\pi(X_1), \dots, \pi(X_L)) \in \mathcal{M}_4(\mathbb{F}_2)^L$ corresponds a vector

$$(x_{10} + fx_{11} + f^2x_{12} + f^3x_{13}, \dots, x_{L0} + fx_{L1} + f^2x_{L2} + f^3x_{L3}).$$

The first level of coding is done using a (L, k_0, d_0) code C_0 that maps k_0 symbols to $x_{10}, \dots, x_{L0}$.

Similarly, the i th level of coding uses a (L, k_i, d_i) code C_i that maps k_i symbols to $x_{1i}, \dots, x_{Li}$, $i = 1, 2, 3$.

Since C_0 has minimum distance d_0 , either all coefficients are zero, or at least d_0 coefficients out of $x_{10}, \dots, x_{L0}$ are non-zero.

- In the latter case, out of $(\pi(X_1), \dots, \pi(X_L))$, d_0 matrices may or may not be invertible, thus having a determinant that may or may not be invertible. An invertible determinant $\det(\pi(X))$ gives the lowest weight, that is, $|\det(X)|^2 \geq \delta$, and all we can say is

$$\min \sum_{i=1}^L |\det(X_i)| \geq d_0 \sqrt{\delta}.$$

Fig. 1. Multilevel encoder for $\mathcal{M}_4(\mathbb{F}_2)$.

- For $x_{10} = \dots = x_{L0} = 0$, we can do the same reasoning for C_1 , except that the situation is more favorable: indeed, all matrices $(\pi(X_1), \dots, \pi(X_L))$ are now not invertible, meaning that their determinant is a multiple of $1 + i$, showing that in this case

$$\min \sum_{i=1}^L |\det(X_i)| \geq \sqrt{2}d_1\sqrt{\delta}.$$

By iterating the same steps for C_2 and C_3 , we get that

$$\min \sum_{i=1}^L |\det(X_i)| \geq \min\{d_0, \sqrt{2}d_1, 2d_2, 2\sqrt{2}d_3\}\sqrt{\delta}.$$

Note that this bound takes only into account the multilevel code, and not the fact that we use a coset code, which gives a further constraint (as shown in the previous subsection), finally yielding:

$$\min \sum_{i=1}^L |\det(X_i)| \geq \min\{4, d_0, \sqrt{2}d_1, 2d_2, 2\sqrt{2}d_3\}\sqrt{\delta}.$$

It is thus enough to guarantee:

$$d_0 = 4, \quad d_1 = 3, \quad d_2 = 2, \quad d_3 = 2.$$

Parity codes can be used for C_3 and C_4 . For example, for $L = 16$, we can choose

$$\begin{aligned}
C_0 &= \text{the } (16, 13, 4) \text{ Reed Solomon code,} \\
C_1 &= \text{the } (16, 14, 3) \text{ Reed Solomon code,} \\
C_2 &= \text{the } (16, 15, 2) \text{ parity check code,} \\
C_3 &= \text{the } (16, 15, 2) \text{ parity check code,}
\end{aligned}$$

for a rate of $47/64 \approx 0.734$. From (4), the normalized redundancy of $\tilde{C}$ when the above outer code C is used is

$$\begin{aligned}
\rho_{\text{norm}} &= \frac{\text{outer code redundancy bits}}{Ln} \\
&= \frac{(1 + 1 + 2 + 3 \text{ symbols in } \mathbb{F}_{16})(4 \text{ bits})}{4L} \\
&= \frac{28}{64} = \frac{7}{16}.
\end{aligned}$$

IV. CODES OVER $\mathcal{M}_2(\mathbb{F}_2)$

In the rest of the paper, we pay a special attention to the case $n = 2$, for which we take as inner code the Golden code $\mathcal{G}$ [2]:

Definition 3: A codeword X belonging to the Golden code $\mathcal{G}$ has the form

$$X = \frac{1}{\sqrt{5}} \begin{pmatrix} \alpha(a + b\theta) & \alpha(c + d\theta) \\ i\bar{\alpha}(c + d\bar{\theta}) & \bar{\alpha}(a + b\bar{\theta}) \end{pmatrix}$$

where a, b, c, d are QAM symbols (that is, $a, b, c, d \in \mathbb{Z}[i]$), $\theta = \frac{1+\sqrt{5}}{2}$, $\bar{\theta} = \frac{1-\sqrt{5}}{2}$, $\alpha = 1 + i - i\theta$ and $\bar{\alpha} = 1 + i - i\bar{\theta}$. Its minimum determinant is given by

$$\delta = \min_{\mathbf{0} \neq X \in \mathcal{G}} |\det(X)|^2 = \frac{1}{5},$$

in particular it is always different from 0, and the Golden code is fully diverse.

The ring structure of the Golden code is best seen if we rewrite

$$\mathcal{G} = \alpha(\mathbb{Z}[i, \theta] \oplus e\mathbb{Z}[i, \theta]),$$

where e is an element of $\mathcal{G}$ such that $e^2 = i$, as already mentioned in Table I. In what follows, we will see $\mathcal{G}$ either formally as above, or as a set of matrices.

A. Codes over $\mathcal{M}_2(\mathbb{F}_2)$ and Hamming weight

We now discuss how codes over $\mathcal{M}_2(\mathbb{F}_2)$ can be obtained from codes over $\mathbb{F}_4$. We start by showing how error correcting codes over $\mathbb{F}_4$ can be expressed as codes over $\mathcal{M}_2(\mathbb{F}_2)$. The starting point is the correspondence between elements in $\mathbb{F}_4$ and matrices in $\mathcal{M}_2(\mathbb{F}_2)$ as given by the lemma below.

Lemma 4: Let $\mathbb{F}_2$ be the finite field with 2 elements, and $\mathbb{F}_4 = \mathbb{F}_2(\omega)$ be the finite field with 4 elements, where $\omega^2 + \omega + 1 = 0$. There is a correspondence between the element $a = a_1 + a_2\omega$ in $\mathbb{F}_4$ and the matrix

$$M_a = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 + a_2 \end{pmatrix}.$$

Proof: The matrix M_a is just the multiplication matrix by a , since

$$(1, \omega) \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 + a_2 \end{pmatrix} = (a, a\omega).$$

■

We thus define a code $\mathcal{C}$ over $\mathcal{M}_2(\mathbb{F}_2)$ as follows: let $(x_1, \dots, x_L)$ be a codeword of an $[L, k, d]$ code over $\mathbb{F}_4$. Then

$$\mathcal{C} = \{(X_1, \dots, X_L) \mid X_i = M_{x_i} \in \mathcal{M}_2(\mathbb{F}_2), i = 1, \dots, L\}.$$

The code is clearly linear since $M_{a-b} = M_a - M_b$.

This allows to easily show that the minimum distance $d_{\min}^H$ of the code over $\mathcal{M}_2(\mathbb{F}_2)$ is d , the minimum distance of the code over $\mathbb{F}_4$. Indeed, we have that

$$d_{\min} = \min_{\mathbf{0} \neq X} w_H((X_1, \dots, X_L))$$

and

$$X_i = M_{x_i} = \mathbf{0} \iff x_i = 0.$$

A code over $\mathbb{F}_4$ easily available for arbitrary values of L is the parity code, the dual of the repetition code of length L . The corresponding code over $\mathcal{M}_2(\mathbb{F}_2)$ is

$$\mathcal{C} = \{(X_1 + X_2 + \dots + X_{L-1}, X_1, X_2, \dots, X_{L-1}) \mid X_i = M_{x_i} \in \mathcal{M}_2(\mathbb{F}_2)\},$$

with minimum distance $d_{\min}^H = 2$.

Example 3: Consider the $[4, 3, 2]$ cyclic linear code over $\mathbb{F}_4$, given by the dual of the repetition code of length $L = 4$. Since the generator matrix of the repetition code is $G = (1, 1, 1, 1)$, its parity check matrix is thus

$$H = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

which is in turn the generator matrix of the dual. Thus a codeword is of the form

$$(x_1 + x_2 + x_3, x_1, x_2, x_3).$$

In words, any coefficient is the sum of the 3 others, thus clearly any shift of such codeword is also a codeword, and we obtain a parity code, which is cyclic, with parameters $[4, 3, 2]$.

The corresponding codeword of length $L = 4$ over $\mathcal{M}_2(\mathbb{F}_2)$ is

$$X = (X_1 + X_2 + X_3, X_1, X_2, X_3)$$

where

$$X_i = \begin{pmatrix} x_{i1} & x_{i2} \\ x_{i2} & x_{i1} + x_{i2} \end{pmatrix} \in \mathcal{M}_2(\mathbb{F}_2)$$

and

$$x_i = x_{i1} + x_{i2}\omega, \quad i = 1, 2, 3, \quad x_{i1}, x_{i2} \in \mathbb{F}_2.$$

Thus

$$\mathcal{C} = \{(X_1 + X_2 + X_3, X_1, X_2, X_3) \mid X_i = M_{x_i} \in \mathcal{M}_2(\mathbb{F}_2)\}$$

is a linear code over $\mathcal{M}_2(\mathbb{F}_2)$ with $d_{\min}^H = 2$.

Now we know from Lemma 3 that

$$\Delta_{\min} \geq \min(4\delta, (d_{\min}^H)^2\delta),$$

so that the parity code is a good candidate, since it satisfies

$$d_{\min}^H = 2.$$

From (4), the normalized redundancy of $\tilde{\mathcal{C}}$ when the dual of the repetition code is used as outer code is

$$\begin{aligned} \rho_{\text{norm}} &= \frac{\text{outer code redundancy bits}}{Ln} \\ &= \frac{(L-1) + 4}{2L} = \frac{L+3}{2L}. \end{aligned}$$

This code has the right minimum distance with respect to the bound of Lemma 3. However its normalized redundancy could be improved, which motivates a second construction. Let

$$\mathcal{C}_M = \{(X_1, X_2, \dots, X_{L-1}, X_1 + X_2 + \dots + X_{L-1}) \mid X_i \in \mathcal{M}_2(\mathbb{F}_2)\}$$

be a code defined directly over $\mathcal{M}_2(\mathbb{F}_2)$ by mimicking our first construction. It encodes $L-1$ elements of $\mathcal{M}_2(\mathbb{F}_2)$ (for a total of $4(L-1)$ bits) into a vector of length L . It is clearly linear, and its minimum distance is $d_{\min}^H = 2$. The normalized redundancy is now

$$\begin{aligned} \rho_{\text{norm}} &= \frac{\text{outer code redundancy bits}}{Ln} \\ &= \frac{4}{2L}. \end{aligned}$$

code	coding gain	ρ_{norm} (bpc)	L
parity code I	4/5	$\frac{L+3}{2L}$	arbitrary
parity code II	4/5	$\frac{4}{2L}$	arbitrary

TABLE II
SUMMARY OF THE PERFORMANCE OF THE PROPOSED CODES.

B. Codes over $\mathcal{M}_2(\mathbb{F}_2)$ and Bachoc weight

So far, we have provided code constructions based on the design criterion of Lemma 3, which is actually a coarse bound, as already noticed during its derivation. Recall from Lemma 2 that for $n = 2$ and $\mathcal{I} = (1 + i)$

$$\begin{aligned} |\det(X)| &\geq 2\sqrt{\delta} \text{ if } \pi(X) = \mathbf{0}, X \neq \mathbf{0} \\ |\det(X)| &\geq \sqrt{2\delta} \text{ if } \mathbf{0} \neq \pi(X) \text{ is not a unit} \\ |\det(X)| &\geq \sqrt{\delta} \text{ if } \pi(X) \text{ is a unit.} \end{aligned}$$

To get the bound of Lemma 3, we use the Hamming weight, that is, we assign a weight of either 1 or 0 on matrices in $\mathcal{M}_2(\mathbb{F}_2)$, to which corresponds a weight of $\sqrt{\delta}$ to each X such that $\pi(X) \neq \mathbf{0}$, and a weight of zero otherwise. We are thus losing a lot of information. In this section, we introduce a new weight to replace the Hamming weight, which will tighten the bound for the minimum determinant.

We consider the new weight w_B on $\mathcal{M}_2(\mathbb{F}_2)$, that we call Bachoc weight, as proposed by C. Bachoc in [1], by setting

$$w_B(Y) = \begin{cases} 0 & \text{if } Y = \mathbf{0} \\ 1 & Y \text{ if is a unit} \\ 2 & \mathbf{0} \neq Y \text{ if is not a unit} \end{cases}. \quad (7)$$

Correspondingly, we get for X the weight

$$\begin{cases} 0 & \text{if } \pi(X) = \mathbf{0} \\ \sqrt{\delta} & \text{if } \pi(X) \text{ is a unit} \\ \sqrt{2\delta} & \text{if } \mathbf{0} \neq \pi(X) \text{ is not a unit} \end{cases}.$$

This weight is clearly finer than the Hamming weight (see Table III for a comparison), since it allows to distinguish non-zero invertible and non-invertible matrices in $\mathcal{M}_2(\mathbb{F}_2)$.

Definition 4: We naturally define the corresponding minimum Bachoc distance

$$d_{\min}^B = \min_{Y \neq Y'} w_B(Y - Y').$$

$\pi(X)$	ideal weight	Bachoc weight	Hamming weight
$\mathbf{0}$	$2\sqrt{\delta}$	0	0
non-unit	$\sqrt{2\delta}$	$\sqrt{2\delta}$	$\sqrt{\delta}$
unit	$\sqrt{\delta}$	$\sqrt{\delta}$	$\sqrt{\delta}$

TABLE III

COMPARISON AMONG WEIGHTS FOR $\mathbf{0} \neq X$, DEPENDING ON $\pi(X)$

Let us now see how we can revisit the original design criterion based on the new weight we have just introduced. Recall that by (2)

$$\begin{aligned} \Delta_{min} &= \min_{\mathbf{X} \neq \mathbf{0}} \det(\mathbf{X}\mathbf{X}^*) \\ &\geq \min_{\mathbf{X} \neq \mathbf{0}} \left(\sum_{i=1}^L |\det(X_i)| \right)^2. \end{aligned}$$

Let us now look at $\sum_{i=1}^L |\det(X_i)|$. It is lower bounded by $(L + i(\sqrt{2} - 1))\sqrt{\delta}$, if i counts the number of non-invertible projections. In particular, the lower bound ranges from L to $\sqrt{2}L$. Now the Bachoc weight, again if i counts the number of non-invertible projections, is $L + i$, which ranges from L to $2L$. Thus $\sum_{j=1}^L |\det(X_j)| \geq (L + i(\sqrt{2} - 1))\sqrt{\delta} \geq ((L + i)/\sqrt{2})\sqrt{\delta}$, and

$$\begin{aligned} \Delta_{min} &\geq \min \left(\sqrt{\delta} w_B / \sqrt{2} \right)^2 \\ &= \frac{(\delta d_{\min}^B)^2}{2}. \end{aligned}$$

Lemma 5: (Bachoc distance bound) We have that

$$\Delta_{min} \geq \min \left(4\delta, \frac{(d_{\min}^B)^2}{2} \delta \right).$$

Let us now see how to construct codes where the Bachoc weight can be controlled. Let again $\mathbb{F}_2$ be the finite field with 2 elements, and $\mathbb{F}_4 = \mathbb{F}_2(\omega)$ be the finite field with 4 elements, where $\omega^2 + \omega + 1 = 0$. As shown in Subsection II-C, we have a ring isomorphism

$$\mathcal{M}_2(\mathbb{F}_2) \simeq \mathbb{F}_2(\omega) + j\mathbb{F}_2(\omega) \tag{8}$$

where $j^2 = 1$ and $\omega j = j\omega^2$, which is explicitly given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto j, \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \mapsto \omega.$$

It in turn induces an isomorphism of $\mathbb{F}_2$ left vector space

$$\phi : \mathbb{F}_4 \times \mathbb{F}_4 \rightarrow \mathcal{M}_2(\mathbb{F}_2).$$

We have that ϕ maps a pair $(a, b) \in \mathbb{F}_4 \times \mathbb{F}_4$ to a matrix in $\mathcal{M}_2(\mathbb{F}_2)$. Since the elements $(a, 0)$ and $(0, b)$ can be identified with $a, bj \in \mathbb{F}_4$ respectively, their image yields an invertible matrix in $\mathcal{M}_2(\mathbb{F}_2)$ whenever $a, b \in \mathbb{F}_4^*$. These 6 elements thus correspond to the 6 invertible matrices of $\mathcal{M}_2(\mathbb{F}_2)$, which establishes a one-to-one correspondence between elements of Hamming weight 1 in $\mathbb{F}_4^2$ and invertible matrices in $\mathcal{M}_2(\mathbb{F}_2)$.

Furthermore, we have that ϕ is actually an *isometry*. This is a one line proof, but due to its importance for our problem, let us repeat it as a lemma.

Lemma 6: The map ϕ as defined above is an isometry.

Proof: We have

$$w_B(Y) = w_B(\phi(y)) = w(y)$$

where w denotes the Hamming weight. ■

Let d be the minimum Hamming distance of a code over $\mathbb{F}_4$, that is

$$d = \min_{0 \neq x} w(x).$$

It follows from the lemma that

$$d_{\min}^B = \min_{0 \neq X} w_B(X) = \min_{0 \neq x} w_B(\phi(x)) = d.$$

Thus, to get a code over $\mathcal{M}_2(\mathbb{F}_2)$ with a suitable Bachoc distance, it is enough to construct a code over $\mathbb{F}_4$ with the same minimum distance. Since the code is brought back from $\mathbb{F}_4$ via ϕ , let us start by being completely explicit:

$$a + \omega b \mapsto \begin{pmatrix} a & b \\ b & a + b \end{pmatrix}, \quad a, b \in \mathbb{F}_2,$$

so that

$$\begin{aligned} (a + b\omega, c + d\omega) &\mapsto \begin{pmatrix} a & b \\ b & a + b \end{pmatrix} + \begin{pmatrix} c & d \\ d & c + d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a + d & b + c \\ b + c + d & a + b + d \end{pmatrix}. \end{aligned} \tag{9}$$

Vice-versa, we have that ϕ^{-1} is given by

$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \mapsto ((y_{11} + y_{12} - y_{21}) + \omega(-y_{11} + y_{22}), (y_{11} + y_{12} - y_{22}) + \omega(-y_{12} + y_{21})).$$

Also, ϕ can be extended to L -tuples

$$\phi : (\mathbb{F}_4 \times \mathbb{F}_4)^L \rightarrow \mathcal{M}_2(\mathbb{F}_2)^L$$

so that if $\mathcal{C}$ is a code of length L over $\mathcal{M}_2(\mathbb{F}_2)$, then $\phi^{-1}(\mathcal{C})$ is a code of length $2L$ over $\mathbb{F}_4$.

Let us now give two examples to illustrate both the map ϕ and ϕ^{-1} .

Example 4: Let us look at the repetition code of length 2. We have that

$$(Y, Y), Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{F}_2)$$

is mapped to

$$(y_1, y_2, y_1, y_2) \in \mathbb{F}_4^4$$

where

$$y_1 = y_{11} + y_{12} - y_{21} + \omega(-y_{11} + y_{22}),$$

$$y_2 = y_{11} + y_{12} - y_{22} + \omega(-y_{12} + y_{21}).$$

We see that if Y is invertible, then $w_B(Y) = 1$, and $w_B((Y, Y)) = 2$. On the other hand, $\phi^{-1}(Y) = (y, 0)$, $0 \neq y \in \mathbb{F}_4$, so that $\phi^{-1}((Y, Y)) = (y, 0, y, 0)$, and $w((y, 0, y, 0)) = 2$. Now, if Y is not invertible, then $w_B(Y) = 2$, and $w_B((Y, Y)) = 4$. Furthermore $\phi^{-1}(Y) = (y_1, y_2)$, $0 \neq y_1, y_2 \in \mathbb{F}_4$, so that $\phi^{-1}((Y, Y)) = (y_1, y_2, y_1, y_2)$, and $w((y_1, y_2, y_1, y_2)) = 4$. Thus the minimum weight is given by

$$d_{\min}^B = \min(2, 4) = 2 = d,$$

where d is the minimum Hamming distance of the code over $\mathbb{F}_4$. Its normalized redundancy is from (4)

$$\rho_{\text{norm}} = \frac{4}{2L} = 1.$$

Example 5: Let us now consider the [6,3,4] hexacode, that is a linear code over $\mathbb{F}_4$ of length 6, dimension 3, and minimum distance 4, whose generator matrix is given by

$$\begin{pmatrix} 1 & 0 & 0 & 1 & \omega & \omega \\ 0 & 1 & 0 & \omega & 1 & \omega \\ 0 & 0 & 1 & \omega & \omega & 1 \end{pmatrix}.$$

A codeword of the hexacode thus has the following form

$$y = (y_1, y_2, y_3, y_1 + \omega(y_2 + y_3), y_2 + \omega(y_1 + y_3), y_3 + \omega(y_1 + y_2)).$$

We now compute $\phi(y)$, using (9). We have that

$$\begin{aligned} (y_1, y_2) &\mapsto y_1 + y_2 j = (y_{11} + y_{12}\omega) + (y_{21} + y_{22}\omega)j \\ &\mapsto \begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{11} + y_{12} \end{pmatrix} + \begin{pmatrix} y_{21} & y_{22} \\ y_{22} & y_{21} + y_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= Y_1, \end{aligned}$$

and

$$\begin{aligned} &(y_3, y_1 + \omega(y_2 + y_3)) \\ &\mapsto y_3 + [y_1 + \omega(y_2 + y_3)]j \\ &= (y_{31} + y_{32}\omega) + [y_{11} + y_{22} + y_{32} + \omega(y_{12} + y_{21} + y_{31} + y_{22} + y_{32})]j \\ &\mapsto \begin{pmatrix} y_{31} & y_{32} \\ y_{32} & y_{31} + y_{32} \end{pmatrix} + \begin{pmatrix} y_{11} + y_{22} + y_{32} & y_{12} + y_{21} + y_{31} + y_{22} + y_{32} \\ y_{12} + y_{21} + y_{31} + y_{22} + y_{32} & y_{11} + y_{12} + y_{21} + y_{31} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= Y_2. \end{aligned}$$

A similar computation holds for $(y_2 + \omega(y_1 + y_3), y_3 + \omega(y_1 + y_2))$ and yields Y_3 . Thus

$$\phi(y) = (Y_1, Y_2, Y_3),$$

a code of length 3 over $\mathcal{M}_2(\mathbb{F}_2)$. Since the hexacode has Hamming distance $d = 4$, the minimum weight $d_{\min}^w$ of the code over $\mathcal{M}_2(\mathbb{F}_2)$ is 4.

If we take for example $y_1 = y_2 = 0$, we get that

$$y = (0, 0, y_3, \omega y_3, \omega y_3, y_3),$$

and

$$\phi(y) = \left(\mathbf{0}, \begin{pmatrix} y_{32} & 0 \\ y_{31} + y_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_{32} \\ 0 & y_{31} + y_{32} \end{pmatrix} \right)$$

which has weight 4, since we have two non-invertible matrices different from $\mathbf{0}$.

Its normalized redundancy from (4) is

$$\rho_{\text{norm}} = \frac{3 \text{ symbols in } \mathbb{F}_4}{2L} = \frac{6}{6} = 1.$$

The next example shows that the minimum Hamming distance and the minimum Bachoc distance yield two different criteria: we will exhibit a code with minimum Bachoc distance of 2, yet of minimum Hamming distance of 1.

Example 6: Consider again the $[4, 3, 2]$ code, the dual of the repetition code as in Example 3, with generator matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

so that a codeword is of the form

$$(x_1 + x_2 + x_3, x_1, x_2, x_3), \quad x_i = x_{i1} + \omega x_{i2} \in \mathbb{F}_4, \quad x_{i1}, x_{i2} \in \mathbb{F}_2.$$

Now

$$\phi((x_1 + x_2 + x_3, x_1, x_2, x_3)) = \left(\begin{pmatrix} x_{11} + x_{21} + x_{31} + x_{12} & x_{12} + x_{22} + x_{32} + x_{11} \\ x_{22} + x_{32} + x_{11} & x_{11} + x_{21} + x_{31} + x_{22} + x_{32} \end{pmatrix}, \begin{pmatrix} x_{21} + x_{32} & x_{22} + x_{31} \\ x_{22} + x_{32} + x_{31} & x_{21} + x_{22} + x_{32} \end{pmatrix} \right).$$

Consider the codeword

$$(0, 1, 1, 0),$$

of Hamming weight 2, we have that

$$\phi((0, 1, 1, 0)) = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right)$$

which is of Hamming weight 1. It is of Bachoc weight 2 though, since the non-zero matrix is not invertible.

V. CODES OVER $\mathcal{M}_2(\mathbb{F}_2[i])$

The bottleneck for lower bounding the performance of coset codes is coming from the codeword whose projection is all zero. In order to increase the lower bound, one has to take a quotient by an ideal of higher norm. This is the goal of this section. Consider the projection

$$\pi : \mathcal{G} \rightarrow \mathcal{G}/(2)\mathcal{G} \simeq \mathcal{M}_2(\mathbb{F}_2[i]), \quad X \mapsto \pi(X),$$

which maps a codeword X in $\mathcal{G}$ to a matrix $\pi(X)$ in $\mathcal{M}_2(\mathbb{F}_2[i])$. Let $\mathbb{F}_4 = \mathbb{F}_2(\omega)$ denote the finite field with 4 elements, where $\omega^2 + \omega + 1 = 0$. Let us first note that the isomorphism (8) can be easily extended:

$$\mathcal{M}_2(\mathbb{F}_2[i]) \simeq \mathbb{F}_2(\omega)[i] + j\mathbb{F}_2(\omega)[i] \simeq \mathbb{F}_4[i] + j\mathbb{F}_4[i]$$

where $j^2 = 1$ and $\omega j = j\omega^2$, and as before, it induces an isomorphism

$$\psi : \mathbb{F}_4[i] \times \mathbb{F}_4[i] \rightarrow \mathcal{M}_2(\mathbb{F}_2[i]).$$

We have that ψ maps a pair $(a, b) \in \mathbb{F}_4[i] \times \mathbb{F}_4[i]$ to a matrix in $\mathcal{M}_2(\mathbb{F}_2[i])$, as described in (9), and ψ can be extended to L -tuples

$$\psi : (\mathbb{F}_4[i] \times \mathbb{F}_4[i])^L \rightarrow \mathcal{M}_2(\mathbb{F}_2[i])^L$$

so that if $\mathcal{C}$ is a code of length L over $\mathcal{M}_2(\mathbb{F}_2[i])$, then $\psi^{-1}(\mathcal{C})$ is a code of length $2L$ over $\mathbb{F}_4[i]$.

A. The structure of $\mathbb{F}_4[i]$

In the following, we may write an element $x \in \mathbb{F}_4[i]$ as $x = a + b\omega$, $a, b \in \mathbb{F}_2[i]$, or $x = a' + b'i$, $a', b' \in \mathbb{F}_4$, depending on the context. The units of $\mathbb{F}_4$ are as usual denoted by $\mathbb{F}_4^*$.

The restriction ϕ of ψ to $\mathbb{F}_4 \times \mathbb{F}_4$ as been studied in Subsection IV-B, where we noticed that ϕ maps a pair $(a, b) \in \mathbb{F}_4 \times \mathbb{F}_4$ to a matrix in $\mathcal{M}_2(\mathbb{F}_2)$, and since the elements $(a, 0)$ and $(0, b)$ can be identified with $a, b \in \mathbb{F}_4$ respectively, their image yields an invertible matrix in $\mathcal{M}_2(\mathbb{F}_2)$ whenever $a, b \in \mathbb{F}_4^*$. These 6 elements thus correspond to the 6 invertible matrices of $\mathcal{M}_2(\mathbb{F}_2)$. We will show below that a similar correspondence holds for $\mathcal{M}_2(\mathbb{F}_2[i])$ via ψ . The correct phrasing which takes into account both $\mathbb{F}_4$ and $\mathbb{F}_4[i]$ is that there is a correspondence between pairs (a, a') and (b', b) where a, b are units, while a', b' are not. In the case of $\mathbb{F}_4$, $a' = b' = 0$, while a', b' are a multiple of $(1 + i)$ for $\mathbb{F}_4[i]$, as shown below.

Lemma 7: The ring $\mathbb{F}_4[i]$ contains exactly 4 non-invertible elements, given by

$$a(1 + i), \quad a \in \mathbb{F}_4.$$

Proof: Let $x \in \mathbb{F}_4[i]$, $x = a + ib$, $a, b \in \mathbb{F}_4$.

- if $a = b = 0$, x is clearly non invertible.
- if $a = 0$, $b \neq 0$, then $x = ib$ and $x^3 = -i = i$, thus x is invertible.
- if $a \neq 0$, $b = 0$, then $x = a$ and $a^3 = 1$, thus x is invertible.
- if $a \neq 0$, $b \neq 0$, then $x = a(1 + iba^{-1})$ and $x^3 = (1 + ic)^3$ with $c \neq 0$. Now $(1 + ic)^2 = 1 - c^2 = 1 + c^2$ and $(1 + c^2)^2 = 1 + c$, so that if $c \neq 1$, x is invertible, and if $c = 1$ (that is $a = b$), then x is not invertible.

To summarize, $\mathbb{F}_4[i]$ has 16 elements, 4 of them non invertible (given by $a(1 + i)$, $a \in \mathbb{F}_4$) and the 12 others being invertible. ■

Proposition 2: If $(a + b\omega, c + d\omega) \in \mathbb{F}_4[i] \setminus \mathbb{F}_4[i]^* \times \mathbb{F}_4[i]^*$ (that is, $a + b\omega$ is not invertible and $c + d\omega$ is, or vice versa), then $\psi((a + b\omega, c + d\omega))$ is invertible.

Proof: Recall by (9) that

$$(a + b\omega, c + d\omega) \mapsto \begin{pmatrix} a + d & b + c \\ b + c + d & a + b + d \end{pmatrix}.$$

Since we have assumed that $a + b\omega$ is not invertible, we know by Lemma 7 that $a + b\omega = a'(1 + i) + b'(1 + i)\omega$ (a', b' possibly 0). Thus

$$\psi(a'(1 + i) + b'(1 + i)\omega, c + d\omega) = \begin{pmatrix} a'(1 + i) + d & b'(1 + i) + c \\ b'(1 + i) + c + d & a'(1 + i) + b'(1 + i) + d \end{pmatrix}$$

whose determinant is $d^2 + c(c + d)$. Since

$$N_{\mathbb{F}_4/\mathbb{F}_2}(c + d\omega) = (c + d\omega)(c + d\omega^2) = c^2 + cd + d^2,$$

the determinant has to be invertible since we have assumed that $c + d\omega$ is invertible. The vice versa case follows similarly. ■

Corollary 1: There is a one to one correspondence between ordered pairs $(a + b\omega, c + d\omega) \in \mathbb{F}_4[i] \times \mathbb{F}_4[i]$ formed by one invertible and one non-invertible element, and invertible matrices in $\mathcal{M}_2(\mathbb{F}_2[i])$.

Proof: There are twice $4 \cdot 12$ ordered pairs formed by one invertible and one non-invertible element, that is 96 pairs. We know by the above proposition that to each of this pair will correspond an invertible matrix. We are thus left to show there are exactly 96 invertible matrices.

Let us then count invertible matrices in $\mathcal{M}_2(\mathbb{F}_2[i])$.

- 1) For the first column, there are a priori 16 choices, from which we have to remove the following pairs, yielding necessarily non-invertible matrices:

$$(0, 0), (0, 1 + i), (1 + i, 0), (1 + i, 1 + i).$$

That let us 12 choices, 4 choices twice for pairing an invertible (1 or i) with a non-invertible (0 or $1 + i$), namely

$$(1, 0), (1, 1 + i), (i, 0), (i, i + 1), (0, 1), (1 + i, 1), (0, i), (i + 1, i)$$

and 4 choices for pairing two invertible elements:

$$(1, 1), (1, i), (i, 1), (i, i).$$

- 2) We now choose the second column, in such a way that we get an invertible determinant. For the 8 choices of first columns where there is an invertible and a non-invertible, the non-invertible can multiply any element in $\mathbb{F}_2[i]$, yielding 4 choices, while the invertible is left to be multiplied by 2 choices. This is thus twice $4 \cdot 4 \cdot 2 = 4 \cdot 8$ choices, that is 64 choices. For the 4 choices with two invertible elements, it is not difficult to see that in each case, we have 4 choices for the first element of the second column, and only 2 choices for the second element, for a total of $4 \cdot 8$.
- 3) Thus the total of invertible matrices is $64 + 32 = 96$.

■

This can be made even more precise. As in the above proof, we use (9), which holds for ψ and $a, b, c, d \in \mathbb{F}_2[i]$ as for ϕ and $a, b, c, d \in \mathbb{F}_2$, to see that

$$(a + b\omega, c + d\omega) \mapsto \begin{pmatrix} a + d & b + c \\ b + c + d & a + b + d \end{pmatrix}$$

and we compute

$$\begin{aligned} \det \begin{pmatrix} a + d & b + c \\ b + c + d & a + b + d \end{pmatrix} &= (a^2 + ab + b^2) + (d^2 + cd + c^2) \\ &= N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a + b\omega) + N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c + d\omega). \end{aligned}$$

Now $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a + b\omega) \in \{0, 1, i\}$ and $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a + b\omega) = 0$ when $a + b\omega$ is not invertible, that is, is a multiple of $1 + i$. Thus, we have three different scenarios for $\det(\phi(a + b\omega, c + d\omega))$:

- $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a + b\omega) = N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c + d\omega)$: this can happen either when both $a + b\omega$ and $c + d\omega$ are not invertible, with then both a norm of zero (no element has norm $1 + i$), or both are invertible, with a norm of either 1 or i .
- $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a + b\omega) \neq N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c + d\omega)$ with $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a + b\omega) \neq 0, N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c + d\omega) \neq 0$: this means that $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a + b\omega) = i, N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c + d\omega) = 1$, or vice-versa.
- Either $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a + b\omega) = 0$ or $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c + d\omega) = 0$, thus the non-zero norm is 1 or i .

B. Weights and codes over $\mathcal{M}_2(\mathbb{F}_2[i])$

First we notice that Lemma 3 can easily be restated here:

Lemma 8: We have that

$$\Delta_{\min} \geq \min(16\delta, (d_{\min}^H)^2\delta),$$

where $\delta = \min |\det(X)|^2$.

This gives a Hamming distance bound. We now derive a new bound based on a bidimensional Lee-like distance. The determinant of a codeword $X \in \mathcal{G}$ can be bounded depending on its projection

$$\pi(X) = \begin{pmatrix} a + d & b + c \\ b + c + d & a + b + d \end{pmatrix}$$

as follows.

Lemma 9: We have that

- 1) $|\det(X)|^2 \geq 4\delta$ if $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a + b\omega) = N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c + d\omega), X \neq \mathbf{0}$.

- 2) $|\det(X)|^2 \geq 2\delta$ if $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a+b\omega) \neq N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c+d\omega)$ with $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a+b\omega) \neq 0, N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c+d\omega) \neq 0$.
- 3) $|\det(X)|^2 \geq \delta$ if $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a+b\omega)$ or $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c+d\omega) = 0$.

Proof:

- 1) If $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a+b\omega) = N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c+d\omega) = 0$, then $\det(\pi(X)) = 0$, thus $\det(X)$ is a multiple of 2 (assuming $X \neq 0$).
- 2) If $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a+b\omega) \neq N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c+d\omega)$ with $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a+b\omega) \neq 0, N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c+d\omega) \neq 0$, then $\det(\pi(X)) = 1+i$ and $\det(X)$ is a multiple of $1+i$.
- 3) If $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a+b\omega)$ or $N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c+d\omega) = 0$, then the non-zero norm is 1 or i , so that $\det(\pi(X)) = 1$ or i , and consequently $\det(X)$ is a multiple of 1 or i . ■

The above suggests to define a weight w_L on $(a+b\omega, c+d\omega) \in \mathbb{F}_4[i]^2$ by looking at their norm as follows:

$$w_L(N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a+b\omega), N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c+d\omega)) = |N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a+b\omega) + N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(c+d\omega)|^2 \in \mathbb{Z}[i],$$

that is, embed each norm in $\mathbb{F}_2[i]$ in $\mathbb{Z}[i]$ and compute the complex module of the sum. This can be seen as some bidimensional Lee weight. It is easy to check that

$$w_L(1, 1) = w_L(i, i) = 4$$

$$w_L(1, i) = w_L(i, 1) = 2$$

$$w_L(0, i) = w_L(0, 1) = 1$$

as desired.

It is not obvious how to translate this weight defined by norms on $\mathbb{F}_2[i]^2$ to $\mathbb{F}_4[i]^2$. To handle it, we first use an inner code that will remove the pairs of lowest weights, in order to have only two weights to distinguish. We propose to use the inner code given by the parity check matrix $H = (1+i, 1+i)$, that is

$$(1+i, 1+i) \begin{pmatrix} a+b\omega \\ c+d\omega \end{pmatrix} = (1+i)(a+b\omega+c+d\omega).$$

Now this parity equation implies that $a+b\omega+c+d\omega$ is of the form $(1+i)\alpha, \alpha = \alpha_1 + w\alpha_2 \in \mathbb{F}_4$, so that pairs satisfying the parity equation are

$$(a+b\omega, a+b\omega+\alpha(1+i)) \in \mathbb{F}_4[i]^2,$$

that is 2 bits encode a , 2 bits encode b , then 2 more bits decide which of the 4 multiples $0, 1+i, (1+i)w, (1+i)w^2$ is used. The corresponding matrix is

$$\begin{pmatrix} a + b + \alpha_2(1+i) & b + a + \alpha_1(1+i) \\ a + (1+i)(\alpha_1 + \alpha_2) & a + \alpha_2(1+i) \end{pmatrix}.$$

The rate is consequently $6/8 = 3/4$ and we have that

$$w_L(a + b\omega, a + b\omega + \alpha(1+i)) = |a\alpha_2(1+i) + b\alpha_1(1+i)|^2 = |1+i|^2 |a\alpha_2 + b\alpha_1|^2 \geq 2,$$

which is what we wanted, since

$$\begin{aligned} & N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a + b\omega + \alpha(1+i)) \\ &= N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a + b\omega + \alpha_1(1+i) + \alpha_2w(1+i)) \\ &= (a + \alpha_1(1+i))^2 + (b + \alpha_2(1+i))^2 + (a + \alpha_1(1+i))(b + \alpha_2(1+i)) \\ &= a^2 + b^2 + ab + a\alpha_2(1+i) + b\alpha_1(1+i) \end{aligned}$$

and

$$N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a + b\omega) + N_{\mathbb{F}_4[i]/\mathbb{F}_2[i]}(a + b\omega + \alpha(1+i)) = a\alpha_2(1+i) + b\alpha_1(1+i).$$

We are now left with designing a multilevel code, made of an outer code C_1 over $\mathbb{F}_4[i]$, with minimum Hamming distance d_1 , and a code C_2 over $\mathbb{F}_4$ for encoding α with minimum Hamming distance d_2 . The total minimum distance is

$$\min\{2d_1, \sqrt{2}d_2\}.$$

The goal is to reach a minimum of 4, for which we can take

- the parity check code $(L, L-1, 2)$ over $\mathbb{F}_4[i]$,
- an (L, k, d) code over $\mathbb{F}_4$ with the same L and $d \geq 3$.

The rate R of the code depends on L and k as follows:

$$R = \frac{L-1}{2L} + \frac{k}{4L}.$$

For example, one could take the code $(4, 2, 3)$ over $\mathbb{F}_4$ and $(4, 3, 2)$ over $\mathbb{F}_4[i]$, where we choose for $(4, 3, 2)$ the parity check code.

For $L \rightarrow \infty$, Gilbert Varshamov bound

$$A_q(L, d) \geq \frac{q^L}{\sum_{i=0}^{d-1} \binom{L}{i} (q-1)^i}$$

Fig. 2. Multilevel encoder for $\mathcal{M}_2(\mathbb{F}_2[i])$.

predicts that

$$A_4(L, 3) \geq \frac{4^L}{\sum_{i=0}^{d-1} 1 + 3L + 9 \frac{L(L-1)}{2}}$$

thus

$$R \geq \frac{1}{2} + \frac{1}{4} \left(1 - \frac{2}{L} \log_4 L + o(1/L) \right).$$

Therefore the rate satisfies $1 \leq R \leq 3/4$. The normalized redundancy is from (4)

$$\begin{aligned} \rho_{\text{norm}} &= \frac{(1+L) \text{ symb in } \mathbb{F}_4[i] + (L-k) \text{ symb in } \mathbb{F}_4}{2L} \\ &= \frac{4(1+L) + 2(L-k)}{2L} = \frac{2+3L-k}{L}. \end{aligned}$$

VI. SUMMARY AND PERSPECTIVES

In this paper, we designed coset codes for quasi-static MIMO fading channels where the inner code comes from a cyclic division algebra. In this case, we showed that the outer code alphabet is a matrix ring over a finite field or ring of the form $\mathcal{M}_n(\mathcal{R})$ where n is the number of transmit antennas and $\mathcal{R}$ is a finite ring in characteristic 2. More precisely, we considered the following cases:

- Codes over $\mathcal{M}_2(\mathbb{F}_2)$ with Hamming distance
- Codes over $\mathcal{M}_2(\mathbb{F}_2)$ with Bachoc distance
- Multilevel codes over $\mathcal{M}_4(\mathbb{F}_2)$ with Hamming distance
- Multilevel concatenated codes over $\mathcal{M}_2(\mathbb{F}_2[i])$ with a bidimensional Lee-like distance.

We established a general framework for designing coset codes via a series of isomorphisms that allows to represent the outer code alphabet in three different ways: an algebra of matrices over a finite ring, a

cyclic algebra over a finite ring, and the Cartesian product of finite rings. Under this framework we can address the following scenarios:

- For $n = 2$, in order to increase the coding gain of the space-time code, we need to consider deeper levels of partitioning giving rise to larger alphabets $\mathcal{M}_2(\mathcal{R})$ with $\mathcal{R} \supset \mathbb{F}_2[i]$.
- For $n = 3$, none of the constructions proposed in this paper properly works over $\mathcal{M}_3(\mathbb{F}_4)$.
- More generally, one may study deeper levels of partitioning in higher dimensions.

For all the above cases, the question of finding a suitable distance and correspondingly designing codes remains open.

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