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## Time dependence of a general class of quantum distribution functions

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We derive an explicit analytic expression for the time development of a generalized class of quantum distribution functions, which include as special cases the Wigner, the normal, and the antinormal distributions. We find that the simplest result is obtained in the case of the Wigner distribution.

### I. INTRODUCTION

Quantum-mechanical distribution functions provide a framework for the treatment of quantum-mechanical problems in terms of classical concepts, and they have enjoyed wide usage in practically every subfield of quantum physics.<sup>1</sup> The initial work on this subject is the paper of Wigner,<sup>2</sup> who introduced a function of position and momentum coordinates. Our considerations here will be restricted to one dimension since it will be clear that they can be generalized to the multidimensional case considered by Wigner. Thus we will denote the Wigner distribution function by  $P_W(q, p)$ , where  $q$  and  $p$  are the position and momentum coordinates, respectively.

Many authors have considered other distribution functions, the most commonly used being those of Glauber<sup>3</sup> and Sudarshan<sup>4</sup> (the normal distribution  $P_n$ ) and Husimi<sup>5</sup> (the antinormal distribution  $P_a$ ). The former has been used extensively in quantum optics. In addition, a more general class of distribution functions,  $P_g$  say, has been introduced by Cahill and Glauber.<sup>6</sup>

In many problems we require a knowledge of the time dependence of the distribution function. For example, dissipative processes frequently appear in many problems of quantum optics. The fluctuation-dissipation theorem establishes the relation between dissipation and the fluctuations of the system in equilibrium<sup>7</sup> and it is this relation that makes the time development of the distribution function of the system of paramount interest.

In Sec. II we consider the Wigner distribution function, with emphasis on how it may be written in terms of characteristic functions.<sup>8,9</sup> The time dependence of  $P_W$  is the simplest among all the distribution functions that we are aware of. In particular,  $P_W$  has the unique property that, in the field-free case, the equation of motion is the classical one.<sup>2</sup> By contrast, the corresponding result for  $P_a$  contains additional  $\hbar$  terms which are not of quantum origin,<sup>10</sup> and similarly for  $P_n$ , as recently pointed out.<sup>11</sup> In this paper we consider, in particular, the time dependence of  $P_a$  and  $P_n$  in the case where a potential is present. This is achieved by introducing, in Sec. III, the  $P_g$  functions,<sup>6</sup> which are characterized by a parameter  $s$ , with  $s = 0, +1, -1$ , corresponding to the Wigner, normal, and antinormal distributions, respectively. Next we derive a relation between  $P_g$  and  $P_W$ . Using the latter result and the known time dependence of  $P_W$ , we derive, in Sec. IV, the time dependence of  $P_g$ . The simplest result is ob-

tained for  $s=0$ , i.e., the Wigner distribution function has the simplest time dependence. In Sec. V, we present a discussion of our results.

### II. THE WIGNER DISTRIBUTION FUNCTION

For simplicity, we treat a system in a pure state  $\psi(q)$  since the case of a mixture does not present any additional essential complications. Then, the Wigner distribution function is given by<sup>2</sup>

$$P_W(q, p) = (\pi\hbar)^{-1} \int [\psi(q+y)]^* \psi(q-y) e^{2ipy/\hbar} dy. \quad (1)$$

If we now introduce the characteristic function

$$C_W(\sigma, \tau) = \langle \psi | \exp[(i/\hbar)(\sigma\hat{q} + \tau\hat{p})] | \psi \rangle, \quad (2)$$

where the carets denote operators, then it follows<sup>8,9</sup> that  $P_W$  is the Fourier transform of  $C_W$ , i.e.,

$$P_W(q, p) = (2\pi\hbar)^{-2} \iint d\sigma d\tau \exp[-(i/\hbar)(\sigma q + \tau p)] \times C_W(\sigma, \tau). \quad (3)$$

Next we introduce the framework of creation and annihilation operators by defining, as usual,

$$\hat{a} = \frac{1}{2} \left[ \frac{\hat{q}}{q_0} + i \frac{\hat{p}}{p_0} \right], \quad (4a)$$

$$\hat{a}^\dagger = \frac{1}{2} \left[ \frac{\hat{q}}{q_0} - i \frac{\hat{p}}{p_0} \right], \quad (4b)$$

where

$$q_0 = (\hbar/2m\omega)^{1/2} \quad (5a)$$

and

$$p_0 = (m\hbar\omega/2)^{1/2} = m\omega q_0 = (\hbar/2q_0). \quad (5b)$$

In addition, we define

$$\alpha = \frac{1}{2} \left[ \frac{q}{q_0} + i \frac{p}{p_0} \right], \quad (6a)$$

$$\alpha^* = \frac{1}{2} \left[ \frac{q}{q_0} - i \frac{p}{p_0} \right], \quad (6b)$$

and

$$\eta = (\sigma q_0 - i\tau p_0)/\hbar, \quad (7a)$$

$$\eta^* = (\sigma q_0 + i\tau p_0)/\hbar, \quad (7b)$$

from which it follows that

$$\hbar^{-1}(\sigma\hat{q} + \tau\hat{p}) = \eta\hat{a} + \eta^*\hat{a}^\dagger \quad (8a)$$

and

$$\hbar^{-1}(\sigma q + \tau p) = \eta\alpha + \eta^*\alpha^*. \quad (8b)$$

Hence

$$\begin{aligned} C_W(\sigma, \tau) &= \langle \psi | \exp[i(\eta\hat{a} + \eta^*\hat{a}^\dagger)] | \psi \rangle \\ &\equiv C_W(\eta, \eta^*). \end{aligned} \quad (9)$$

Substituting this result in Eq. (3), and using Eq. (8b) and the fact [see Eq. (7)] that  $d\sigma d\tau = \hbar d\eta d\eta^*$ , it follows that

$$\begin{aligned} P_W(q, p) &= \hbar(2\pi\hbar)^{-2} \int \int d\eta d\eta^* \exp[-i(\eta\alpha + \eta^*\alpha^*)] \\ &\quad \times C_W(\eta, \eta^*) \\ &\equiv \hbar^{-1} P_W(\alpha, \alpha^*). \end{aligned} \quad (10)$$

In summary,  $P_W(\alpha, \alpha^*)$  may be considered as the Fourier transform of  $C_W(\eta, \eta^*)$ , the latter being defined explicitly in Eq. (9). As we shall see in Sec. III, this is a convenient starting point for obtaining a generalized distribution function  $P_g$ —we simply generalize the characteristic function first and define  $P_g$  as the Fourier transform of the generalized characteristic function  $C_g$ .

For later purposes, it is convenient to write here the time dependence of  $P_W$ . It may be decomposed into two parts:<sup>2</sup>

$$\frac{\partial P_W}{\partial t} = \frac{\partial_k P_W}{\partial t} + \frac{\partial_v P_W}{\partial t}, \quad (11)$$

the first part resulting from the  $(i\hbar/2m)\partial^2/\partial q^2$ , the second from the potential energy  $V/i\hbar$  part of the expression for  $\partial\psi/\partial t$ . Also it has been shown that<sup>2</sup>

$$\frac{\partial_k P_W(q, p)}{\partial t} = -\frac{(p/m)\partial P_W(q, p)}{\partial q}, \quad (12)$$

i.e., the field-free case corresponds to the classical result. In addition

$$\frac{\partial_v P_W(q, p)}{\partial t} = \sum_{\lambda} \frac{1}{\lambda!} \left[ \frac{\hbar}{2i} \right]^{\lambda-1} \frac{\partial^\lambda V(q)}{\partial q^\lambda} \frac{\partial^\lambda P_W(q, p)}{\partial p^\lambda}, \quad (13)$$

where the summation over  $\lambda$  is to be extended over all odd positive integers.

### III. A GENERAL CLASS OF DISTRIBUTION FUNCTIONS

A generalized distribution function  $P_g(q, p, s)$ , where  $s$  is a parameter, is defined by replacing  $C_W(\eta, \eta^*)$  in Eq. (10) by  $C_g(\eta, \eta^*)$ , defined as follows:<sup>6</sup>

$$C_g(\eta, \eta^*) \equiv \langle \psi | \exp[\frac{1}{2}s |\eta|^2 + i(\eta\hat{a} + \eta^*\hat{a}^\dagger)] | \psi \rangle. \quad (14)$$

It follows that for  $s=0$  we get the Wigner (or as oftentimes called, the symmetric) characteristic function  $C_W$  given in Eq. (9). Also, making use of the Baker-Hausdorff theorem:

$$\exp(\hat{A})\exp(\hat{B}) = \exp(\frac{1}{2}[\hat{A}, \hat{B}])\exp(\hat{A} + \hat{B}) \quad (15)$$

provided  $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$ , and the fact that  $[\hat{a}, \hat{a}^\dagger] = 1$ , it follows that

$$C_a(\eta, \eta^*) = \langle \psi | \exp(i\eta\hat{a})\exp(i\eta^*\hat{a}^\dagger) | \psi \rangle \quad (16)$$

and

$$C_n(\eta, \eta^*) = \langle \psi | \exp(i\eta^*\hat{a}^\dagger)\exp(i\eta\hat{a}) | \psi \rangle, \quad (17)$$

where  $C_a$  and  $C_n$  are the so-called antinormal and normal characteristic functions, corresponding to taking  $s = -1$  and  $s = 1$ , respectively.

We now turn to a determination of the relation between  $P_g$  and  $P_W$ . For this purpose, it is convenient to express  $P_g$  in terms of  $p$  and  $q$ . First, we note that Eq. (7) implies that

$$|\eta|^2 = \hbar^{-2}(\sigma^2 q_0^2 + \tau^2 p_0^2). \quad (18)$$

Hence, using Eqs. (9), (14), and (18), it follows that

$$\begin{aligned} C_g(\eta, \eta^*) &= \exp[\frac{1}{2}s\hbar^{-2}(\sigma^2 q_0^2 + \tau^2 p_0^2)] C_W(\sigma, \tau) \\ &\equiv C_g(\sigma, \tau). \end{aligned} \quad (19)$$

Thus  $P_g(q, p)$  may be obtained from the expression for  $P_W(q, p)$  given in Eq. (3) by simply replacing  $C_W(\sigma, \tau)$  by  $C_g(\sigma, \tau)$  given in Eq. (19). Noting that  $\sigma^2$  and  $\tau^2$  inside the integrand may be replaced by  $-\hbar^2(\partial^2/\partial q^2)$  and  $-\hbar^2(\partial^2/\partial p^2)$  outside the integral sign, it follows that

$$P_g(q, p) = \exp \left[ -\frac{s}{2} q_0^2 \frac{\partial^2}{\partial q^2} - \frac{s}{2} p_0^2 \frac{\partial^2}{\partial p^2} \right] P_W(q, p). \quad (20)$$

This is a basic result giving the relation between the generalized distribution function  $P_g$  and the Wigner distribution function  $P_W$ . For the case where  $s = \pm 1$  (corresponding to the normal and antinormal distributions, respectively) this result has been presented already (without proof) by one of us.<sup>11</sup> For the specific choices of  $s=0, 1$ , and  $-1$ , *integral relations* between  $P_W$ ,  $P_n$ , and  $P_a$  are given by Cahill and Glauber<sup>6</sup> and also by Agarwal and Wolf.<sup>12</sup> It is also noteworthy that  $P_a$  as defined above is identical to the Husimi distribution function,<sup>13</sup> but since this is not obvious and since we have seen no proof in the literature, we derive this result in Appendix A.

### IV. TIME DEPENDENCE OF THE GENERAL CLASS OF DISTRIBUTION FUNCTIONS

It is convenient to define the following operators which act on functions in phase space:

$$C \equiv \exp \left[ -\frac{s}{2} q_0^2 \frac{\partial^2}{\partial q^2} \right], \quad (21)$$

$$D \equiv \exp \left[ -\frac{s}{2} p_0^2 \frac{\partial^2}{\partial p^2} \right]. \quad (22)$$

Thus Eq. (20) may be written in either of the following forms:

$$P_g = CDP_W \quad (23a)$$

and

$$P_W = C^{-1} D^{-1} P_g. \quad (23b)$$

Also, we note that  $C$  and  $D$  commute with each other and they are independent of time. Hence

$$\frac{\partial}{\partial t} P_g = CD \frac{\partial}{\partial t} P_W. \quad (24)$$

Our goal is to write the right-hand side of Eq. (24) in terms of  $P_g$ , as distinct from  $P_W$ . Using Eqs. (12) and (23), and noting that  $C$  commutes with  $p$ , it immediately follows that

$$\begin{aligned} \frac{\partial P_g}{\partial t} &= -\frac{1}{m} \left[ CDp \frac{\partial P_W}{\partial q} \right] \\ &= -\frac{1}{m} \left[ CDpC^{-1}D^{-1} \frac{\partial P_g}{\partial q} \right] \\ &= -\frac{1}{m} \left[ DpD^{-1} \frac{\partial P_g}{\partial q} \right]. \end{aligned} \quad (25)$$

We now write an important identity, which is proved in Appendix B, viz.,

$$\begin{aligned} Df(p)D^{-1} &= \sum_{\mu=0}^{\infty} \frac{\partial^{\mu} f(p)}{\partial p^{\mu}} \sum_{k=0}^{[\mu/2]} \frac{(-s)^{\mu-k} p_0^{2(\mu-k)}}{2^k k! (\mu-2k)!} \\ &\quad \times \frac{\partial^{\mu-2k}}{\partial p^{\mu-2k}}, \end{aligned} \quad (26)$$

where  $[\mu/2]$  is  $\mu/2$  for  $\mu$  even and  $(\mu-1)/2$  for  $\mu$  odd, and where  $f(p)$  is an arbitrary function of  $p$  containing, in general, derivatives of arbitrary order. In the case where  $f(p)=p$ , only the  $\mu=0$  and 1 terms contribute so that

$$DpD^{-1} = \left[ p - sp_0^2 \frac{\partial}{\partial p} \right] \quad (27)$$

and hence

$$\frac{\partial P_g(q,p)}{\partial t} = \sum_{\lambda} \sum_{\mu=0}^{\infty} \sum_{k=0}^{[\mu/2]} \frac{(i\hbar)^{\lambda-1} q_0^{2(\mu-k)} (-s)^{\mu-k}}{2^{\lambda+k-1} \lambda! k! (\mu-2k)!} \frac{\partial^{\lambda+\mu} V(q)}{\partial q^{\lambda+\mu}} \frac{\partial^{\lambda}}{\partial p^{\lambda}} \frac{\partial^{\mu-2k}}{\partial q^{\mu-2k}} P_g(q,p), \quad (31)$$

and we recall that  $\lambda$  assumes all odd—and only odd—values from 1 to infinity. This is our desired result which, when combined with Eq. (28), gives the time dependence of the generalized distribution functions. Again we have a check on the correctness of this result since for  $s=-1$  it reduces to Eq. (29) of Ref. 10, if we note that  $q_0^2 = \alpha/2$  where  $\alpha = \hbar/m\omega$  is a quantity appearing in Ref. 10. For  $s=0$ , the only contribution is for  $\mu=k=0$  and then Eq. (31) reduces to the result given in Eq. (13). We conclude that, in general, the time dependence of the Wigner distribution is simpler in structure than that of any other distribution function, and that in the field-free case it is the only distribution function whose time dependence corre-

$$\frac{\partial P_g}{\partial t} = -\frac{1}{m} \left[ p - sp_0^2 \frac{\partial}{\partial p} \right] \frac{\partial P_g}{\partial q}. \quad (28)$$

Thus, only for  $s=0$  does the field-free time dependence coincide with the classical result. For  $s \neq 0$  we get additional  $\hbar$  terms which are not of quantum origin. As a check, we note that for  $s=-1$  (the Husimi distribution), Eq. (28) agrees with Eq. (16) of Ref. 10, if we note that  $p_0^2 = \hbar^2/2\alpha$ , where  $\alpha = \hbar/m\omega$  is a quantity appearing in Ref. 10.

We next derive the extra contribution due to the presence of a potential. Making use of Eqs. (13) and (23) it follows that

$$\begin{aligned} \frac{\partial P_g(q,p)}{\partial t} &= CD \frac{\partial P_W(q,p)}{\partial t} \\ &= CD \sum_{\lambda} \frac{1}{\lambda!} \left[ \frac{\hbar}{2i} \right]^{\lambda-1} \frac{\partial^{\lambda} V(q)}{\partial q^{\lambda}} \frac{\partial^{\lambda} C^{-1} D^{-1} P_g(q,p)}{\partial p^{\lambda}} \\ &= \sum_{\lambda} \frac{1}{\lambda!} \left[ \frac{\hbar}{2i} \right]^{\lambda-1} C \frac{\partial^{\lambda} V(q)}{\partial q^{\lambda}} C^{-1} \frac{\partial^{\lambda} P_g(q,p)}{\partial p^{\lambda}}. \end{aligned} \quad (29)$$

To proceed further we again make use of Eq. (26), but now with the variable  $p$  replaced by  $q$ . Next we choose

$$f(q) = \frac{\partial^{\lambda} V(q)}{\partial q^{\lambda}}$$

so that

$$\begin{aligned} C \frac{\partial^{\lambda} V(q)}{\partial q^{\lambda}} C^{-1} &= \sum_{\mu=0}^{\infty} \frac{\partial^{\lambda+\mu} V(q)}{\partial q^{\lambda+\mu}} \sum_{k=0}^{[\mu/2]} \frac{(-s)^{\mu-k} q_0^{2(\mu-k)}}{2^k k! (\mu-2k)!} \\ &\quad \times \frac{\partial^{\mu-2k}}{\partial q^{\mu-2k}}. \end{aligned} \quad (30)$$

Substituting this result in Eq. (29) and noting that, since  $\lambda$  is odd, we can replace  $(\hbar/2i)^{\lambda-1}$  by  $(i\hbar/2)^{\lambda-1}$ , it follows that

sponds to the classical case.

Generalization to the case of several dimensions is achieved in the following manner: the indices  $k, \lambda, \mu$  must be replaced by as many sets of indices  $k_n, \lambda_n, \mu_n$  as there are space (or momentum) dimensions and the summations extended over all these indices. The restrictions are then that the sum of all the  $\lambda$  must be odd and that all the  $\mu_n - 2k_n$  must be non-negative, as are also all  $k_n, \mu_n, \lambda_n$ .

## V. DISCUSSION OF OUR RESULTS

There is a vast array of problems in many different branches of physics (see, for example, Refs. 14 and 15) in-

volving a potential which is a function of coordinates only. For such problems, we conclude that the optimum classical-quantum correspondence is that given by Wigner,<sup>2</sup> since the corresponding quantum distribution function has the simplest time dependence, as we have shown above.

On the other hand, in the areas of quantum optics and synergetics one often encounters momentum-dependent forces (which, for example, arise from  $\hat{a}^\dagger \hat{b}^\dagger$  and  $\hat{a} \hat{b}$  terms in the interaction Hamiltonian for the two coupled modes of a parametric amplifier<sup>16</sup>) or fluctuating forces which cannot be represented by a potential. In such cases it is clear that other criteria must be brought to bear on the question of choosing the optimum distribution function to be used in solving a particular problem, but a complete investigation of such a question remains in the future.

In quantum optics, the Glauber<sup>3</sup>-Sudarshan<sup>4</sup>  $P$  distribution (which is identical to  $P_n$ ) has been widely used, but it has also been recognized that the corresponding Fokker-Planck equations often have nonpositive definite diffusion coefficients, particularly in the case of phenomena involving nonclassical effects, such as photon antibunching. For such nonclassical photon fields,  $P_n$  does not exist as a well-behaved function and so, in order to avoid such problems, Drummond and Gardiner<sup>17</sup> introduced a class of

generalized  $P$  representations. These functions have found wide application in the area of quantum optics. However, it should also be pointed out that the Wigner function is never singular and it also has been used extensively in this area, a recent example being the investigations of Lugiato *et al.*,<sup>18</sup> for which the Wigner function was found to be more preferable than the Glauber-Sudarshan distribution. However, in the case of dispersive bistability and two-photon absorption, Walls and Milburn<sup>19</sup> concluded that the generalized  $P$  representation of Drummond and Gardiner is preferable to the use of the Wigner function because the latter gives rise to equations containing third-order derivatives. Thus a question arises as to why the generalized  $P$  representation is clearly the best choice for certain applications. Further investigations of this question are clearly warranted.

*Note added in proof.* Some other manifestations of the relationships between the quantum distribution functions discussed above as well as some others, have recently been considered.<sup>20</sup>

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#### APPENDIX A: PROOF OF THE EQUIVALENCE BETWEEN THE ANTINORMAL AND HUSIMI DISTRIBUTION FUNCTIONS

Husimi's distribution function,  $P_s$ , say, is a smoothed Wigner function and it is everywhere non-negative. It is defined as follows:<sup>5,10</sup>

$$P_s(q, p) = (\pi\hbar)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_W(q', p') \exp[-(q' - q)^2/\alpha] \exp[-\alpha(p' - p)^2/\hbar^2] dq' dp', \quad (A1)$$

where  $\alpha = \hbar/m\omega$ . Using a Taylor expansion we can write

$$P_W(q', p') = \sum_{m,n=0}^{\infty} \frac{\partial^{m+n} P_W(q, p)}{\partial q^m \partial p^n} \frac{(q' - q)^m (p' - p)^n}{m! n!}. \quad (A2)$$

We substitute Eq. (A2) into Eq. (A1) to get

$$P_s(q, p) = (\pi\hbar)^{-1} \sum_{m,n=0}^{\infty} \frac{\partial^{m+n} P_W(q, p)}{\partial q^m \partial p^n} \frac{1}{m! n!} \int_{-\infty}^{\infty} \exp[-(q' - q)^2/\alpha] (q' - q)^m dq' \int_{-\infty}^{\infty} \exp[-\alpha(p' - p)^2/\hbar^2] (p' - p)^n dp'. \quad (A3)$$

It is clear that only terms with even  $m$  and  $n$  contribute. Thus carrying out the integrations we obtain

$$\begin{aligned} P_s(q, p) &= (\pi\hbar)^{-1} \sum_{m,n=0}^{\infty} \frac{\alpha^{m-n} \hbar^{2n+1}}{(2m)!(2n)!} \Gamma(m + \frac{1}{2}) \Gamma(n + \frac{1}{2}) \frac{\partial^{2m+2n} P_W(q, p)}{\partial q^{2m} \partial p^{2n}} \\ &= \exp \left[ + \frac{\alpha}{4} \frac{\partial^2}{\partial q^2} + \frac{\hbar^2}{4\alpha} \frac{\partial^2}{\partial p^2} \right] P_W. \end{aligned} \quad (A4)$$

Since  $q_0^2 = \alpha/2$  and  $p_0^2 = \hbar^2/2\alpha$  we conclude that

$$P_s(q, p) = P_a(q, p), \quad (A5)$$

i.e., the Husimi and antinormal distributions are equivalent.

It may be of interest to prove Eq. (A5) in a different

way. By setting  $s = -1$  in Eq. (19) one gets

$$\begin{aligned} C_a(\sigma, \tau) &= \exp \left[ - \frac{1}{2\hbar^2} (\sigma^2 q_0^2 + \tau^2 p_0^2) \right] C_W(\sigma, \tau) \\ &\equiv f(\sigma, \tau) C_W(\sigma, \tau), \end{aligned} \quad (A6)$$

where

$$f(\sigma, \tau) = \exp \left[ -\frac{1}{2\hbar^2} (\sigma^2 q_0^2 + \tau^2 p_0^2) \right]. \quad (\text{A7})$$

We note that  $P_a(q, p)$  is the Fourier inverse of  $C_a(\sigma, \tau)$ . Explicitly,

$$C_a(\sigma, \tau) = \mathcal{F}(P_a(q, p)) \quad (\text{A8})$$

and

$$P_a(q, p) = \mathcal{F}^{-1}(C_a(\sigma, \tau)), \quad (\text{A9})$$

where  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  denote Fourier transform and Fourier inverse, respectively. The convolution theorem states that if

$$g(q, p) = \mathcal{F}^{-1}(f(\sigma, \tau)) \quad (\text{A10})$$

and

$$P_W(q, p) = \mathcal{F}^{-1}(C_W(\sigma, \tau)), \quad (\text{A11})$$

then

$$\mathcal{F}^{-1}[f(\sigma, \tau)C_W(\sigma, \tau)] = \int \int g(q - q', p - p') P_W(q', p') dq' dp'. \quad (\text{A12})$$

In our case

$$g(q, p) = \mathcal{F}^{-1}[f(\sigma, \tau)] = (\pi\hbar)^{-1} \exp \left[ -\frac{q^2}{2q_0^2} - \frac{p^2}{2p_0^2} \right]. \quad (\text{A13})$$

Substituting Eq. (A13) into Eq. (A12) gives Eq. (A5) immediately.

## APPENDIX B: PROOF OF EQ. (26)

We denote the left-hand side of Eq. (26) by  $A$  and the right-hand side by  $B$  and we denote  $-2sp_0^2$  by  $\beta$ . Thus

$$A = \exp \left[ \frac{\beta}{4} \frac{\partial^2}{\partial p^2} \right] f(p) \exp \left[ -\frac{\beta}{4} \frac{\partial^2}{\partial p^2} \right] \quad (\text{B1})$$

and

$$B = \sum_{\mu=0}^{\infty} \frac{\partial^{\mu} f(p)}{\partial p^{\mu}} \sum_{k=0}^{[\mu/2]} \frac{\beta^{\mu-k}}{2^{\mu} k! (\mu-2k)!} \frac{\partial^{\mu-2k}}{\partial p^{\mu-2k}}. \quad (\text{B2})$$

Since the direct proof of  $A=B$  is difficult, we first obtain the differential equation satisfied by  $A$  and  $B$ , with  $\beta$  as a variable. Thus

$$\frac{dA}{d\beta} = \frac{1}{4} \left[ \frac{\partial^2}{\partial p^2} A - A \frac{\partial^2}{\partial p^2} \right] \quad (\text{B3})$$

and

$$\frac{dB}{d\beta} = \sum_{\mu=1}^{\infty} \frac{\partial^{\mu} f(p)}{\partial p^{\mu}} \sum_{k=0}^{[\mu/2]} \frac{(\mu-k)\beta^{\mu-k-1}}{2^{\mu} k! (\mu-2k)!} \frac{\partial^{\mu-2k}}{\partial p^{\mu-2k}}. \quad (\text{B4})$$

Furthermore

$$\frac{1}{4} \left[ \frac{\partial^2}{\partial p^2} B - B \frac{\partial^2}{\partial p^2} \right] = \sum_{\mu=0}^{\infty} \frac{\partial^{\mu+1} f(p)}{\partial p^{\mu+1}} \sum_{k=0}^{[\mu/2]} \frac{\beta^{\mu-k}}{2^{\mu+1} k! (\mu-2k)!} \frac{\partial^{\mu-2k+1}}{\partial p^{\mu-2k+1}} + \sum_{\mu=0}^{\infty} \frac{\partial^{\mu+2} f(p)}{\partial p^{\mu+2}} \sum_{k=0}^{[\mu/2]} \frac{\beta^{\mu-k}}{2^{\mu+2} k! (\mu-2k)!} \frac{\partial^{\mu-2k}}{\partial p^{\mu-2k}}. \quad (\text{B5})$$

We now carry out some manipulations on the right-hand side of Eq. (B4). First, we split each term into two parts by writing  $\mu-k = (\mu-2k) + k$ . Then in the first term we change the variable from  $\mu$  to  $\mu+1$  and in the second term from  $\mu$  to  $\mu+2$ . Also, in the second term, it is clear that the  $k=0$  term does not contribute. Thus

$$\frac{dB}{d\beta} = \sum_{\mu=0}^{\infty} \frac{\partial^{\mu+1} f(p)}{\partial p^{\mu+1}} \sum_{k=0}^{[(\mu+1)/2]} \frac{\beta^{\mu-k}}{2^{\mu+1} k! (\mu-2k)!} \frac{\partial^{\mu-2k+1}}{\partial p^{\mu-2k+1}} + \sum_{\mu+1=0}^{\infty} \frac{\partial^{\mu+2} f(p)}{\partial p^{\mu+2}} \sum_{k=1}^{[(\mu+2)/2]} \frac{\beta^{\mu-k+1}}{2^{\mu+2} (k-1)! (\mu-2k+2)!} \frac{\partial^{\mu-2k+2}}{\partial p^{\mu-2k+2}}. \quad (\text{B6})$$

Also, the upper limit of the  $k$  summation, in the first term, i.e.,  $[(\mu+1)/2]$ , may be replaced by  $[\mu/2]$  since when  $\mu$  is even the two terms are the same, and when  $\mu$  is odd the extra term  $k=(\mu+1)/2$  gives no contribution, again because of the  $(\mu-2k)!$  term in the denominator. Thus the first term on the right-hand sides of Eqs. (B5) and (B6) are the same.

In the second term on the right-hand side of Eq. (B6) we change the variable from  $k$  to  $k+1$  to get

$$\sum_{\mu+1=0}^{\infty} \frac{\partial^{\mu+2} f(p)}{\partial p^{\mu+2}} \sum_{k=0}^{[\mu/2]} \frac{\beta^{\mu-k}}{2^{\mu+2} k! (\mu-2k)!} \frac{\partial^{\mu-2k}}{\partial p^{\mu-2k}}.$$

However, the  $(\mu-2k)!$  term in the denominator ensures that we may start the  $\mu$  summation at  $\mu=0$ . Thus the second terms—in addition to the first terms—on the right-hand sides of Eqs. (B5) and (B6) are the same.

We conclude that  $A$  and  $B$  satisfy the same first-order differential equation. In addition, when  $\beta=0$ ,

$$A(\beta=0) = B(\beta=0) = f(p). \quad (\text{B7})$$

Thus

$$A = B \quad (\text{B8})$$

for any  $\beta$ . Substituting  $-2sp_0^2$  for  $\beta$  we obtain Eq. (26).

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