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A comparison study on TDOA based localization algorithms for sensor networks

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Abstract—For the purpose of source localization, we have proposed two recursive algorithms in our companion paper, which use time difference of arrival (TDOA) measurements received from sensors by accounting for random uncertainties in sensor positions. This paper is devoted to presenting a comparative analysis on the two recursive localization algorithms. The first algorithm is called recursive localization algorithm, which uses the current estimate of source position to form a new measurement equation of the unknown source position. The second algorithm firstly estimates an auxiliary variable and then rearranges the nonlinear TDOA equation into a linear measurement equation. By employing the update covariance of the update localization of the two algorithms, it is shown that the second algorithm outperforms the first one. An illustrative example is included to validate our theoretic results.

Index Terms—Source localization, time difference of arrival, weighted least-squares, recursive localization algorithm.

I. INTRODUCTION

In modern source localization applications, passive sensors could be born by airplanes or unmanned aerial vehicles (UAVs) whose positions will change dynamically and the information of their positions can be available with high quality GPS [1]. So the position of source emitter may be tracked over time from multiple TDOA measurements. Motivated by this, in our companion paper [2], we have proposed two recursive algorithms to deal with the stationary source localization problem by using TDOA measurements received from mobile sensor network.

The problem of tracking stationary or moving source using multiple TDOA measurements collected over time was tackled by Okello and Musicki [3]. In their approach, they consider only two UAVs and the TDOA defines a hyperboloid on which the emitter must be located. The hyperbolic measurement error region is approximated by a sum of weighted Gaussian distributions. They applied unscented Kalman filters (UKFs) initiated with Gaussian mixture measurement (GMM) components. This method has

been extended to include frequency difference of arrival in addition to TDOA in [4]. Fletcher et al. [5] have investigated the recursive localization problem using a simple extended Kalman filter (EKF) from two UAVs. A comparison of these recursive algorithms from a pair of UAVs is proposed in [6].

All of the aforementioned methods are based on dealing with the nonlinear TDOA measurement equations directly. In our companion paper [2], we proposed two recursive algorithms to update the source localization, both of which rearrange the nonlinear TDOA measurement equations into a set of linear equations. The first recursive algorithm is referred to as “recursive localization algorithm” and the second as “improved recursive localization algorithm”.

In this paper, we mainly focus on a comparative analysis between the recursive localization algorithm and the improved recursive localization algorithm. The result shows that the update error covariance of the improved recursive localization algorithm at each sampling time step is smaller than that of the recursive localization algorithm. We note that in each update of both the two recursive algorithms, the update error covariance is based on the corresponding approximation of the measurement noise covariance, so generally the update error covariance is not the actually error covariance. However, the update error covariance establishes a lower bound on the true error covariance and we can use it as a performance benchmark of the comparative analysis. Our Monte Carlo simulation results corroborate the theoretical results and the better performance of the improved recursive localization algorithm.

The rest of the paper is organized as follows. Section II formulates the recursive source localization problem with mobile sensors. Section III reviews two recursive source localization algorithms. Section IV gives a performance analysis of the two recursive algorithms and Section V contains the simulation results and Section VI is the conclusion.

II. PROBLEM FORMULATION

Consider the source localization scenario where a network of n mobile passive sensors collaborate to localize a stationary emitter source with unknown position $\mathbf{u} = [x, y, z]'$. The

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sensor positions change dynamically in each step to recursively determine the unknown source position. We assume that the true position of the i th sensor at sampling time k is $\mathbf{s}_i^{k0} = [x_i^{k0}, y_i^{k0}, z_i^{k0}]'$, where $i = 1, \dots, n$. However, in practice the true sensor positions $\mathbf{s}_i^{k0}$ are not known and only their noisy versions $\mathbf{s}_i^k = [x_i^k, y_i^k, z_i^k]'$ are available, where

$$\mathbf{s}_i^k = \mathbf{s}_i^{k0} + \Delta \mathbf{s}_i^k, \quad (1)$$

with $\Delta \mathbf{s}_i^k$ being the position error in $\mathbf{s}_i^k$. We collect the available sensor positions as

$$\mathbf{s}^k = \mathbf{s}^{k0} + \Delta \mathbf{s}^k,$$

where $\mathbf{s}^k = [\mathbf{s}_1^{k'}, \mathbf{s}_2^{k'}, \dots, \mathbf{s}_n^{k'}]'$ and the corresponding error vector $\Delta \mathbf{s}^k = [\Delta \mathbf{s}_1^{k'}, \Delta \mathbf{s}_2^{k'}, \dots, \Delta \mathbf{s}_n^{k'}]'$ is zero-mean Gaussian with covariance matrix $\mathbf{Q}_s^k$.

Assuming line-of-sight signal propagation at each sampling step, the TDOA measurements of the received signals with respect to the signal at reference sensor are available. Without loss of generality, let the first sensor be the reference and the TDOA measurements model is given by

$$t_{i1}^k = t_{i1}^{k0} + \Delta t_{i1}^k, \quad (2)$$

where t_{i1}^k is the estimated TDOA between sensor pair i and 1 at step k , t_{i1}^{k0} is the corresponding true TDOA and $\Delta \mathbf{t}^k = [\Delta t_{21}^k, \dots, \Delta t_{n1}^k]'$ is zero-mean Gaussian noise with covariance $\mathbf{Q}_t^k$. The TDOA measurements can be easily converted to range difference of arrival (RDOA) measurements given the propagation speed c , which are modeled as:

$$r_{i1}^k = r_{i1}^{k0} + c \Delta t_{i1}^k, \quad (3)$$

where $r_{i1}^k = ct_{i1}^k$, $r_{i1}^{k0} = ct_{i1}^{k0}$,

$$r_{i1}^{k0} = r_i^{k0} - r_1^{k0}, \quad (4)$$

and the distances between the source to the true position and available position of sensor i ($i = 1, \dots, n$) are defined as

$$r_i^{k0} = \|\mathbf{u} - \mathbf{s}_i^{k0}\|, \quad (5)$$

$$r_i^k = \|\mathbf{u} - \mathbf{s}_i^k\|. \quad (6)$$

For notation simplicity, we collect r_{i1}^k , $i = 2, 3, \dots, n$ to form the $(n-1) \times 1$ RD measurement vector as

$$\mathbf{r}^k = [r_{21}^k, r_{31}^k, \dots, r_{n1}^k]' = \mathbf{r}^{k0} + \Delta \mathbf{r}^k,$$

where the RDOA error vector $\Delta \mathbf{r}^k = c \Delta \mathbf{t}^k$ is zero-mean Gaussian noise with covariance $c^2 \mathbf{Q}_t^k$. In this paper, TDOA and RDOA will be used interchangeably because they are equivalent.

The sensor position noise $\Delta \mathbf{s}^k$ is assumed to be independent of the TDOA noise $\Delta \mathbf{t}^k$. The objective of recursive source localization is to update the estimation of the source location $\mathbf{u}$ using the available noisy TDOA measurements $\mathbf{r}^k$ together with the noisy sensor positions $\mathbf{s}^k$ recursively in each sampling time step.

III. RECURSIVE LOCALIZATION ALGORITHMS FOR MOBILE SENSOR NETWORK

A. The recursive localization algorithm

At the first sampling time step $k = 1$, we need to get an initial estimate of the unknown source position and its estimation error covariance. The constrained weighted least squares (CWLS) method in [7] is applicable which follows the reorganization idea [8] to rearrange the nonlinear TDOA equations into a set of linear equations. The resulted linear equations are with respect to unknown parameter $\mathbf{u}_1^1 = [x, y, z, r_1^1]'$ where the auxiliary variable $r_1^1 = \|\mathbf{u} - \mathbf{s}_1^1\|$. The CWLS source localization method incorporates the relationship between $\mathbf{u}$ and r_1^1 as a second order equality constraint in the weighted LS estimator. The resulted initial estimate of the unknown source position and its estimation error covariance are denoted as $\hat{\mathbf{u}}^1$ and $\hat{\mathbf{P}}^1$.

At sampling time step $k = 2, 3, \dots$, because the available TDOA measurements $\mathbf{r}^k$ are nonlinear measurements on $\mathbf{u}$, it is impossible to use them directly to update source localization. In this recursive source localization algorithm, we firstly use the current TDOA measurements $\mathbf{r}^k$ and noisy sensor positions $\mathbf{s}^k$ to get an current estimation of source position $\hat{\mathbf{u}}^*$ and then treat it as a linear measurement on $\mathbf{u}$, so the optimal recursive WLS algorithm is applicable.

We use the WLS estimation method to get the current localization $\hat{\mathbf{u}}^*$ and its error covariance $\mathbf{P}^k$ at sampling time step $k \geq 2$. The rearrangement of the nonlinear RDOA measurement equations into linear equations (4) at sampling time step k with respect to $\mathbf{u}$ and r_1^{k0} is

$$r_{i1}^{k02} - R_i^{k02} - R_1^{k02} = -2(\mathbf{s}_i^{k0} - \mathbf{s}_1^{k0})^T \mathbf{u} - 2r_{i1}^{k0} r_1^{k0}, \quad (7)$$

where $R_i^{k0} = \sqrt{x_i^{k02} + y_i^{k02} + z_i^{k02}}$, ($i = 2, \dots, n$).

In the presence of TDOA measurement noise in r_{i1}^k , the sensor position errors $\Delta \mathbf{s}_i^k$ and the first-order approximation of r_1^{k0} , the equation (7) can be viewed as a linear measurement equation on unknown parameter $\mathbf{u}_1^k = [x, y, z, r_1^k]'$ as:

$$r_{i1}^{k2} - R_i^{k2} - R_1^{k2} = -2(\mathbf{s}_i^k - \mathbf{s}_1^k)^T \mathbf{u} - 2r_{i1}^k r_1^k + \epsilon_i^k, \quad (8)$$

where $R_i^k = \sqrt{x_i^{k2} + y_i^{k2} + z_i^{k2}}$ and ϵ_i^k is the measurement error. The corresponding linear vector equation of (8) is

$$\epsilon^k = \mathbf{h}^k - \mathbf{G}^k \mathbf{u}_1^k \quad (9)$$

where

$$\mathbf{h}^k = \begin{pmatrix} r_{21}^{k2} - R_2^{k2} + R_1^{k2} \\ \vdots \\ r_{n1}^{k2} - R_n^{k2} + R_1^{k2} \end{pmatrix},$$

$$\mathbf{G}^k = -2 \begin{pmatrix} (\mathbf{s}_2^k - \mathbf{s}_1^k)^T r_{21}^k \\ \vdots \\ (\mathbf{s}_n^k - \mathbf{s}_1^k)^T r_{n1}^k \end{pmatrix},$$

$$\epsilon^k = [\epsilon_2^k, \dots, \epsilon_n^k]'$$

The measurement noise vector ϵ^k can be written in terms of $\Delta \mathbf{t}^k$ and $\Delta \mathbf{s}^k$ as

$$\begin{aligned} \epsilon^k &= c\mathbf{B}^k \Delta \mathbf{t}^k + c^2 \Delta \mathbf{t}^k \otimes \Delta \mathbf{t}^k + \mathbf{D}^k \Delta \mathbf{s}^k + \Delta \mathbf{s}^k \otimes \Delta \mathbf{s}^k \\ &\approx c\mathbf{B}^k \Delta \mathbf{t}^k + \mathbf{D}^k \Delta \mathbf{s}^k, \end{aligned}$$

where

$$\mathbf{B}^k = 2 \begin{pmatrix} r_2^{k0} & 0 & \dots & 0 \\ 0 & r_3^{k0} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_n^{k0} \end{pmatrix}, \quad (10)$$

and $\mathbf{D}^k$ is shown in (11).

Equation (9) is a set of linear equations with respect to $\mathbf{u}_1^k$. By considering $\mathbf{u}$ and r_1^k as independent variables, the WLS solution of $\mathbf{u}_1^k$ that minimizes $\epsilon^{k'} \mathbf{W}^k \epsilon^k$ is

$$\hat{\mathbf{u}}_1^k = (\mathbf{G}^{k'} \mathbf{W}^k \mathbf{G}^k)^{-1} \mathbf{G}^{k'} \mathbf{W}^k \mathbf{h}^k, \quad (12)$$

where $\mathbf{W}^k = E[\epsilon^k \epsilon^{k'}]^{-1}$.

We denote the estimation error of $\hat{\mathbf{u}}_1^k$ as $\Delta \mathbf{u}^k = \hat{\mathbf{u}}_1^k - \mathbf{u}_1^k$. By substituting $\mathbf{h}^k = \mathbf{G}^k \mathbf{u}_1^k + \epsilon^k$ into (12), we have

$$\Delta \mathbf{u}^k = (\mathbf{G}^{k'} \mathbf{W}^k \mathbf{G}^k)^{-1} \mathbf{G}^{k'} \mathbf{W}^k \epsilon^k. \quad (13)$$

Hence, the WLS estimate $\hat{\mathbf{u}}_1^k$ has a covariance matrix given by

$$\text{cov}(\hat{\mathbf{u}}_1^k) = (\mathbf{G}^{k'} \mathbf{W}^k \mathbf{G}^k)^{-1}. \quad (14)$$

Finally, the current localization $\hat{\mathbf{u}}^*$ and its error covariance $\mathbf{P}^*$ are obtained as follows:

$$\hat{\mathbf{u}}^* = \hat{\mathbf{u}}_1^k(1:3), \quad (15)$$

$$\mathbf{P}^k = \text{cov}(\hat{\mathbf{u}}_1^k)(1:3, 1:3). \quad (16)$$

The following Algorithm 1 summarizes the recursive localization algorithm:

B. The improved recursive localization algorithm

In the improved recursive localization algorithm, the initial estimate of the unknown source position and its estimation error covariance are the same as in the recursive localization algorithm. At sampling time step $k \geq 2$, instead of treating the current localization based on only the current TDOA measurements as a new measurement of $\mathbf{u}$, we use the current TDOA measurements to estimate the auxiliary variable r_1^k using WLS method first, and then substitute the estimation into the TDOA equation (8) to rearrange it as a linear measurement equation of the unknown source position $\mathbf{u}$ which can be used to update the source localization directly.

Similar with the rearrangement of TDOA measurement equation in (8), we get a linear equation with respect to $\mathbf{u}_1^k = [x, y, z, r_1^k]'$. The WLS estimate of $\mathbf{u}_1^k$ in (12) contains the corresponding estimate of r_1^k which we denote as

$$\hat{r}_1^k = \hat{\mathbf{u}}_1^k(4). \quad (20)$$

Algorithm 1: Recursive localization algorithm

- 1) Let $k = 1$, and calculate the initial localization $\hat{\mathbf{u}}^1$ and the initial localization error covariance $\hat{\mathbf{P}}^1$ with CWLS method.
- 2) Let $k = k + 1$, and calculate the current localization denoted as $\hat{\mathbf{u}}^*$ and its error covariance denoted as $\mathbf{P}^k$ with WLS method in (14)-(16).
- 3) Taking the current localization $\hat{\mathbf{u}}^*$ as a measurement of the unknown source position $\mathbf{u}$ at time step k :

$$\hat{\mathbf{u}}^* = \mathbf{u} + \Delta \mathbf{u}^k,$$

where the covariance of measurement noise $\Delta \mathbf{u}^k$ is $\mathbf{P}^k$.

- 4) Update the source localization and its error covariance as follows:

$$\hat{\mathbf{u}}^k = \hat{\mathbf{u}}^{k-1} + \mathbf{K}^k(\mathbf{y}^k - \hat{\mathbf{u}}^{k-1}), \quad (17)$$

$$\mathbf{K}^k = \hat{\mathbf{P}}^{k-1}(\hat{\mathbf{P}}^{k-1} + \mathbf{P}^k)^{-1}, \quad (18)$$

$$\hat{\mathbf{P}}^k = (\mathbf{I} - \mathbf{K}^k)\hat{\mathbf{P}}^{k-1}. \quad (19)$$

Go back to step 2).

Meanwhile, the covariance of the estimation error $\Delta r_1^k = \hat{r}_1^k - r_1^k$ is denoted as

$$Q_r^k = \text{cov}(\hat{\mathbf{u}}_1^k)(4, 4). \quad (21)$$

Now we substitute $\hat{r}_1^k$ into (8) and rearrange the linear equation only with respect to $\mathbf{u}$ as:

$$r_{i1}^{k2} - R_i^{k2} - R_1^{k2} + 2r_{i1}^k \hat{r}_1^k = -2(\mathbf{s}_i^k - \mathbf{s}_1^k)^T \mathbf{u} + \tilde{\epsilon}_i^k. \quad (22)$$

The corresponding linear vector equation is

$$\tilde{\epsilon}^k = \tilde{\mathbf{h}}^k - \tilde{\mathbf{G}}^k \mathbf{u} \quad (23)$$

where

$$\begin{aligned} \tilde{\mathbf{h}}^k &= \begin{pmatrix} r_{21}^{k2} - R_2^{k2} + R_1^{k2} + 2r_{21}^k \hat{r}_1^k \\ \vdots \\ r_{n1}^{k2} - R_n^{k2} + R_1^{k2} + 2r_{n1}^k \hat{r}_1^k \end{pmatrix}, \\ \tilde{\mathbf{G}}^k &= -2 \begin{pmatrix} (\mathbf{s}_2^k - \mathbf{s}_1^k)^T \\ \vdots \\ (\mathbf{s}_n^k - \mathbf{s}_1^k)^T \end{pmatrix}, \\ \tilde{\epsilon}^k &= [\tilde{\epsilon}_2^k, \dots, \tilde{\epsilon}_n^k]'. \end{aligned}$$

The measurement noise vector $\tilde{\epsilon}^k$ can be written in terms of $\Delta \mathbf{t}^k$ and $\Delta \mathbf{s}^k$ as

$$\begin{aligned} \tilde{\epsilon}^k &= c\mathbf{B}^k \Delta \mathbf{t}^k + c^2 \Delta \mathbf{t}^k \otimes \Delta \mathbf{t}^k + \mathbf{D}^k \Delta \mathbf{s}^k + \Delta \mathbf{s}^k \otimes \Delta \mathbf{s}^k \\ &\quad + \mathbf{E}^k \Delta r_1^k + c\Delta \mathbf{t}^k \Delta r_1^k \\ &\approx c\mathbf{B}^k \Delta \mathbf{t}^k + \mathbf{D}^k \Delta \mathbf{s}^k + \mathbf{E}^k \Delta r_1^k, \end{aligned}$$

where

$$\mathbf{E}^k = 2[r_{21}^k, \dots, r_{n1}^k]', \quad (24)$$

$\mathbf{B}_k$ and $\mathbf{D}_k$ are the same as in (10) and (11).

Generally, Δr_1^k is correlated with $\Delta \mathbf{t}^k$ and $\Delta \mathbf{s}^k$, here we neglect this correlation and the corresponding covariance of measurement noise $\tilde{\epsilon}^k$ can be approximated as

$$\mathbf{Q}_e^k = c^2 \mathbf{B}^k \mathbf{Q}_t^k \mathbf{B}^{k'} + \mathbf{D}^k \mathbf{Q}_s^k \mathbf{D}^{k'} + Q_{r^k} \mathbf{E}^k \mathbf{E}^{k'}. \quad (25)$$

The resulted measurement equation (23) is a linear model only with respect to the unknown parameter $\mathbf{u}$, so the optimal Kalman filtering can be applied to recursively update the source localization. The following Algorithm 2 summarizes the improved recursive localization algorithm.

Algorithm 2: Improved recursive localization algorithm

- 1) Let $k = 1$, and calculate the initial localization $\tilde{\mathbf{u}}^1$ and the initial localization error covariance $\tilde{\mathbf{P}}^1$ with CWLS method.
- 2) Let $k = k + 1$, and use the WLS method to calculate the estimate of current auxiliary variable r_1^k as $\hat{r}_1^k$ and its error covariance denoted as Q_{r^k} with current TDOA measurements.
- 3) Rearrange the nonlinear TDOA measurement equations into a set of linear equations only with respect to the unknown source position $\mathbf{u}$ as in (22)-(23).
- 4) Update the source localization and its error covariance with Kalman filtering as follows:

$$\tilde{\mathbf{u}}^k = \tilde{\mathbf{u}}^{k-1} + \tilde{\mathbf{K}}^k (\tilde{\mathbf{h}}^k - \tilde{\mathbf{G}}^k \tilde{\mathbf{u}}^{k-1}), \quad (26)$$

$$\tilde{\mathbf{K}}^k = \tilde{\mathbf{P}}^{k-1} \tilde{\mathbf{G}}^{k'} (\tilde{\mathbf{G}}^k \tilde{\mathbf{P}}^{k-1} \tilde{\mathbf{G}}^{k'} + \mathbf{Q}_e^k)^{-1}, \quad (27)$$

$$\tilde{\mathbf{P}}^k = (\mathbf{I} - \tilde{\mathbf{K}}^k \tilde{\mathbf{G}}^k) \tilde{\mathbf{P}}^{k-1}. \quad (28)$$

Go back to step 2).

IV. PERFORMANCE ANALYSIS

It is notable that in both of the two proposed recursive source localization algorithms, the measurement equations (9) and (23) are approximations of the actual TDOA measurement equation. Meanwhile, in comparison with the standard recursive WLS algorithm, $\mathbf{P}^k$ in (16) of Algorithm 1 is only an estimate of the covariance of $\Delta \mathbf{u}^k$, so the update in step

4) of Algorithm 1 is only a sub-optimal update based on all these approximations. Similarly, in step 4) of Algorithm 2, in comparison with the standard Kalman filtering, $\mathbf{Q}_e^k$ in (25) is not the true covariance of measurement noise $\tilde{\epsilon}^k$, so generally the update of the improved recursive localization algorithm is not the optimal update with all available TDOA and sensor position information. However, the two algorithms 1 and 2 provide an efficient sub-optimal recursive method to overcome the nonlinearity of TDOA measurement equation. In our recent paper [2], the simulation results indicate that the performance of Algorithm 2 is more accurate than that of Algorithm 1. In this subsection, we will theoretically analysis the performance improvement of Algorithm 2.

Under the same source localization scenario, the following result shows that the improved recursive localization algorithm 2 outperforms the recursive localization algorithm 1 if the same amount of TDOA information is used.

Theorem 1: Assume that $\hat{\mathbf{u}}^1 = \tilde{\mathbf{u}}^1$ and $\hat{\mathbf{P}}^1 = \tilde{\mathbf{P}}^1$, then $\hat{\mathbf{P}}^k > \tilde{\mathbf{P}}^k$ for $k = 2, 3, \dots$.

Proof: We first prove that $\hat{\mathbf{P}}^2 > \tilde{\mathbf{P}}^2$.

Substituting the innovation $\mathbf{K}^k$ (18) and $\tilde{\mathbf{K}}^k$ (27) into (19) and (28) respectively, we have

$$\hat{\mathbf{P}}^2 = \hat{\mathbf{P}}^1 - \hat{\mathbf{P}}^1 (\hat{\mathbf{P}}^1 + \mathbf{P}^2)^{-1} \hat{\mathbf{P}}^1, \quad (29)$$

$$\tilde{\mathbf{P}}^2 = \tilde{\mathbf{P}}^1 - \tilde{\mathbf{P}}^1 \tilde{\mathbf{G}}^{2'} (\tilde{\mathbf{G}}^2 \tilde{\mathbf{P}}^1 \tilde{\mathbf{G}}^{2'} + \mathbf{Q}_e^2)^{-1} \tilde{\mathbf{G}}^2 \tilde{\mathbf{P}}^1. \quad (30)$$

By matrix inversion Lemma we have

$$\begin{aligned} & (\tilde{\mathbf{G}}^2 \tilde{\mathbf{P}}^1 \tilde{\mathbf{G}}^{2'} + \mathbf{Q}_e^2)^{-1} \\ &= (\mathbf{Q}_e^2)^{-1} + (\mathbf{Q}_e^2)^{-1} \tilde{\mathbf{G}}^2 (\hat{\mathbf{P}}^1)^{-1} + \tilde{\mathbf{G}}^{2'} \hat{\mathbf{P}}^1 \tilde{\mathbf{G}}^2)^{-1} \tilde{\mathbf{G}}^{2'} (\mathbf{Q}_e^2)^{-1}, \end{aligned} \quad (31)$$

$$\begin{aligned} & \tilde{\mathbf{G}}^{2'} (\tilde{\mathbf{G}}^2 \tilde{\mathbf{P}}^1 \tilde{\mathbf{G}}^{2'} + \mathbf{Q}_e^2)^{-1} \tilde{\mathbf{G}}^2 \\ &= \mathbf{T}^2 + \mathbf{T}^2 ((\hat{\mathbf{P}}^1)^{-1} + \mathbf{T}^2)^{-1} \mathbf{T}^2 \\ &= (\hat{\mathbf{P}}^1 + (\mathbf{T}^2)^{-1})^{-1}, \end{aligned} \quad (32)$$

where $\mathbf{T}^k = \tilde{\mathbf{G}}^{k'} (\mathbf{Q}_e^k)^{-1} \tilde{\mathbf{G}}^k$. Combining (32) and (30) yields

$$\tilde{\mathbf{P}}^2 = \tilde{\mathbf{P}}^1 - \tilde{\mathbf{P}}^1 (\hat{\mathbf{P}}^1 + (\mathbf{T}^2)^{-1})^{-1} \tilde{\mathbf{P}}^1. \quad (33)$$

Thus the only difference between $\tilde{\mathbf{P}}^2$ and $\hat{\mathbf{P}}^2$ is $(\mathbf{T}^2)^{-1}$ and $\mathbf{P}^2$ in the right side of (33) and (29) respectively. Note that

$$\mathbf{Q}_e^2 = (\mathbf{W}^2)^{-1} + Q_{r^2} \mathbf{E}^2 \mathbf{E}^{2'},$$

$$\mathbf{D}^k = 2 \times \begin{pmatrix} -r_{21}^k \mathbf{g}'_{\mathbf{u}, \mathbf{s}_1^k} - (\mathbf{u} - \mathbf{s}_1^k)' & (\mathbf{u} - \mathbf{s}_2^k)' & \mathbf{0}' & \dots & \mathbf{0}' \\ -r_{31}^k \mathbf{g}'_{\mathbf{u}, \mathbf{s}_1^k} - (\mathbf{u} - \mathbf{s}_1^k)' & \mathbf{0}' & (\mathbf{u} - \mathbf{s}_3^k)' & \dots & \mathbf{0}' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -r_{n1}^k \mathbf{g}'_{\mathbf{u}, \mathbf{s}_1^k} - (\mathbf{u} - \mathbf{s}_1^k)' & \mathbf{0}' & \mathbf{0}' & \dots & (\mathbf{u} - \mathbf{s}_n^k)' \end{pmatrix} \quad (11)$$

so

$$\mathbf{T}^2 = \tilde{\mathbf{G}}^2 \left(\mathbf{W}^2 - \mathbf{W}^2 \mathbf{E}^2 \left(\frac{1}{Qr^2} \mathbf{I} + \mathbf{E}^{2'} \mathbf{W}^2 \mathbf{E}^2 \right)^{-1} \mathbf{E}^{2'} \mathbf{W}^2 \right) \tilde{\mathbf{G}}^2. \quad (34)$$

On the other hand, from (14) and (16), we have

$$\mathbf{P}^2 = (\mathbf{G}^{2'} \mathbf{W}^2 \mathbf{G}^2)^{-1} (1 : 3, 1 : 3).$$

Because $\mathbf{G}^2 = [\tilde{\mathbf{G}}^2 \ \mathbf{E}^2]$, it is easy to verify that

$$\mathbf{G}^{2'} \mathbf{W}^2 \mathbf{G}^2 = \begin{pmatrix} \tilde{\mathbf{G}}^{2'} \mathbf{W}^2 \tilde{\mathbf{G}}^2 & \tilde{\mathbf{G}}^{2'} \mathbf{W}^2 \mathbf{E}^2 \\ \mathbf{E}^{2'} \mathbf{W}^2 \tilde{\mathbf{G}}^2 & \mathbf{E}^{2'} \mathbf{W}^2 \mathbf{E}^2 \end{pmatrix}. \quad (35)$$

Invoking the partitioned matrix inversion formula [9], we have

$$\mathbf{P}^2 = (\tilde{\mathbf{G}}^{2'} \mathbf{W}^2 \tilde{\mathbf{G}}^2 - \tilde{\mathbf{G}}^{2'} \mathbf{W}^2 \mathbf{E}^2 (\mathbf{E}^{2'} \mathbf{W}^2 \mathbf{E}^2)^{-1} \mathbf{E}^{2'} \mathbf{W}^2 \tilde{\mathbf{G}}^2)^{-1}. \quad (36)$$

Comparing (34) and (36), we have $(\mathbf{T}^2)^{-1} < \mathbf{P}^2$, which leads to $\tilde{\mathbf{P}}^2 > \mathbf{P}^2$.

Now assume $\tilde{\mathbf{P}}^{k-1} > \mathbf{P}^{k-1}$, we prove $\tilde{\mathbf{P}}^k > \mathbf{P}^k$ for $k > 2$. Similar with the above derivation, it is easy to verify that $(\mathbf{T}^k)^{-1} < \mathbf{P}^k$. Because

$$\begin{aligned} \hat{\mathbf{P}}^k &= \hat{\mathbf{P}}^{k-1} - \hat{\mathbf{P}}^{k-1} (\hat{\mathbf{P}}^{k-1} + \mathbf{P}^k)^{-1} \hat{\mathbf{P}}^{k-1} \\ &= ((\hat{\mathbf{P}}^{k-1})^{-1} + (\mathbf{P}^k)^{-1})^{-1}, \end{aligned} \quad (37)$$

and

$$\begin{aligned} \tilde{\mathbf{P}}^k &= \tilde{\mathbf{P}}^{k-1} - \tilde{\mathbf{P}}^{k-1} (\tilde{\mathbf{P}}^{k-1} + (\mathbf{T}^k)^{-1})^{-1} \tilde{\mathbf{P}}^{k-1} \\ &= ((\tilde{\mathbf{P}}^{k-1})^{-1} + \mathbf{T}^k)^{-1}, \end{aligned} \quad (38)$$

we have

$$\begin{aligned} ((\hat{\mathbf{P}}^{k-1})^{-1} + (\mathbf{P}^k)^{-1})^{-1} &> ((\tilde{\mathbf{P}}^{k-1})^{-1} + (\mathbf{P}^k)^{-1})^{-1} \\ &> ((\tilde{\mathbf{P}}^{k-1})^{-1} + \mathbf{T}^k)^{-1}, \end{aligned} \quad (39)$$

which completes the proof. ■

V. NUMERICAL SIMULATIONS

This section contains simulation results of the two recursive source localization algorithms. The simulation scenario contains $n = 8$ mobile sensors, and their nominal positions at sampling time step $k = 1$ are given in Table I. The sensor position noises at different coordinates and for different receivers are assumed to be independent Gaussian noises with variance $\sigma_s^{k^2}$, i.e. $\mathbf{Q}_s^k = \sigma_s^{k^2} \mathbf{I}$, where $\mathbf{I}$ is the $3n \times 3n$ identity matrix. We consider that the emitter source is far away from the sensor network and is located at $[20000, 50000, 0]'$ m. The eight sensors move towards the target with a step size of $[40, 100, 0]$ m for each step. The TDOA measurements are obtained by adding Gaussian noise with covariance matrix $\mathbf{Q}_t^k$ to the true values, where $\mathbf{Q}_t^k = \sigma_t^2 \mathbf{T}$ and $\mathbf{T}$ is the $(n-1) \times (n-1)$ matrix with 1 in the diagonal elements and 0.5 otherwise. The TDOA noise and sensor position noise are independent.

We implement the proposed recursive localization algorithm and improved recursive algorithm following the steps

TABLE I
NOMINAL POSITIONS (IN METERS) OF SENSORS

sensor no.i	x_i	y_i	z_i
1	5000	5000	6000
2	5000	-5000	6000
3	-5000	-5000	6000
4	-5000	5000	6000
5	5000	5000	1000
6	5000	-5000	1000
7	-5000	-5000	1000
8	-5000	5000	1000

as described in the Algorithm 1 and Algorithm 2. The localization accuracy is evaluated by the average range error (ARE) and the standard deviation (SD) of localization error which are defined as

$$\text{ARE}(\mathbf{u}) = \sum_{l=1}^L \|\hat{\mathbf{u}}_l - \mathbf{u}\| / L, \quad \text{SD}(\mathbf{u}) = \sqrt{\sum_{l=1}^L \|\hat{\mathbf{u}}_l - \mathbf{u}\|^2 / L},$$

where $\hat{\mathbf{u}}_l$ denotes the unknown source position estimate at ensemble l and $L = 1000$ is the number of ensemble runs.

Fig. 1 shows the localization accuracy of the two recursive localization algorithms for 500 sampling time steps when $\sigma_t = 50$ nano-second and $\sigma_s = 5/\sqrt{3}$ m. For comparison purpose, the simulated performance of the recursive localization in Algorithm 1 is shown in dashed line, and that of the improved recursive localization in Algorithm 2 is shown in solid line. In the right figure, the update standard deviation of the recursive localization in Algorithm 1 is shown in dashed-dotted line and that of the improved recursive localization in Algorithm 2 is shown in dotted line.

It is evident from the figure that both the two algorithms can significantly improve the location accuracy of the initial localization. As expected, the update SD of the improved recursive algorithm outperforms that of the recursive algorithm. Meanwhile, both the simulated ARE and simulated SD of the improved recursive algorithm is smaller than that of the recursive algorithm. For each algorithm, although the simulated SD is larger than the update SD, especially in first a few sampling time steps, it converges to the update SD as sampling time step increases.

Fig. 2 is the corresponding result for 500 sampling time steps when $\sigma_t = 100$ nano-second and $\sigma_s = 10/\sqrt{3}$ m. As expected, the localization accuracy is generally worse in the case of a larger noisy level. However, the proposed recursive localization algorithms can significantly improve the accuracy, especially in the first a few steps. Generally, when the TDOA noise and sensor position noise are larger, the recursive localization algorithms need more steps to reach the location accuracy. Also, the performance improvement of the improved recursive localization algorithm is more significant than that of recursive localization algorithm as in Fig. 1. Meanwhile, the simulated SDs of both the two

Fig. 1. Comparison of the localization accuracy of the two recursive localization algorithms. The left is the average range error of location when $\sigma_t = 100$ nano-second and $\sigma_x = \sigma_y = \sigma_z = 10/\sqrt{3}$ m, and the right is the corresponding SD.

Fig. 2. Comparison of the localization accuracy of the proposed recursive localization algorithms. The left is the average range error of location when $\sigma_t = 200$ nano-second and $\sigma_x = \sigma_y = \sigma_z = 20/\sqrt{3}$ m, and the right is the corresponding SD.

algorithms converge to the update SDs as sampling time step increases.

VI. CONCLUSION

In this work, we have provided a comparative analysis of two recursive source localization algorithms by using TDOA measurements received from mobile sensor network. We proved that the update covariance of the improved recursive algorithm is smaller than that of the recursive localization algorithm. The simulation results corroborate the theoretical results and the good performance of the improved recursive algorithm.

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