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Iterated MIDO Space-Time Code Constructions

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Abstract—We consider the problem of designing codes for multiple antenna channels, with 4 transmit antennas and two receive antennas, using as code design full diversity and fast decodability. We present an iterated construction, which allows us to start with any algebraic 2×2 space-time code, and map it to a 4×4 MIDO code, for which a criterion for full diversity is available. We illustrate our method with two MIDO codes, built upon the Silver Code and the Alamouti code, for which the decoding complexity is analyzed. Simulation results show that the proposed codes exhibit very good behavior compared to existing codes.

Index Terms—Fast Decodability, Full Diversity, MIDO Code, Space-Time Code.

I. INTRODUCTION

After many years of research on space-time code design for coherent multiple antenna systems, many good codes are available, in particular many linear dispersion codes [8], which also include algebraic constructions coming from division algebras [15]. Most of these space-time codes have a lattice structure, which in particular simplifies the full diversity condition. Maximum likelihood decoding of lattice codes is, however, typically done using a sphere decoder [16], which is costly in complexity. This triggered a new line of research focusing on designing codes with reduced sphere decoding complexity, called *fast decodable codes*, initiated in [2]. The case of 4 transmit and 2 receive antennas, sometimes referred to as MIDO channel (for multiple input double output), has been of special interest, motivated by possible applications to digital TV broadcast, where the end user carries a portable TV device. Several 4-dimensional fast decodable MIDO codes have subsequently been studied, attempting to improve on the MIDO code in [2], which combines a quasi-orthogonal code with a twisted unitary transformation of another quasi-orthogonal code, and gets a lower decoding complexity by sacrificing the full diversity property. The idea of combining two 2-dimensional fast decodable codes to obtain a (hopefully) fast decodable 4-dimensional code is recurrent, mainly because designing a fast decodable 4-dimensional code from scratch is difficult. Recent fast decodable code constructions include [14], and MIDO codes based on crossed product algebras [12], [10]. Analysis of conditions under which a code satisfies fast decodability can be found in e.g. [6], [7], [9], where the notions of fast decodable, group decodable and conditionally group decodable are detailed.

In this paper, we focus on codes having an algebraic structure. More precisely, we start with 2×2 algebraic space-time block codes, and propose an iterated construction that generates 4×4 MIDO codes. We give a criterion which determines when a code obtained with this method is fully diverse. To illustrate our technique, we consider the 2×2 Silver code, and show how it can be used to obtain new MIDO codes, all with decoding complexity at most $O(|S|^{13})$. We simulate one code instance which has decoding complexity $O(|S|^{10})$, where S denotes the signal constellation, and $O(|S|^{16})$ is the complexity of exhaustive search. We give another example of a MIDO code, inspired by the Alamouti code [1], with complexity $O(|S|^8)$. The quasi-orthogonal code introduced by Jafarkhani [5] can be seen as a particular case of our method.

The paper is organized as follows: in Section II, we recall the system model considered and the two main code design criteria that we take into account, namely diversity and fast decodability. In Section III, the general construction is presented, together with a criterion for full diversity. The Silver code is utilized as an example in Section IV, where one new MIDO code is proposed, and its decoding complexity is computed. Finally, a new MIDO code inspired by the Alamouti code is given in Section V.

II. SYSTEM MODEL AND FAST DECODABILITY

We consider transmission over a coherent Rayleigh fading channel with 4 Tx antennas, 2 Rx antennas and perfect channel state information at the receiver (CSIR):

$$Y_{2 \times 4} = H_{2 \times 4} X_{4 \times 4} + V_{2 \times 4}, \quad (1)$$

where H is the channel matrix and V is the Gaussian noise at the receiver. The space-time code $X_{4 \times 4}$ is sometimes referred to as MIDO space-time, where MIDO stands for multiple input double output. A 4×4 MIDO code can transmit up to 8 complex (say QAM) information symbols, or equivalently 16 real (say PAM) information symbols. Maximum-likelihood (ML) decoding consists of finding the codeword X that achieves the minimum of the squared Frobenius norm

$$d(X) = \|Y - HX\|_F^2. \quad (2)$$

By describing the space-time code X in terms of basis matrices $B_1, \dots, B_{16}$ and a PAM vector $\mathbf{g} = (g_1, \dots, g_{16})^T$

as

$$X = \sum_{i=1}^{16} g_i B_i,$$

we can rewrite in Euclidean norm

$$d(X) = \|\mathbf{y} - B\mathbf{g}\|_E^2, \quad (3)$$

where $\mathbf{y}$ is the channel output which has been vectorized, with real and imaginary parts separated, and

$$B = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_{16}) \in M_{16 \times 16}(\mathbb{R}),$$

also obtained by vectorizing and separating the real and imaginary parts of HB_i to obtain $\mathbf{b}_i : i = 1, \dots, 16$. This search can then be performed using a real sphere decoder [16].

We will focus on two aspects of MIDO space-time code design: diversity and fast decodability.

Full diversity. It is well known that the first code design criterion for a space-time codebook $\mathcal{C}$ is full diversity, that is, we require

$$\det(X - X') \neq 0, \quad X \neq X' \in \mathcal{C},$$

which simplifies to

$$\det(X) \neq 0, \quad X \neq \mathbf{0}, \quad (4)$$

when $\mathcal{C}$ has the property that $X \pm X' \in \mathcal{C}$ for all $X, X' \in \mathcal{C}$.

Fast decodability. Using a QR decomposition $B = QR$ of B , with $Q^\dagger Q = I$, computing (3) reduces to computing

$$d(X) = \|\mathbf{y} - QR\mathbf{g}\|_E^2 = \|Q^\dagger \mathbf{y} - R\mathbf{g}\|_E^2 \quad (5)$$

where R is an upper right triangular matrix. The number and position of nonzero elements in the upper right part of R will determine the complexity of the sphere decoding process [2]: if S is the real alphabet in use, and κ is the number of independent real information symbols from S within one code matrix, then the *ML decoding complexity* is the minimum number of values of $d(X)$ in (5) that should be computed while performing ML decoding. The worst case is given when the matrix R is a full upper right triangular matrix, yielding the complexity of the exhaustive-search ML decoder, that is here $O(|S|^{16})$, with $\kappa = 16$ and $|S|$ is the number of PAM symbols in use. If the structure of the code is such that the decoding complexity has an exponent of $|S|$ smaller than 16, we say that the code is *fast-decodable*.

Notation. In what follows, x^* denotes the complex conjugate of x or, in case x is a matrix, the Hermitian transpose of x , $\mathbf{I}_n$ is used for the n -dimensional identity matrix, and $M_n(K)$ for the set of $n \times n$ matrices with coefficients in a field K .

III. A GENERAL ITERATED CODE CONSTRUCTION

We start by presenting the iterated construction which maps a pair of 2×2 algebraic space-time codewords to a 4×4 MIDO space-time codeword.

Let F be a number field, i.e., a finite extension of the field $\mathbb{Q}$ of rational numbers. A well studied way of getting

fully diverse space-time codes is to consider division algebras over F , i.e., algebras in which every nonzero element is invertible. In particular, when the division algebra happens to be a generalized quaternion division algebra over F [17], it contains a quadratic extension $K = F(\sqrt{a})$ of F , and the corresponding space-time codebook $\mathcal{C}$ contains codewords of the form

$$\begin{bmatrix} c & \gamma\sigma(d) \\ d & \sigma(c) \end{bmatrix}, \quad c, d \in K, \quad (6)$$

where c and d can be written more explicitly as $c = c_0 + \sqrt{a}c_1$, $d = d_0 + \sqrt{a}d_1$, $\sigma : \sqrt{a} \mapsto -\sqrt{a}$, and $\gamma \in F$ is not a norm in K/F , that is, there exists no element $b \in K$ such that $N(b) := b\sigma(b) = \gamma$. Codewords in $\mathcal{C}$ are, by definition of division algebra, either $\mathbf{0}$ or invertible. The fact that this algebra is division is a consequence of γ not being a norm. This can be checked immediately here, since the non-norm condition can be simply restated as saying that every nonzero matrix has nonzero determinant:

$$\begin{aligned} \det \begin{bmatrix} c & \gamma\sigma(d) \\ d & \sigma(c) \end{bmatrix} &= c\sigma(c) - \gamma d\sigma(d) = 0 \\ \iff \gamma &= \frac{c\sigma(c)}{d\sigma(d)} = N(c/d). \end{aligned}$$

Full diversity as stated in (4) is thus satisfied, since $\mathcal{C}$ has the property that $X \pm X' \in \mathcal{C}$ for all $X, X' \in \mathcal{C}$, and we have just shown that $\det(X) \neq 0$ for any nonzero $X \in \mathcal{C}$.

By abuse of notation we will also write σ for the map acting componentwise by

$$\sigma : \begin{bmatrix} c & \gamma\sigma(d) \\ d & \sigma(c) \end{bmatrix} \mapsto \begin{bmatrix} \sigma(c) & \gamma d \\ \sigma(d) & c \end{bmatrix}.$$

Given an element θ of K , let $\alpha_\theta : M_2(K) \times M_2(K) \rightarrow M_4(K)$ be the map defined by

$$\alpha_\theta : (A, B) \mapsto \begin{bmatrix} A & \theta\sigma(B) \\ B & \sigma(A) \end{bmatrix},$$

so that in particular

$$\begin{aligned} \alpha_\theta : & \left(\begin{bmatrix} c & \gamma\sigma(d) \\ d & \sigma(c) \end{bmatrix}, \begin{bmatrix} e & \gamma\sigma(f) \\ f & \sigma(e) \end{bmatrix} \right) \\ \mapsto & \begin{bmatrix} c & \gamma\sigma(d) & \theta\sigma(e) & \theta\gamma f \\ d & \sigma(c) & \theta\sigma(f) & \theta e \\ e & \gamma\sigma(f) & \sigma(c) & \gamma d \\ f & \sigma(e) & \sigma(d) & c \end{bmatrix}. \end{aligned} \quad (7)$$

As a special case of this iterative construction we recall the quasi-orthogonal construction introduced by Jafarkhani [5].

Example 1. Take $F = \mathbb{Q}$ and $K = \mathbb{Q}(i)$, with $\sigma : i \mapsto -i$ the complex conjugation, and $\gamma = -1$. Then $\mathcal{C}$ corresponds to the celebrated Alamouti code [1], whose codewords are matrices

$$\begin{bmatrix} c & -d^* \\ d & c^* \end{bmatrix} : \quad c, d \in \mathbb{Z}[i].$$

Now, for $\theta = -1$, we get

$$\alpha_\theta : \left(\begin{bmatrix} c & -d^* \\ d & c^* \end{bmatrix}, \begin{bmatrix} e & -f^* \\ f & e^* \end{bmatrix} \right)$$

$$\mapsto \begin{bmatrix} c & -d^* & -e^* & f \\ d & c^* & -f^* & -e \\ e & -f^* & c^* & -d \\ f & e^* & d^* & c \end{bmatrix}.$$

However, this code carries only 4 complex symbols, while 8 are preferred for a MIDO code. Furthermore, it is not fully-diverse, as already noticed in [5].

In this paper, we will study the family $\mathcal{C}$ of 4×4 MIDO codewords of the form (7). We begin by giving a criterion for full diversity.

Lemma 1. *Let $K = F(\sqrt{a})$. Suppose $\gamma, \theta \in F$ satisfy the following conditions:*

1) *for any $u \in K, v \in \sqrt{a}F$,*

$$\theta \neq \det \begin{bmatrix} u & \gamma\sigma(v) \\ v & \sigma(u) \end{bmatrix},$$

2) $\theta \neq \gamma(\text{mod } K^{\times 2})$, *where $K^{\times 2}$ denotes the squares in $K^\times$.*

Then nonzero codewords of $\mathcal{C}$ have nonzero determinant.

Proof: Consider a nonzero codeword

$$\begin{bmatrix} X & \theta\sigma(Y) \\ Y & \sigma(X) \end{bmatrix} \in \mathcal{C},$$

where the entries X, Y are 2×2 matrices of the form (6), that is, in particular, either $\mathbf{0}$ or invertible with coefficients in K . We demonstrate that the determinant of the codeword is nonzero. If $X = \mathbf{0}$ (resp. $Y = \mathbf{0}$), the matrix is clearly invertible. Hence we assume that X and Y are both nonzero, and thus, both invertible. The determinant of a block matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

when A is invertible, is given by $\det(A) \det(D - CA^{-1}B)$. Then

$$\det \begin{bmatrix} X & \theta\sigma(Y) \\ Y & \sigma(X) \end{bmatrix} = \det(X) \det(\sigma(X) - YX^{-1}\theta\sigma(Y)).$$

Now since $\sigma(X) - YX^{-1}\theta\sigma(Y)$ is again a codeword of the form (6), and hence invertible when nonzero, we have

$$\det(\sigma(X) - YX^{-1}\theta\sigma(Y)) \neq 0 \iff$$

$$\sigma(X) - YX^{-1}\theta\sigma(Y) \neq \mathbf{0}.$$

It thus suffices to demonstrate

$$\sigma(X) - YX^{-1}\theta\sigma(Y) \neq \mathbf{0}.$$

Since Y is invertible (it is nonzero), and noting that $\sigma(X^{-1}) = \sigma(X)^{-1}$, the latter inequality is equivalent to

$$XY^{-1}\sigma(XY^{-1}) \neq \theta\mathbf{I}_2.$$

Let $Z = XY^{-1}$ and write it as

$$Z = \begin{bmatrix} u & \gamma\sigma(v) \\ v & \sigma(u) \end{bmatrix},$$

where $u, v \in K$. For the sake of contradiction, suppose $Z\sigma(Z) = \theta\mathbf{I}_2$. This gives

$$\begin{bmatrix} u\sigma(u) + \gamma\sigma(v)^2 & u\gamma v + \gamma\sigma(v)u \\ v\sigma(u) + \sigma(u)\sigma(v) & v\gamma v + \sigma(u)u \end{bmatrix} = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}.$$

Suppose $u \neq 0$. Comparing entries (1, 2) of the above matrices gives $v + \sigma(v) = 0$, which implies that $v = v_1\sqrt{a}$, for some $v_1 \in F$, since $K = F(\sqrt{a})$. Comparing entries (1, 1) gives

$$u\sigma(u) + \gamma\sigma(v^2) = \theta.$$

Equivalently, recalling the previous condition that $v = v_1\sqrt{a}$, we have

$$\theta = \det \begin{bmatrix} u & \gamma\sigma(v) \\ v & \sigma(u) \end{bmatrix}.$$

The first assumption on θ gives us a contradiction.

Now we consider the case $u = 0$. Comparing entries (1, 1) gives us the equality

$$\gamma\sigma(v^2) = \theta$$

By the second assumption that $\theta \neq \gamma(\text{mod } K^{2\times})$, we obtain a contradiction. $\blacksquare$

IV. AN ITERATED MIDO SILVER CODE

We now illustrate the general iterated construction using the Silver code, to obtain new MIDO space-time codes.

The Silver code, discovered in [4], and re-discovered in [13], is given by codewords of the form

$$\begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{bmatrix},$$

where

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \frac{1}{\sqrt{7}} \begin{bmatrix} 1+i & -1+2i \\ 1+2i & 1-i \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

and $x_1, x_2, x_3, x_4 \in \mathbb{Z}[i]$ are the information symbols. Consider the number fields $F = \mathbb{Q}(\sqrt{-7})$ and $K = F(i)$, with $\sigma : i \mapsto -i$. Note that σ is not the complex conjugation since it fixes $\sqrt{-7}$. It was shown in [3] that Silver codewords can alternatively be viewed as scaled matrices over K looking like

$$\begin{bmatrix} c & -\sigma(d) \\ d & \sigma(c) \end{bmatrix},$$

where $c, d \in \mathbb{Z}[i] \oplus \mathbb{Z}[i](\frac{1+\sqrt{-7}}{2})$. Pick $\theta \in K$, so that

$$\begin{aligned} \alpha_\theta : & \left(\begin{bmatrix} c & -\sigma(d) \\ d & \sigma(c) \end{bmatrix}, \begin{bmatrix} e & -\sigma(f) \\ f & \sigma(e) \end{bmatrix} \right) \\ \mapsto & \begin{bmatrix} c & -\sigma(d) & \theta\sigma(e) & -\theta f \\ d & \sigma(c) & \theta\sigma(f) & \theta e \\ e & -\sigma(f) & \sigma(c) & -d \\ f & \sigma(e) & \sigma(d) & c \end{bmatrix}. \end{aligned} \quad (8)$$

We start with a general lemma which holds for any choice of θ .

Lemma 2. *The complexity of an iterated Silver MIDO code is at most $O(|S|^{13})$, no matter the choice of θ .*

Proof: Since $x_1 = a_1 + ib_1, x_2 = a_2 + ib_2$, $a_1, b_1, a_2, b_2 \in \mathbb{Z}$, we may choose a basis for the Silver code so that it contains the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

so that

$$B_1 = \alpha_\theta \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 0 \right), B_2 = \alpha_\theta \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, 0 \right),$$

$$B_3 = \alpha_\theta \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, 0 \right), B_4 = \alpha_\theta \left(\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, 0 \right)$$

are four basis matrices for an iterated MIDO Silver code, independently of θ . It is a direct computation to check that

$$B_k B_l^* + B_l B_k^* = 0, \text{ for all } k \neq l \in [1, \dots, 4].$$

It was shown in [6, Lemma2] that if the matrix $M := (M_{k,l})$, where $M_{k,l} = \|B_k B_l^* + B_l B_k^*\|_F$, has the structure

$$M = \begin{bmatrix} \Delta & B_1 \\ B_2 & B_3 \end{bmatrix},$$

where Δ is diagonal, then the matrix R of the QR decomposition in the sphere decoder as in (5) satisfies

$$R = \begin{bmatrix} \Delta & B \\ 0 & R_1 \end{bmatrix}.$$

By choosing an ordering on the basis of an iterated MIDO Silver code such that the first four elements are B_1, B_2, B_3, B_4 , the above computations show that Δ is a 4×4 diagonal matrix. Hence the decoding complexity reduces from $O(|S|^{16})$ to $O(|S|^{13})$. ■

It turns out that when $\theta = -1$, the decoding complexity is reduced further:

$$\begin{aligned} \alpha_{-1} : \left(\begin{bmatrix} c & -\sigma(d) \\ d & \sigma(c) \end{bmatrix}, \begin{bmatrix} e & -\sigma(f) \\ f & \sigma(e) \end{bmatrix} \right) \\ \mapsto \begin{bmatrix} c & -\sigma(d) & -\sigma(e) & f \\ d & \sigma(c) & -\sigma(f) & -e \\ e & -\sigma(f) & \sigma(c) & -d \\ f & \sigma(e) & \sigma(d) & c \end{bmatrix}. \end{aligned} \quad (9)$$

We demonstrate this further reduction in complexity in the following lemma.

Recall from [7] that a code is called *conditionally g-group decodable* if there exists a partition of $\{1, \dots, K\}$ into $g+1$ disjoint subsets $\Gamma_1, \dots, \Gamma_g, \Gamma^C$ such that

$$\|B_l B_m^* + B_m B_l^*\|_F = 0 \quad \forall l \in \Gamma_i, \forall m \in \Gamma_j, i \neq j.$$

In this case, the sphere decoding complexity order reduces to $|S|^{\Gamma^C + \max_{1 \leq i \leq g} \Gamma_i}$.

Lemma 3. *The complexity of the iterated MIDO Silver code (9) with $\theta = -1$ is $O(|S|^{10})$.*

Proof: We show that in fact the code is conditionally 4-group decodable. Fix the natural basis of the Silver code, and consider its image by α_θ

$$\begin{aligned} \alpha_\theta \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 0 \right), \alpha_\theta \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, 0 \right), \alpha_\theta \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, 0 \right), \\ \alpha_\theta \left(\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, 0 \right), \alpha_\theta \left(\frac{1}{\sqrt{7}} \begin{bmatrix} 1+i & -1+2i \\ -1-2i & -1+i \end{bmatrix}, 0 \right), \\ \alpha_\theta \left(\frac{1}{\sqrt{7}} \begin{bmatrix} -1+i & 2+i \\ 2-i & 1+i \end{bmatrix}, 0 \right), \\ \alpha_\theta \left(\frac{1}{\sqrt{7}} \begin{bmatrix} -1+2i & -1-i \\ -1+i & 1+2i \end{bmatrix}, 0 \right), \\ \alpha_\theta \left(\frac{1}{\sqrt{7}} \begin{bmatrix} -2-i & -1+i \\ -1-i & 2-i \end{bmatrix}, 0 \right), \alpha_\theta \left(0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), \\ \alpha_\theta \left(0, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \right), \alpha_\theta \left(0, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right), \\ \alpha_\theta \left(0, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right), \alpha_\theta \left(0, \frac{1}{\sqrt{7}} \begin{bmatrix} 1+i & -1+2i \\ -1-2i & -1+i \end{bmatrix} \right), \\ \alpha_\theta \left(0, \frac{1}{\sqrt{7}} \begin{bmatrix} -1+i & 2+i \\ 2-i & 1+i \end{bmatrix} \right), \\ \alpha_\theta \left(0, \frac{1}{\sqrt{7}} \begin{bmatrix} -1+2i & -1-i \\ -1+i & 1+2i \end{bmatrix} \right), \\ \alpha_\theta \left(0, \frac{1}{\sqrt{7}} \begin{bmatrix} -2-i & -1+i \\ -1-i & 2-i \end{bmatrix} \right), \end{aligned}$$

which forms a basis for the iterated MIDO Silver code. Using the basis above, in this exact order, define the partitions

$$\begin{aligned} \Gamma_1 &= \{1, 11\}, \Gamma_2 = \{3, 9\}, \Gamma_3 = \{4, 10\}, \Gamma_4 = \{2, 12\}, \\ \Gamma^C &= \{5, 6, 7, 8, 13, 14, 15, 16\}. \end{aligned}$$

It can be verified by direct computation that the R matrix of the iterated MIDO Silver code when $\theta = -1$ has the shape

$$\begin{bmatrix} t & t & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t & t & t & t & t \\ 0 & t & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t & t & t & t & t \\ 0 & 0 & t & t & 0 & 0 & 0 & 0 & t & t & t & t & t & t & t & t \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & t & t & t & t & t & t & t & t \\ 0 & 0 & 0 & 0 & t & t & 0 & 0 & t & t & t & t & t & t & t & t \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & t & t & t & t & t & t & t & t \\ 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t & t & t & t & t & t & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t & t & t & t & t & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & t & t & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & t & 0 & t & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \end{bmatrix}$$

where t denotes any nonzero coefficient. From the above matrix, it can be seen that the iterated Silver code is

conditionally 4-group decodable: the four groups of two symbols are clearly distinct, and the meaning of conditionally group decodable can also be well understood. Conditioned on decoding the 8 last information symbols (for a cost of $O(|S|^8)$), the other half of the symbols is decodable by pair (for a cost of $O(|S|^2)$) independently, resulting in a complexity of $O(|S|^{10})$. ■

Note that other values of $\theta \in F$ could have been chosen, but we favored a root of unity, since it yields a cubic shaping (see for example [2] for more details on the notion of cubic shaping). Furthermore, for the sake of experiment, we tried the choice of $\theta = i$, another root of unity which seems very natural, even though it belongs to K and thus is not included in the description of our iterated code construction. An explicit computation of the matrix R in (5) in the particular case when $\theta = i$ shows that the 8×8 upper left corner of R looks like

$$\begin{bmatrix} t & 0 & 0 & 0 & t & t & t & t \\ 0 & t & 0 & 0 & t & t & t & t \\ 0 & 0 & t & 0 & t & t & t & t \\ 0 & 0 & 0 & t & t & t & t & t \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \end{bmatrix}$$

where again t denotes any nonzero coefficient. Consequently, the decoding complexity of the code (8) when $\theta = i$ is $O(|S|^{13})$, which can be clearly seen from the shape of the above matrix R .

The performance of the two versions of the iterated MIDO Silver code is illustrated on Fig. 1 and 2. In both figures, the x -axis corresponds to the signal-to-noise ratio (SNR) in dB, and the y -axis to the Block Error Rate (BLER). The signal constellation S used is a 4-QAM constellation. On Fig. 1, codes with decoding complexity $O(|S|^{12})$ and $O(|S|^{13})$ are displayed: we see that the difference among the codes is tiny up to 18 dB. The BHV code [2] has complexity $O(|S|^{12})$ and is known not to be fully diverse. It has very good performance, and this is only starting from 18 dB that the loss in diversity starts to be noticed. The SPCOM code [11] has worse decoding complexity than the BHV code, but is fully diverse, and thus gives a way of comparing the gain in diversity with respect to the BHV code, which is best seen at higher SNR. The iterated MIDO Silver code with $\theta = i$ also starts to exhibit a loss in diversity around 18 dB. The iterated MIDO Silver code with $\theta = -1$ is also shown for the sake of comparison. The fact that this code is not fully diverse can be seen much earlier than for the BHV code, though its final performance in this range of SNR is quite similar. To get a fairer comparison, the iterated MIDO Silver code which has decoding complexity $O(|S|^{10})$ is compared with two codes of same complexity, based on crossed product algebras [10]. The code C_5 is not fully diverse, but thanks to good shaping, has very good performance from low to medium SNR. The code C_4 is fully diverse, but its performance suffers from

Fig. 1. Comparison among codes with decoding complexity $O(|S|^{12})$ and $O(|S|^{13})$.

Fig. 2. Comparison among codes with decoding complexity $O(|S|^{10})$.

the lack of cubic shaping. The newly proposed code behaves closely to the code C_5 .

V. AN ITERATED MIDO ALAMOUTI CODE

In this section, we propose another MIDO code, which can be seen as an iterated MIDO Alamouti Code. Pick $F = \mathbb{Q}(\sqrt{b})$, $b > 0$, and $K = F(i)$, with $\sigma : i \mapsto -i$.

Then we have

$$\alpha_\theta : \left(\begin{bmatrix} c & \gamma d^* \\ d & c^* \end{bmatrix}, \begin{bmatrix} e & \gamma f^* \\ f & e^* \end{bmatrix} \right) \mapsto \begin{bmatrix} c & \gamma d^* & \theta e^* & \theta \gamma f \\ d & c^* & \theta f^* & \theta e \\ e & \gamma f^* & c^* & \gamma d \\ f & e^* & d^* & c \end{bmatrix}.$$

Since $c, d, e, f \in F(i)$, we can write $c = c_0 + ic_1, d = d_0 + id_1, e = e_0 + ie_1, f = f_0 + if_1$, with $c_0, c_1, d_0, d_1, e_0, e_1, f_0, f_1 \in F$. Now $F = \mathbb{Q}(\sqrt{b})$, and thus

$$c = c_0 + ic_1 = (c_{00} + \sqrt{b}c_{01}) + i(c_{10} + \sqrt{b}c_{11}),$$

with $c_{00}, c_{01}, c_{10}, c_{11} \in \mathbb{Q}$. Alternatively

$$c = (c_{00} + ic_{10}) + \sqrt{b}(c_{01} + i\sqrt{b}c_{11}),$$

showing that c can encode 2 QAM symbols $c_{00} + ic_{10}$ and $c_{01} + i\sqrt{b}c_{11}$. In turn, the whole MIDO code contains 8 QAM symbols. We can guarantee full diversity for such a code, e.g. parameters $\gamma = -1, b = 5, \theta = -\frac{1+\sqrt{5}}{2}$ satisfy the full diversity criterion of Lemma 1.

Lemma 4. *Let $\theta = \gamma = -1$. Then the complexity of the iterated MIDO Alamouti code is $O(|S|^8)$.*

Proof: The proof relies on the notion of group decodability, a special case of conditional group decodability recalled earlier, where the group Γ^C is empty. It was shown in [6] that the reduction in decoding complexity will follow if we can subdivide the basis of the code into two groups $\Gamma_1 \cup \Gamma_2$ so that $AB^* + BA^* = 0$ for all $A \in \Gamma_1, B \in \Gamma_2$. Thus let

$$\Gamma_1 = \{\alpha_\theta(D, 0), \alpha_\theta(0, J)\}, \Gamma_2 = \{\alpha_\theta(J, 0), \alpha_\theta(0, D)\},$$

where

$$D = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{b} & 0 \\ 0 & \sqrt{b} \end{bmatrix}, \begin{bmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{bmatrix}, \begin{bmatrix} \sqrt{a}\sqrt{b} & 0 \\ 0 & -\sqrt{a}\sqrt{b} \end{bmatrix} \right\}$$

and

$$J = \left\{ \begin{bmatrix} 0 & \gamma \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \gamma\sqrt{b} \\ \sqrt{b} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\gamma\sqrt{a} \\ \sqrt{a} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\gamma\sqrt{a}\sqrt{b} \\ \sqrt{a}\sqrt{b} & 0 \end{bmatrix} \right\}.$$

It is only left to verify that $AB^* + BA^* = 0$ for

$$A \in \alpha_\theta(D, 0), \quad B \in \alpha_\theta(J, 0) \quad (10)$$

$$A \in \alpha_\theta(0, J), \quad B \in \alpha_\theta(0, D) \quad (11)$$

$$A \in \alpha_\theta(D, 0), \quad B \in \alpha_\theta(0, D) \quad (12)$$

$$A \in \alpha_\theta(0, J), \quad B \in \alpha_\theta(J, 0) \quad (13)$$

Now showing that $AB^* + BA^* = 0$ for (10) and (11) reduces, to verifying that

$$UV^* + VU^* = 0 : U \in D, V \in J. \quad (14)$$

Similarly, we have that $AB^* + BA^* = 0$ for (12) and (13) reduces to showing

$$VU^* - \sigma(UV^*) = 0 : U, V \in J \text{ or } U, V \in D. \quad (15)$$

We can verify by direct computation that both (14) and (15) hold. ■

VI. CONCLUSIONS

We presented an iterative method of designing 4×4 space-time codes made for 4 transmit and 2 receive antennas starting from a 2×2 algebraic space-time code. We give a criterion for full diversity for the resulting iterated codes. Any 2×2 algebraic space-time code can be chosen as base code, in particular we illustrated our technique giving two new MIDO codes, based on the Silver code and Alamouti code. Simulation results of these codes show good behavior. The sphere decoding complexity of all the new codes is computed. Current and future work will involve understanding the limits and trade-offs of this method, in terms of

- performance,
- least decoding complexity: what is the least possible decoding complexity that can be gained using this technique, by choosing a clever code structure,
- full diversity: what would be easier code design criterion to guarantee full diversity.

It would be of interest to know what are the fundamental limits of this problem. Finally, other research directions are naturally the generalization to other asymmetric space-time codes in higher dimension, and the study of these codes using a suboptimal decoder rather than maximum likelihood decoder.

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