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2009

Li, Q. M., Ye, Z. Q., Ma, G., Jones, N. & Zhou, H. (2009). The influence of elastic shear deformation on the transverse shear failure of a fully clamped beam subjected to idealized blast loading. *International Journal of Mechanical Sciences*, 51(6), 413–423.

<https://hdl.handle.net/10356/94855>

<https://doi.org/10.1016/j.ijmecsci.2009.03.004>

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The influence of elastic shear deformation on the transverse shear failure of a fully clamped beam subjected to idealized blast loading

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Abstract: The influence of elastic shear deformation on the transverse shear response of a fully clamped beam is investigated in the present paper. The beam is made from a rigid, perfectly plastic material and subjected to a uniformly distributed pressure pulse loading. The elastic shear deformation is idealized by an elastic, perfectly plastic spring with a constant spring coefficient. Analytical solutions are obtained for the transverse shear response, which are then used to predict the occurrence of a transverse shear failure. The method presented in the paper may be extended to study the blast-induced shear failure of other structural elements when the elastic shear deformation needs to be considered.

Keywords: blast load, elastic effect, shear response and failure, rigid-plastic beam, dynamic plastic response

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Notation	
H	Thickness of the beam
I_0	Loading impulse per unit length
K	Shear stiffness of the beam at the support
k	$k = \frac{K}{\mu P_b Q_0 / I_0^2}$
$\bar{k}$	$\bar{k} = \frac{KH}{Q_0}$
L	Half length of a beam
M and Q	Bending moment and shear force
M_0 and Q_0	Plastic bending and shear capacities
P(t)	Loading
P_b	$P_b = 4M_0/L^2$
t	Time
V_a	$V_a = \sqrt{\frac{4M_0 H}{\mu L^2}}$
V_c	Critical impulsive velocity for the beam
$w_0, \dot{w}_0, \ddot{w}_0$	Transverse displacement, velocity, acceleration at the mid-span
$w_1, \dot{w}_1, \ddot{w}_1$	Transverse shear displacement, velocity, acceleration at the support
$\dot{w}_z, \ddot{w}_z$	Transverse velocity, acceleration at the interface between the plastic and rigid regions
$w_{1\max}$	Maximum transverse shear displacement
$\Delta w_{1\max}$	Maximum plastic transverse shear displacement
x	Coordinate along the beam
z	Position of the interface between the plastic and rigid regions

β	Coefficient of shear failure criterion
η	transverse shear coefficient
μ	Mass per unit length of a beam
$\bar{\mu}$	Poisson's ratio
λ	$\lambda = \sqrt{k\nu/2}$
$m, q, \bar{w}, \dot{\bar{w}}, \ddot{\bar{w}}, \xi, \tau, p, \nu, z$	Non-dimensional quantities defined by Eqs. (3a~j)

1. Introduction

As first observed experimentally by Menkes and Opat(1973), three response and failure modes may appear with the increasing impulse of an intensive pressure loading distributed uniformly over the entire span. These are (i) large inelastic deformations; (ii) tearing (tensile failure) of outer fibers, at the supports, and (iii) transverse shear failure at the supports. Further experiments have confirmed the existence of these failure modes in beams [Ross et al.(1977)], square plates [Ross et al.(1977), Olson et al.(1993)], circular plates [Teeling-Smith and Nurick(1991)], and cylindrical shells [Ross et al.(1977), Opat and Menkes(1974), Strickland et al.(1976)]. Because of the widespread applications of these structural elements in various engineering fields, the importance of these failure modes has received considerable attention in recent years.

The structural response corresponding to a mode III failure can be predicted by using the rigid-plastic idealization with the consideration of transverse shear localization [Symonds(1968), Li(2000), Li and Jones(2000a)]. The concept of a localized shear deformation has been successfully applied in the dynamic plastic response analyses of various structural members [e.g. Jones(1976), Nonaka(1977), Jones(1989a,b),

Duffey(1989), Li and Jones(1994, 1995a,b)], when neglecting the elastic deformation in the beam.

Among the three failure modes observed by Menkes and Opat(1973), transverse shear failure becomes dominant failure mode with the increase of loading rates and intensities. It is also the most complicated one because different failure mechanisms may be involved during the failure process [Kalthoff and Winkler(1987), Kalthoff(1990), Li and Jones(1999), Chen and Li(2003, 2005)]. Duffey(1989) pointed out that the transverse shear failure occurs at the hard-points of structural elements when subjected to intensive dynamic loads. The so-called hard points represent those points (or lines), such as a rigid support or the contacting periphery of projectile and target, across which significant contrast of deformability of the structural material exists.

When a rigid-plastic model is employed, the idealization is based on neglecting elastic deformations. Elastic deformation in structural elements plays a role of reducing the rigidity of the structural element, and thus, reducing the maximum deformation predicted according to rigid-plastic assumption. The elastic effects on the dynamic plastic response of a beam originate from three sources, i.e. the elastic bending, membrane and shear effects. When transverse shear failure becomes the dominant failure mode in a beam, the membrane deformation is normally less significant than the bending and shear deformations as shown by Menks and Opat(1973). The elastic effect on the bending response of a rigid, perfectly plastic cantilever beam was studied by introducing a rotational spring at the root of the cantilever beam [Wang and Yu(1991), Yu(1993), Wang(1994)], in which it was found that the interactions between the root spring and the travelling plastic hinge cause the change of rigid-plastic response mode in the cantilever beam. The tip velocity increases with the decrease of the root spring stiffness [Wang and Yu(1991)].

The elastic shear effect will be considered in the present paper for a fully clamped beam. The elastic shear effect throughout the beam is concentrated in the elastic transverse deformation of an elastic, perfectly plastic spring with a constant spring coefficient. Further discussion about this simplification is given in Section 5. The model proposed in this paper can also be used directly to study the influence of the supporting flexibility on the transverse shear hard-point failure in a rigid, perfectly plastic beam, which, although will not be referred to in the following discussion, may become important when flexible supports are used. Theoretical solutions are obtained for the transverse shear-sliding phase, which can be used together with a shear failure criterion to predict the occurrence of shear failure in a metal beam. The method proposed in this paper can also be extended to explore the importance of elastic shear effects in the dynamic plastic response of other structural elements subjected to blast type loads.

2. Basic equations

Consider the fully clamped beam shown in Figs.(1a,b), which is subjected to an idealized blast pressure loading [Fig.(1c)] distributed uniformly over the entire span. The negative phase of the blast loading is neglected. In Sections 2 and 3, the blast loading is formulated as a general descending function of time. The effect of elastic shear deformations is modeled by an elastic, perfectly plastic spring [Fig.(1d)] with spring constant K at each end of the beam.

The equilibrium equations for the shear and bending response of a beam can be expressed in the form [Li and Jones(1995a)]

$$\frac{\partial q}{\partial \xi} = 2 \frac{\ddot{w} - p(\tau)}{\nu} \quad (1a)$$

$$\text{and} \quad 2\nu q + \frac{\partial m}{\partial \xi} = 0 \quad (1b)$$

with continuity conditions

$$[m] = 0, \quad [q] = 0, \quad [\dot{w}] = 0, \quad [\bar{w}] = 0 \quad (2a\sim d)$$

at a bending interface, where the non-dimensional quantities are defined as

$$\begin{aligned} m &= M/M_0, \quad q = Q/Q_0, \quad \bar{w} = \frac{w}{I_0^2/\mu P_b}, \quad \dot{\bar{w}} = \frac{\dot{w}}{I_0/\mu}, \quad \ddot{\bar{w}} = \frac{\ddot{w}}{P_b/\mu} \\ \xi &= \frac{x}{L}, \quad \tau = \frac{t}{I_0/P_b}, \quad p = \frac{P}{P_b}, \quad \nu = \frac{Q_0 L}{2M_0}, \quad z = \frac{Z}{L} \end{aligned} \quad (3a\sim j)$$

where $P_b = 4M_0/L^2$ is the plastic collapse pressure for a fully clamped beam; μ is the mass per unit length of a beam; I_0 is the loading impulse per unit length; M_0 and Q_0 are plastic bending and shear capacities of the beam, respectively.

The boundary conditions at the supports are

$$Q = \begin{cases} -Kw_1, & \text{for } w_1 < Q_0/K \\ -Q_0, & \text{for } w_1 \geq Q_0/K \end{cases} \quad (4a,b)$$

in which, K is the spring constant, w_1 is the transverse shear displacement at the supports.

The non-dimensional forms of Eqs.(4a,b) are

$$q = \begin{cases} -k\bar{w}_1, & \text{for } \bar{w}_1 < 1/k \\ -1, & \text{for } \bar{w}_1 \geq 1/k \end{cases} \quad (5a,b)$$

$$\text{where } \bar{w}_1 = \frac{w_1}{I_0^2/\mu P_b}, \quad k = \frac{K}{\mu P_b Q_0/I_0^2}.$$

An independent yield condition between the bending moment and transverse shear force, as shown in Fig.2, is used in the following analysis in order to obtain analytical solution of the transverse shear response. The independent yield criterion between shear and bending deformations is a simplification of the plastic behaviour of beams. The influence of this simplification on the rigid-plastic response of beams, plates and shells have been discussed in Jones(1989b) where it was concluded that the use of an independent yield condition is adequate when the maximum transverse displacements are of interest. On the other hand, Yu and Chen(2000) showed that the transverse shear sliding based on an independent yield curve is 20-25% lower than the predictions based

on other interactive yield curves (circular and Hodge's yield curves). This is due to the fact that the independent yield curve circumscribes other yield curves and offers a lower bound solution of the transverse shear displacement, which should be noted in the application of the present analytical results. Furthermore, the independent square yield condition may be scaled down to obtain an inscribing yield curve of the actual interactive yield curve to offer an upper bound solution [Jones(1989b)].

3. Theoretical analysis

Similar to the results in Li and Jones(1995a), three flow patterns may exist, which will be discussed in this section. In each case, plastic shear deformation occurs when $\max\{\bar{w}_1\} \geq \frac{1}{k}$. Otherwise, i.e. when $\max\{\bar{w}_1\} < \frac{1}{k}$, only an elastic shear response exists.

3.1 Case I, $0 < \nu \leq 2$ [Fig.(3a)]

The transverse velocity profile for case (a) in Fig.3 is uniform across the whole beam. The non-dimensional transverse velocity field may be obtained by solving Eq.(1a) together with the boundary conditions (5a,b) and a zero shear force at the mid-span of the beam, i.e. $q = 0$ at $\xi=0$, which leads to

$$\ddot{\bar{w}}_1 + \frac{k\nu}{2} \bar{w}_1 = p \quad \text{for } \bar{w}_1 < \frac{1}{k} \quad (6a)$$

$$\text{and} \quad \ddot{\bar{w}}_1 = p - \frac{\nu}{2} \quad \text{when } \bar{w}_1 \geq \frac{1}{k} \quad (6b)$$

The initial conditions at $\tau=0$ are

$$\bar{w}_1 = 0 \quad \text{and} \quad \dot{\bar{w}}_1 = 0. \quad (7a,b)$$

3.1.1 Phase 1, $0 \leq \tau \leq \tau_0$

The analytical solution for Eqs.(6a) and (7a,b) is

$$\bar{w}_1 = \frac{1}{\lambda} \left[\sin(\lambda\tau) \int_0^\tau p(\tau) \cos(\lambda\tau) d\tau - \cos(\lambda\tau) \int_0^\tau p(\tau) \sin(\lambda\tau) d\tau \right] \quad (8a)$$

$$\text{and} \quad \dot{\bar{w}}_1 = \cos(\lambda\tau) \int_0^\tau p(\tau) \cos(\lambda\tau) d\tau + \sin(\lambda\tau) \int_0^\tau p(\tau) \sin(\lambda\tau) d\tau \quad (8b)$$

for $\bar{w}_1 \leq \frac{1}{k}$ or $\tau \leq \tau_0$ before any plastic shear deformation occurs at the beam support,

where $\lambda = \sqrt{\frac{kV}{2}}$ and τ_0 is determined by $\bar{w}_1|_{\tau=\tau_0} = \frac{1}{k}$, i.e.,

$$\sin(\lambda\tau_0) \int_0^{\tau_0} p(\tau) \cos(\lambda\tau) d\tau - \cos(\lambda\tau_0) \int_0^{\tau_0} p(\tau) \sin(\lambda\tau) d\tau = \frac{\lambda}{k}. \quad (9)$$

Equation (9) has a solution when $\max\{\bar{w}_1\} \geq \frac{1}{k}$ is satisfied.

3.1.2 Phase 2, $\tau_0 \leq \tau \leq \tau_1$

Equations (8a,b) give the initial conditions for $\tau \geq \tau_0$

$$\bar{w}_{10} = \bar{w}_1|_{\tau=\tau_0} = \frac{1}{k} \quad (10a)$$

$$\text{and} \quad \dot{\bar{w}}_{10} = \dot{\bar{w}}_1|_{\tau=\tau_0} = \cos(\lambda\tau_0) \int_0^{\tau_0} p(\tau) \cos(\lambda\tau) d\tau + \sin(\lambda\tau_0) \int_0^{\tau_0} p(\tau) \sin(\lambda\tau) d\tau. \quad (10b)$$

Now, Eq.(6b) with Eqs.(10a,b) predicts the non-dimensional transverse shear velocity and displacement

$$\dot{\bar{w}}_1 = \dot{\bar{w}}_{10} + \int_{\tau_0}^\tau p(\tau) d\tau - \frac{V}{2}(\tau - \tau_0) \quad (11a)$$

$$\bar{w}_1 = \bar{w}_{10} + \dot{\bar{w}}_{10}(\tau - \tau_0) + \int_{\tau_0}^\tau \left(\int_{\tau_0}^\tau p(\tau) d\tau \right) d\tau - \frac{V}{4}(\tau^2 - \tau_0^2) + \frac{V\tau_0}{2}(\tau - \tau_0) \quad (11b)$$

for $\tau \geq \tau_0$.

The maximum transverse shear displacement occurs at $\tau = \tau_1$, which is determined by

$$\dot{\bar{w}}_1|_{\tau=\tau_1} = 0 \text{ from Eq.(11a), i.e.}$$

$$\dot{\bar{w}}_{10} + \int_{\tau_0}^{\tau_1} p(\tau) d\tau - \frac{\nu}{2}(\tau_1 - \tau_0) = 0. \quad (12)$$

Beam responses beyond the maximum transverse shear displacement will not be considered. The static admissibility of the corresponding generalized stress field is proved in the appendix.

3.2 Case II, $2 < \nu \leq 3$ [Fig.3(b)]

It may be shown that the beam response is the same as Phase 1 in Section 3.1 when

$\bar{w}_1 < \frac{2}{k\nu}$. Therefore, we only present the solution when $\bar{w}_1 \geq \frac{2}{k\nu}$ (or $\tau \geq \tau_c$), where τ_c

is the time when $\bar{w}_1 = \frac{2}{k\nu}$ according to Eq.(8a).

3.2.1 Phase 1, $0 \leq \tau \leq \tau_c$: Solution is given in Section 3.1.1, as mentioned above.

3.2.2 Phase 2, $\tau_c \leq \tau \leq \tau_0$

In this case, the non-dimensional velocity profile in Fig.3(b) is assumed, or

$$\dot{\bar{w}} = \dot{\bar{w}}_1 + (\dot{\bar{w}}_0 - \dot{\bar{w}}_1)(1 - \xi). \quad (13)$$

From Eq.(13), it is easy to show that the distribution of the acceleration field is linear. By

using Eqs.(1a, b) as well as $q|_{\xi=0} = 0$ and $m|_{\xi=0} = -1$, the generalized stresses are

$$q = \frac{2}{\nu} \left[(\ddot{\bar{w}}_0 - p)\xi - \frac{1}{2}(\ddot{\bar{w}}_0 - \ddot{\bar{w}}_1)\xi^2 \right] \quad (14a)$$

$$\text{and } m = -1 - 2(\ddot{\bar{w}}_0 - p)\xi^2 + \frac{2}{3}(\ddot{\bar{w}}_0 - \ddot{\bar{w}}_1)\xi^3. \quad (14b)$$

Therefore, we have

$$\ddot{\bar{w}}_0 + \ddot{\bar{w}}_1 + \nu k \bar{w}_1 = 2p, \quad \text{for } \bar{w}_1 < \frac{1}{k} \quad (15a)$$

$$\text{and } \ddot{\bar{w}}_0 + \ddot{\bar{w}}_1 = 2p - \nu, \quad \text{for } \bar{w}_1 \geq \frac{1}{k} \quad (15b)$$

according to Eqs.(5a,b); and

$$\ddot{\bar{w}}_1 + 2\ddot{\bar{w}}_0 = 3p - 3 \quad (15c)$$

according to $m\Big|_{\xi=1} = 1$. Thus, the differential equations to control the non-dimensional

transverse shear displacement $\bar{w}_1$ are obtained

$$\ddot{\bar{w}}_1 + 2\nu k \bar{w}_1 = p + 3, \quad \text{for } \bar{w}_1 < \frac{1}{k} \quad (16a)$$

from Eqs. (15a) and (15c);

$$\text{and } \ddot{\bar{w}}_1 = p + 3 - 2\nu, \quad \text{for } \bar{w}_1 \geq \frac{1}{k}, \quad (16b)$$

from Eqs. (15b) and (15c).

Equation (16a) with the initial conditions at $\tau=\tau_c$ in the first phase gives

$$\bar{w}_1 = \frac{1}{2\lambda} \left[\sin \theta \int_{\tau_c}^{\tau} p(\tau) \cos \theta d\tau - \cos \theta \int_{\tau_c}^{\tau} p(\tau) \sin \theta d\tau + \frac{3}{2\lambda} [1 - \cos \theta] \right] + \frac{\sin \theta}{2\lambda} \dot{\bar{w}}_{1c} + \bar{w}_{1c} \cos \theta \quad (17a)$$

$$\text{and } \dot{\bar{w}}_1 = \cos \theta \int_{\tau_c}^{\tau} p(\tau) \cos \theta d\tau + \sin \theta \int_{\tau_c}^{\tau} p(\tau) \sin \theta d\tau + \frac{3}{2\lambda} \sin \theta + \dot{\bar{w}}_{1c} \cos \theta - 2\lambda \bar{w}_{1c} \sin \theta \quad (17b)$$

for $\frac{2}{k\nu} \leq \bar{w}_1 \leq \frac{1}{k}$, where $\theta = 2\lambda(\tau - \tau_c)$.

τ_0 is determined using $\bar{w}_1 = \frac{1}{k}$ in Eq.(17a).

3.2.3 Phase 3, $\tau_0 \leq \tau \leq \tau_1$

The initial conditions at $\tau=\tau_0$ in the third phase are determined from Eqs.(17a,b) in the second phase. From Eq.(16b), the non-dimensional transverse shear displacement and velocity at supports in the third phase are,

$$\bar{w}_1 = \bar{w}_{10} + \dot{\bar{w}}_{10}(\tau - \tau_0) + \int_{\tau_0}^{\tau} \left(\int_{\tau_0}^{\tau} p(\zeta) d\zeta \right) d\tau + \frac{1}{2} (3 - 2\nu)(\tau - \tau_0)^2 \quad (18a)$$

$$\dot{\bar{w}}_1 = \dot{\bar{w}}_{10} + \int_{\tau_0}^{\tau} p(\tau) d\tau + (3-2\nu)(\tau - \tau_0), \quad (18b)$$

where $\bar{w}_{10}$ and $\dot{\bar{w}}_{10}$ are given by Eqs.(17a,b) at $\tau = \tau_0$. For the rigid perfectly plastic case, i.e. $k \rightarrow \infty$, Eq.(18b) is consistent with Eq.(16) in Li and Jones(1995a).

The non-dimensional maximum transverse shear displacement is reached at τ_1 when $\dot{\bar{w}}_1 = 0$, or

$$\dot{\bar{w}}_{10} + \int_{\tau_0}^{\tau_1} p(\tau) d\tau + (3-2\nu)(\tau_1 - \tau_0) = 0. \quad (19)$$

3.3 Case III, $\nu > 3$ [Fig.3(c)]

It may be shown that the beam response is the same as in Section 3.2 when $\bar{w}_1 < \frac{3}{k\nu}$.

Therefore, we only present the solutions when $\bar{w}_1 \geq \frac{3}{k\nu}$ (or $\tau \geq \tau_a$) where τ_a is the time

when $\bar{w}_1 = \frac{3}{k\nu}$, which can be obtained from Section 3.2.2.

3.3.1 Phase 1, $0 \leq \tau \leq \tau_c$: Solution is given in Section 3.2.1.

3.3.2 Phase 2, $\tau_c \leq \tau \leq \tau_a$: Solution is given in Section 3.2.2.

3.3.3 Phase 3, $\tau_a \leq \tau \leq \tau_0$

The velocity profile in Fig. (3c), which is assumed in order to satisfy the requirement of static admissibility, consists of a central plastic deformation region $0 \leq \xi \leq z$ and an outer rigid region $z \leq \xi \leq 1$. The transverse velocity distribution in the rigid region is linear, i.e.

$$\dot{\bar{w}} = \dot{\bar{w}}_1 + \frac{1-\xi}{1-z} (\dot{\bar{w}}_z - \dot{\bar{w}}_1), \quad z \leq \xi \leq 1, \quad (20)$$

where z is the position of the interface between plastic and rigid regions and $\dot{\bar{w}}_z$ is the transverse velocity of the local material at the interface. The central plastic region

expands during this phase, i.e., the rigid-plastic interface travels outwards and the transverse velocity ($\dot{\bar{w}}_z$) at the interface will be determined numerically.

Taking $q=0$ and $m=-1$ in the central plastic region, Eq.(1a) gives

$$\ddot{\bar{w}} = p(\tau) \text{ for } 0 \leq \xi \leq z. \quad (21)$$

The distribution of the material acceleration in the rigid region can be obtained by differentiating Eq.(20), i.e.

$$\ddot{\bar{w}} = \ddot{\bar{w}}_1 + \frac{1-\xi}{1-z}(\ddot{\bar{w}}_z - \ddot{\bar{w}}_1) + \frac{1-\xi}{(1-z)^2} \dot{z}(\dot{\bar{w}}_z - \dot{\bar{w}}_1), \quad z \leq \xi \leq 1, \quad (22)$$

which may be rewritten as

$$\ddot{\bar{w}} = \frac{1}{(1-z)^2} \left\{ (1-z) \left[(\ddot{\bar{w}}_1 - \ddot{\bar{w}}_z) \xi + \ddot{\bar{w}}_z - \ddot{\bar{w}}_1 z \right] + (1-\xi) \dot{z}(\dot{\bar{w}}_z - \dot{\bar{w}}_1) \right\}, \quad z \leq \xi \leq 1, \quad (23)$$

therefore, Eq.(1a) with $q=0$ at $\xi=z$ predicts

$$q = \frac{2}{v} \int_z^\xi (\ddot{\bar{w}} - p) d\xi$$

or

$$q = \frac{2}{v(1-z)^2} \int_z^\xi \left\{ (1-z) \left[(\ddot{\bar{w}}_1 - \ddot{\bar{w}}_z) \xi + \ddot{\bar{w}}_z - \ddot{\bar{w}}_1 z \right] + (1-\xi) \dot{z}(\dot{\bar{w}}_z - \dot{\bar{w}}_1) - p(1-z)^2 \right\} d\xi$$

$$\text{giving, } q = \frac{\xi - z}{v(1-z)^2} \left[(1-z)(\ddot{\bar{w}}_1 - \ddot{\bar{w}}_z)(\xi + z) + 2(1-z)(\ddot{\bar{w}}_z - \ddot{\bar{w}}_1 z) + \right.$$

$$\left. 2\dot{z}(\dot{\bar{w}}_z - \dot{\bar{w}}_1) - \dot{z}(\dot{\bar{w}}_z - \dot{\bar{w}}_1)(\xi + z) - 2p(1-z)^2 \right] \quad \text{for } z \leq \xi \leq 1 \quad (24)$$

By using $q = -k\bar{w}_1$ at $\xi=1$, we have

$$-k\bar{w}_1 = (1-z)(\ddot{\bar{w}}_1 + \ddot{\bar{w}}_z) - \dot{z}(\dot{\bar{w}}_1 - \dot{\bar{w}}_z) - 2p(1-z). \quad (25)$$

Equations (1b, 24) with $m=-1$ at $\xi=z$ give

$$m = -1 - \frac{2}{(1-z)^2} \int_z^\xi (\xi - z) \left\{ (1-z)(\ddot{\bar{w}}_1 - \ddot{\bar{w}}_z) - \dot{z}(\dot{\bar{w}}_z - \dot{\bar{w}}_1) \right\} (\xi + z) + \\ 2(1-z)(\ddot{\bar{w}}_z - \ddot{\bar{w}}_1 z) + 2\dot{z}(\dot{\bar{w}}_z - \dot{\bar{w}}_1) - 2p(1-z)^2 \} d\xi$$

$$\text{or } m = -1 - \frac{2(\xi - z)^2}{3(1 - z)^2} \left\{ \left[(1 - z)(\ddot{\bar{w}}_1 - \ddot{\bar{w}}_z) - \dot{z}(\dot{\bar{w}}_z - \dot{\bar{w}}_1) \right] \xi + 2z \right) + 3 \left[(1 - z)(\ddot{\bar{w}}_z - \ddot{\bar{w}}_1 z) + \dot{z}(\dot{\bar{w}}_z - \dot{\bar{w}}_1) - p(1 - z)^2 \right] \right\} \quad \text{for } z \leq \xi \leq 1. \quad (26)$$

m=1 at $\xi=1$ predicts

$$-3 = (\ddot{\bar{w}}_1 + 2\ddot{\bar{w}}_z)(1 - z)^2 - 2(\dot{\bar{w}}_1 - \dot{\bar{w}}_z)\dot{z}(1 - z) - 3p(1 - z)^2. \quad (27)$$

It is noted from Eq.(21) that the transverse acceleration ($\ddot{\bar{w}}$) of the local material at the interface is equal to $p(\tau)$. Thus, by equating the right hand side of Eq. (22) to p when $\xi=z$, we have

$$\dot{z}(\dot{\bar{w}}_1 - \dot{\bar{w}}_z) = (\ddot{\bar{w}}_z - p)(1 - z). \quad (28)$$

Considering Eq.(28), Eqs.(25, 27) can be respectively rewritten as

$$-k\nu\bar{w}_1 = (1 - z)(\ddot{\bar{w}}_1 - p) \quad (29a)$$

$$\text{and } -3 = (1 - z)^2(\ddot{\bar{w}}_1 - p) \quad (29b)$$

The time history of $\bar{w}_1$ and the position of the interface can be derived from Eqs.(29a,b)

to give

$$\ddot{\bar{w}}_1 - p = -\frac{(k\nu\bar{w}_1)^2}{3}. \quad (30)$$

$$z = 1 - \frac{3}{k\nu\bar{w}_1}. \quad (31)$$

From Eqs.(22) and (28), the distribution of the acceleration field may be expressed as

$$\ddot{\bar{w}} = \ddot{\bar{w}}_1 + \frac{1 - \xi}{1 - z}(p - \ddot{\bar{w}}_1), \quad z \leq \xi \leq 1 \quad (32)$$

in the rigid portion. Equation (32) shows that the distribution of the acceleration field is linear in Section 3.2.2 for case II.

It may be shown that the sufficient and necessary condition to maintain $0 \leq z \leq 1$ is $k\nu\bar{w}_1 \geq 3$, which is satisfied in this phase when $\tau \geq \tau_a$. It can be proved that z is an

increasing function of the non-dimensional time τ in this case, which implies that the central plastic region expands outwards. The elastic response at the supports stops when $\tau = \tau_0$, which is determined from $k\bar{w}_1 = 1$. At the end of this phase, the interface reaches its farthest position, i.e.,

$$z_{\max} = 1 - \frac{3}{\nu} . \quad (33)$$

The initial conditions for the next response phase of motion are determined from the end of this phase, i.e.

$$\bar{w}_{10} = \bar{w}_1 \Big|_{\tau=\tau_0} = \frac{1}{k} \quad \text{and} \quad \dot{\bar{w}}_{10} = \dot{\bar{w}}_1 \Big|_{\tau=\tau_0} \quad (34)$$

3.3.4 Phase 4, $\tau_0 \leq \tau \leq \tau_1$

When $k\bar{w}_1 \geq 1$, the shear force at the support $q = -1.0$ and it is assumed that z is fixed as $(1 - 3/\nu)$ throughout this phase. This is a reasonable assumption since the shear force at the support is constant.

From Eq. (28) and considering $\dot{z} = 0$, we have

$$\ddot{\bar{w}}_z = p(\tau) . \quad (35)$$

By using $q = -1$ at $\xi = 1$ in Eq. (24) and considering Eqs. (33) and (35), we have

$$\ddot{\bar{w}}_1 - p = -\nu^2 / 3 . \quad (36)$$

Plastic shear displacement occurs at the supports during this phase, and reaches a maximum value at $\tau = \tau_1$ when $\dot{\bar{w}}_1 = 0$. This response phase is different from the corresponding response phase in the rigid, perfectly plastic solution of Li and Jones(1995a) due to the existence of a central plastic deformation region developed during the elastic response phase at the supports. However, Eq.(33) is identical to Eq.(26) in Li and Jones(1995a) when the elastic response phase at the supports does not exist (i.e. $\dot{\bar{w}}_0 = \dot{\bar{w}}_1 = 0$ at $\tau = \tau_0$).

The static admissibility of the generalized stress fields in case III is illustrated in the appendix.

4. Results for impulsive loading

In this case, $p(\tau) = \delta(\tau)$ is a Dirac delta function and the corresponding results in Section 3 may be simplified as follows.

4.1 $0 < \nu \leq 2$

4.1.1 Phase 1, $0 \leq \tau \leq \tau_0$

The non-dimensional transverse displacement and velocity at the supports are

$$\bar{w}_1 = \frac{1}{\lambda} \sin(\lambda \tau) \quad \text{and} \quad \dot{\bar{w}}_1 = \cos(\lambda \tau) \quad (37a,b)$$

in which, τ_0 is determined from $\bar{w}_1 = \frac{1}{k}$, or

$$\sin(\lambda \tau_0) = \frac{\lambda}{k} \quad (38)$$

If $\frac{\lambda}{k} > 1$, Eq.(38) has no solution for τ_0 , which means that $\bar{w}_1$ can not reach $\frac{1}{k}$.

Therefore, the condition for the existence of a transverse plastic shear response is $\frac{\lambda}{k} \leq 1$,

or

$$I_0 \geq \sqrt{\frac{\mu P_b Q_0 \nu}{2K}}. \quad (39)$$

4.1.2 Phase 2 $\tau_0 < \tau \leq \tau_1$

The solutions for the non-dimensional transverse shear displacement and velocity from Eq.(11) are

$$\bar{w}_1 = \frac{1}{k} + (\tau - \tau_0) \cos(\lambda \tau_0) - \frac{\nu}{4} (\tau^2 - \tau_0^2) + \frac{\nu \tau_0}{2} (\tau - \tau_0) \quad (40a)$$

and $\dot{\bar{w}}_1 = \cos(\lambda\tau_0) - \frac{\nu}{2}(\tau - \tau_0)$ (40b)

and τ_1 is given by $\dot{\bar{w}}_1|_{\tau=\tau_1} = 0$, i.e.,

$$\tau_1 = \tau_0 + \frac{2}{\nu} \cos(\lambda\tau_0) . \quad (41)$$

Equation (41), when $\tau_0 \rightarrow 0$, is identical to Eq.(11) in Li and Jones(1995a) for impulsive loading.

4.2 $2 < \nu \leq 3$

4.2.1 Phase 1, $0 \leq \tau \leq \tau_c$: Solutions are given in Section 3.1.1.

The transverse displacement and velocity at supports are

$$\bar{w}_1 = \frac{1}{\lambda} \sin(\lambda\tau) \quad \text{and} \quad \dot{\bar{w}}_1 = \cos(\lambda\tau) \quad (42a,b)$$

$$\bar{w}_{1c} = \frac{1}{\lambda} \sin(\lambda\tau_c) = \frac{2}{k\nu} \quad \text{and} \quad \dot{\bar{w}}_{1c} = \cos(\lambda\tau_c) = \sqrt{1 - \frac{2}{k\nu}} \quad (43a,b)$$

4.2.2 Phase 2, $\tau_c \leq \tau \leq \tau_0$

Equations (17a,b) have the following expressions when $p(\tau) = \delta(\tau)$

$$\bar{w}_1 = \frac{1}{4\lambda^2} \left[3 + 2\sqrt{\lambda^2 - 1} \sin(2\lambda(\tau - \tau_c)) + \cos(2\lambda(\tau - \tau_c)) \right] \quad (44a)$$

$$\text{and} \quad \dot{\bar{w}}_1 = \frac{1}{2\lambda} \left[2\sqrt{\lambda^2 - 1} \cos(2\lambda(\tau - \tau_c)) - \sin(2\lambda(\tau - \tau_c)) \right], \quad (44b)$$

and τ_0 is reached when $\bar{w}_1 = \frac{1}{k}$.

4.2.3 Phase 3, $\tau_0 \leq \tau \leq \tau_1$

Equations (18a,b) may be simplified in this case as

$$\bar{w}_1 = \frac{1}{k} + \dot{\bar{w}}_{10}(\tau - \tau_0) + \frac{1}{2}(3 - 2\nu)(\tau - \tau_0)^2 \quad (45a)$$

$$\text{and} \quad \dot{\bar{w}}_1 = \dot{\bar{w}}_{10} + (3 - 2\nu)(\tau - \tau_0) \quad (45b)$$

which is valid during $\tau_0 \leq \tau \leq \tau_1$, where τ_1 is determined from $\dot{\bar{w}}_1 = 0$.

4.3 $\nu > 3$

According to the results in Section 3.3, there are four response phases in this case

4.3.1 Phase 1, $0 \leq \tau \leq \tau_c$, where τ_c is reached when $\bar{w}_1 = \frac{2}{k\nu}$

4.3.2 Phase 2, $\tau_c \leq \tau \leq \tau_a$, where τ_a is reached when $\bar{w}_1 = \frac{3}{k\nu}$

4.3.3 Phase 3, $\tau_a < \tau \leq \tau_0$: The transverse displacement and velocity fields are determined from Eq.(31) together with $\bar{w}_1|_{\tau=\tau_a} = \frac{3}{k\nu}$ and the solutions in Section 4.3.2.

This process should be solved numerically.

4.3.4 Phase 4, $\tau_0 < \tau \leq \tau_1$: $\bar{w}_1$ and $\dot{\bar{w}}_1$ may be obtained from Eq.(36) together with initial condition $\bar{w}_1|_{\tau=\tau_0} = \frac{1}{k}$.

The above results for the impulsive loading case agree with the corresponding results in Li and Jones(1995a) for the rigid perfectly plastic case when $k \rightarrow \infty$.

5. Discussion

Numerical calculations for impulsive loading are conducted to demonstrate the validity of the model and to present parametric analyses. Figures 4-6 show the variations of the non-dimensional transverse shear displacement at the supports with non-dimensional time from the start of the response to the time when the maximum transverse shear displacement is reached. They show that the non-dimensional time required for the beam to reach its maximum transverse shear displacement increases with a decrease of the elastic shear stiffness, i.e. it takes a longer time to reach the peak transverse displacement of a beam having a flexible shear stiffness. Moreover, the maximum transverse shear displacement of a beam with a small elastic shear stiffness is larger than

that for a beam having a large elastic shear stiffness. Effects of the elastic shear deformation are demonstrated by changing the value of non-dimensional number k from 10 to 1000¹ for a range of shear and bending capacity ratios represented by the non-dimensional number v , as shown in Fig.7. In these figures, the range of the k value is selected to cover a wide span from an order representing real structure (i.e. 10¹) to the order which is sufficient to represent a rigid-plastic model (i.e. 10³). An explicit expression of k will be given later. It shows that neglecting the elastic shear deformation in a rigid-plastic beam analysis generally underestimates the maximum transverse shear displacement in the beam. However, the influence of elastic shear deformation on the maximum transverse shear displacement depends on the value of v . The influence is smaller for smaller values of v . When $v=2.0$, the relative difference of the maximum transverse shear displacement between $k=10$ and $k=1000$ is about 10%, which increases to 58% when $v=4.0$.

The maximum transverse plastic shear displacement is closely related to the maximum plastic shear strain within the localized shear deformation zone, and therefore, has been frequently used as an elementary transverse shear failure criterion [Jones(1976,1989a)]. The maximum transverse plastic shear displacement is

$$\Delta w_{1\max} = w_{1\max} - \frac{Q_0}{K} \quad (46a)$$

or, in non-dimensional form,

$$\Delta \bar{w}_{1\max} = \bar{w}_{1\max} - \frac{1}{k} \quad (46b)$$

where Q_0/K represents the maximum elastic shear displacement for an elastic, perfectly plastic model; $w_{1\max}$ is the maximum transverse shear displacement. Thus, the elementary transverse shear failure criterion can be expressed as

$$\Delta w_{1\max} = \beta H \quad (47)$$

¹ $k=1000$ is essentially the rigid, perfectly plastic case.

in which, $0 < \beta \leq 1$ and H is the beam thickness. It has been shown that β is dependent on the material properties and geometries of the cross-section of the beam [Jouri and Jones(1988)]. In the following calculation examples, the failure coefficient β is approximately taken to be 0.2 and 0.4 for a titanium-alloy beam and a mild steel beam, respectively. However, the exact value of the failure coefficient β should be determined experimentally for the materials and cross-sectional shapes of the investigated beams.

Figure 8 illustrates the influence of the elastic shear stiffness on the non-dimensional maximum transverse plastic shear displacements for different values of ν . It shows consistently that the maximum transverse plastic shear displacement increases with an increase of the elastic shear stiffness K , which is contrary to the influence of K on the maximum transverse shear displacement shown in Fig. 7. It may be concluded that when the shear flexibility in the beam is considered, shear failure is more difficult. Thus, failure analyses based on rigid-plastic idealization are conservative. When $k \rightarrow \infty$, the corresponding curves in Figs.7 and 8 approach to the same asymptotic line, i.e. the rigid, perfectly plastic solution.

According to Li and Jones(1995a), the critical initial impulsive velocities for transverse shear failures using Eq.(47) without considering elastic shear deformation are

$$\frac{V_c}{V_a} = \sqrt{\beta\nu}, \quad \nu \leq 2 \quad (48a)$$

$$\frac{V_c}{V_a} = \sqrt{2\beta(2\nu-3)}, \quad 2 < \nu \leq 3 \quad (48b)$$

$$\frac{V_c}{V_a} = \sqrt{\frac{2\beta\nu^2}{3}}, \quad \nu > 3 \quad , \quad (48c)$$

where $V_a = \sqrt{\frac{4M_0 H}{\mu L^2}}$. For an impulsive loading, the critical loading impulse $I_c = \mu V_c$.

It is evident that parameter ν plays a very important role in governing the response mode of the beam. For a solid beam with rectangular cross-section, $\nu=L/H$ according to

Eq.(3i), which implies that only $\nu > 3$ is practically meaningful, and thus, some previous studies, such as Wen et al. (1995) and Yu and Chen (2000), only considered the case when ν is greater than 3. However, cases for $\nu < 3$ are met frequently for wide-flanged I-beams and beams with sandwiched cross-sections [Jones(1989b)]. The example given in Jones(1989b) (pp.272) shows that $\nu=1$ may correspond to a sandwiched beam with $2L/H=11$.

The solutions in Section 4.1 can be used to obtain an explicit critical impulsive velocity, i.e.

$$V_c = \sqrt{\frac{2M_0\nu}{\mu L^2} \left(\frac{Q_0}{K} + 2\beta H \right)} \quad \text{or} \quad \frac{V_c}{V_a} = \sqrt{\nu \left(\frac{1}{2\bar{k}} + \beta \right)} \quad (49a,b)$$

for $\nu \leq 2$, where $\bar{k} = \frac{KH}{Q_0} = \frac{k}{(V_c/V_a)^2}$. Equation (49b) is identical to Eq.(48a) when

$\bar{k} \rightarrow \infty$.

For $\nu > 2$, numerical calculations are necessary to predict the critical impulsive velocities. The non-dimensional critical velocity (V_c/V_a) is obtained using an iterative procedure, in which the maximum transverse plastic shear displacement is first calculated based on given values of $\bar{k}$ and ν and an initial trial value of the non-dimensional impulsive velocity. The maximum transverse plastic shear displacement is then compared with the failure criterion and then the non-dimensional impulsive velocity is adjusted until Eq.(47) is satisfied. Variations of the non-dimensional impulsive velocity with $\bar{k}$ are illustrated in Figure 9 for a titanium-alloy beam and Figure 10 for a mild steel beam. It is seen that the non-dimensional impulsive velocity decreases with an increase of $\bar{k}$ and then approaches a constant value when $\bar{k}$ is very large. The influence of the elastic shear stiffness on the critical impulsive velocity is significant when $\bar{k}$ is less than 100. It is seen again that the rigid-plastic analysis for the shear failure of a beam is conservative when the elastic shear flexibility at the supports is considered.

The above analysis is based on an assumption that the elastic deformation in the plastic- dominated response of a beam can be represented by an elastic spring at the support, which needs to be rationalized. Meanwhile, the value of K (and $\bar{k}$) introduced in the model should be related to the material and geometric parameters of the beam with the aid of an elastic analysis.

Considering a fully supported beam subjected to a uniformly-distributed pressure load, as shown in Fig.1(a), the shear force distribution during the shear sliding period is independent of time and can be determined from a plastic response analysis [e.g. Fig.5 in Li and Jones(1995a)]. For quasi-static deformation and for $v \leq 2$ in a dynamic plastic response, the shear force distribution is linear, i.e. from zero at mid-span to maximum at the support. When $v > 2$, a zero shear force zone expands, which implies that the corresponding elastic shear influence is reduced. In the following analysis, a linear shear force distribution

$$Q = -Q_0 \frac{x}{L}, \quad (50)$$

where Q_0 is the maximum shear force at the support, will be employed to demonstrate the method. Readers should use the actual shear force distribution obtained from a dynamic plastic response analysis [Li and Jones(1995a)] in a practical problem.

The elastic shear stress in the beam and the corresponding average elastic shear strain in the beam cross-section is [Timoshenko(1921)]

$$\tau = \frac{Q}{A} \quad \text{and} \quad \gamma = \frac{\tau}{\eta G} \quad (51a,b)$$

where A is the cross-sectional area of the beam; $G = \frac{E}{2(1+\bar{\mu})}$ is the shear modulus; E and $\bar{\mu}$ are Young's modulus and Poisson's ratio; η is the transverse shear coefficient, which depends on the shape of the beam cross-section.

Therefore, the total transverse elastic shear energy in the beam is

$$W^e = \int_0^L A \left(\int_0^\gamma \tau d\gamma \right) dx = \int_0^L \frac{Q^2}{2\eta GA} dx = \frac{Q_0^2 L}{6A\eta G}. \quad (52)$$

According to the energy equivalence method [Biggs(1964)], the elastic transverse shear deformation energy in the beam is equivalent to the elastic energy of a transversely-placed spring with spring constant K , i.e.

$$\frac{1}{2} K \Delta^2 = W^e \quad (53)$$

where the maximum transverse shear displacement Δ is determined by

$$\Delta = \left| \int_0^L \gamma dx \right| = \int_0^L \frac{Q}{\eta AG} dx = \frac{Q_0 L}{2\eta AG}, \quad (54)$$

and therefore,

$$K = \frac{4\eta AG}{3L}. \quad (55)$$

On the other hand, if the beam is supported by flexible supports, K may represent the stiffness of the support, which should be determined using an alternative method.

The non-dimensional parameter $\bar{k}$ is,

$$\bar{k} = \frac{KH}{Q_0} = \frac{8}{3} \eta \left(\frac{G}{\sigma_0} \right) \frac{1}{(L/H)} = \frac{4}{3} \eta \left(\frac{E}{\sigma_0} \right) \left(\frac{1}{1+\bar{\mu}} \right) \left(\frac{1}{L/H} \right) \quad (56)$$

when the shear capacity $Q_0 = A \tau_y = \frac{A \sigma_0}{2}$ (under Tresca yield condition, $\tau_y = \sigma_0/2$) is

employed. σ_0 is the uniaxial yield stress of the beam material.

For example, a mild steel beam having $E=300\text{GPa}$, $\bar{\mu} = 0.30$, $\eta=2/3$ and $\sigma_0=400\text{MPa}$, $\bar{k}$ varies between 76.9 and 256.4 when L/H varies between 3 and 10. However, for a titanium-alloy beam having $E=150\text{GPa}$, $\bar{\mu} = 0.33$, $\eta=2/3$ and $\sigma_0=900\text{MPa}$, $\bar{k}$ varies between 16.7 and 55.7 for the same range of L/H . Thus, the influence of elastic shear deformation on the transverse shear failure of a titanium beam may be important. The analysis neglecting the elastic shear deformation in a rigid-plastic titanium-alloy beam

may cause an error between 1.3% and 5.3% when $v=4.0$ and an error between 2.0% and 7.9% when $v=1.5$. However the same effect can be neglected in a steel beam where the error is between 0.1% and 0.6% when $v=4.0$ and between 0.1% and 0.8% when $v=1.5$. It should be mentioned that the elastic effect on the shear failure becomes less significant when v is larger than 4.0. This conclusion is enhanced when the diminished transverse shear force distribution for large value of v is considered.

The explicit expression of another non-dimensional number k used in Figs.4-8 is

$$k = \frac{4\eta}{3(1+\bar{\mu})} \left(\frac{E}{\sigma_0} \right) \left(\frac{L}{H} \right) \left(\frac{\rho V_0^2}{\sigma_0} \right). \quad (57)$$

Take steel beam as an example, $\bar{\mu}=0.3$, $\eta=2/3$, $\frac{E}{\sigma_0} \approx 10^3$, $\frac{L}{H} \approx 10$ and Johnson's

damage number $\frac{\rho V_0^2}{\sigma_0} \approx 10^{-3}$ (using $\rho=7800\text{kg/m}^3$, $\sigma_0=400\text{MPa}$, $V_0=10\text{m/s}$), then

$k \approx 10^1$, which is considered as the order of the value to represent a real structure in previous discussion. These non-dimensional numbers have actually been included in a more general dimensional analysis for the dynamic structural response under impact and blast loadings [Li and Jones(2000b)].

The analytical model used in this study is based on rigid, perfectly plastic beam theory, whose accuracy decreases with the decrease of beam slenderness ratio which is associated with the non-dimensional number v for a beam with solid cross-section (i.e. $v=L/H$). In order to support the conclusions drawn from the analytical modeling, numerical simulations using finite difference method through ANSYS AUTODYN V11.0.00 are conducted to investigate the influence of elastic shear deformation on the transverse shear response of the beam. The overall length and thickness of the mild steel beam are 4m and 2m, respectively. 40 horizontal and 4 vertical meshes were used in the simulation. Mesh sensitivity has been conducted for higher mesh density (i.e. 80

horizontal and 8 vertical meshes), and only 0.6% difference was found on maximum transverse displacement. Transverse shear displacement is defined as the relative transverse displacement across the shear hinge. Subjective factor may influence the determination of the boundary of a shear hinge. However, its influence on the calculation of the transverse shear displacement is systematic and small because shear hinge zone can be easily identified in a numerical simulation [Li and Jones(2000a)]. The material model used is an elastic, perfectly plastic von Mises model. Material property parameters used in numerical simulation are $\bar{\mu} = 0.30$ and $\sigma_0 = 400 \text{ MPa}$ while three different Young's moduli, i.e. $E = 300 \text{ GPa}$, 600 GPa and 30000 GPa , are used corresponding to $k = 10$, 20 and 1000 according to Eq.(57) when $\eta = 2/3$, $\rho = 7800 \text{ kg/m}^3$ and $V_0 = 22.36 \text{ m/s}$. Comparisons between numerical and analytical results from $k = 10$ to 1000 are shown in Fig.11. It is found that the analytical variation of the maximum transverse shear displacement with the shear stiffness is consistent with the numerical one although a relative difference of less than 15.0% between analytical and numerical predictions is observed which may be attributed to the rigid, perfectly plastic simplification used in the analytical model.

6. Conclusions

A theoretical model for the transverse shear response of a rigid-plastic beam with consideration of the influence of elastic shear deformations is proposed in this paper. The elastic shear deformation is taken into account by introducing elastic shear stiffness at the supports of a fully clamped beam. Analytical solutions are obtained for different ranges of the system parameters.

The transverse shear failure of a fully clamped beam is studied for the impulsive loading case. By using the elementary failure criterion, i.e. a failure criterion controlled by the value of the transverse shear displacement, the critical impulsive velocities are calculated for different values of the other system parameters. It demonstrated that the

elastic shear stiffness has a significant influence on the maximum transverse shear displacement and on the critical impulsive velocity. Generally speaking, the smaller the non-dimensional shear stiffness of the beam, the larger the influence of the elastic shear deformation. Although the maximum shear displacement decreases with an increase of the elastic shear stiffness, the maximum plastic shear displacement increases with the increase of the elastic shear stiffness. Therefore, the shear failure of the beam under blast loading becomes more likely if the elastic shear stiffness of the beam is large. For an impulsive loading, the critical impulsive velocity to initiate a shear failure in the beam decreases with an increase of the elastic shear stiffness of the beam. When the elastic shear stiffness approaches infinity, the present predictions reduce to the results in Li and Jones(1995a) for a rigid, perfectly plastic beam model without any elastic shear effects. It is concluded that the rigid-plastic model generally gives conservative predictions for the critical loading parameters for a transverse shear failure. The proposed model can be extended to study the shear failures of other structural elements (e.g. plates and shells) when the effects of the elastic shear stiffness on the transverse shear responses of these elements become important.

Appendix: Static Admissibility of the Generalized Stress Fields

A.1 $0 < \nu \leq 2$

For $\tau < \tau_0$, the generalized stress fields are given as follows according to Eqs.(1a,b) and Eq.(6a) together with $q=0$ at $\xi=0$ and $m=1$ at $\xi=1$.

$$q = \frac{2}{\nu} (\ddot{w}_1 - p) = -k\bar{w}_1\xi \quad (A1)$$

$$m = 1 - \nu k\bar{w}_1(1 - \xi^2). \quad (A2)$$

$-1 \leq q \leq 0$ requires $\bar{w}_1 \leq \frac{1}{k}$ which is satisfied for $\tau < \tau_0$. $-1 \leq m \leq 1$ requires

$\bar{w}_1 \leq \frac{2}{\nu k}$, which is satisfied automatically when $\nu \leq 2$ and $\bar{w}_1 \leq \frac{1}{k}$. However, when

$\nu > 2$, condition $\bar{w}_1 \leq \frac{2}{\nu k}$ must be satisfied if the same velocity field is applicable.

For $\tau \geq \tau_0$ (or $\bar{w}_1 \geq \frac{1}{k}$), the static admissibility of the generalized stress field has

been discussed by Li and Jones(1995a).

A.2 $2 < \nu \leq 3$

The static admissibility of the generalized stress fields for $\tau < \tau_c$ (or $\bar{w}_1 < \frac{2}{\nu k}$) has

been shown in Section A.1. For $\tau_c \leq \tau < \tau_0$ (or $\frac{2}{\nu k} \leq \bar{w}_1 < \frac{1}{k}$), Eqs.(15a,c) give

$$\ddot{\bar{w}}_1 = p + 3 - 2\nu k \bar{w}_1$$

$$\text{and } \ddot{\bar{w}}_0 = \nu k \bar{w}_1 + p - 3,$$

by which, Eq.(14a) may be rewritten as

$$q = \frac{2}{\nu} \left[(\nu k \bar{w}_1 - 3)\xi - \frac{3}{2}(\nu k \bar{w}_1 - 2)\xi^2 \right]. \quad (\text{A3})$$

Thus, $q|_{\xi=0} = 0$ and $q|_{\xi=1} = -k\bar{w}_1$. In order to satisfy $-1 \leq q \leq 0$, we need to ensure

that $q|_{\xi=1} > -1$, which requires $\bar{w}_1 < \frac{1}{k}$, and $0 \geq q|_{\xi=\xi_c} > -1$, where the extreme value of

q at ξ_c is determined from $\frac{\partial q}{\partial \xi} \Big|_{\xi=\xi_c} = 0$, i.e.

$$\xi_c = \frac{\nu k \bar{w}_1 - 3}{3(\nu k \bar{w}_1 - 2)}. \quad (\text{A4})$$

The existence of ξ_c requires $0 < \frac{3 - \nu k \bar{w}_1}{3(2 - \nu k \bar{w}_1)} < 1$, which leads to $\nu k \bar{w}_1 < \frac{3}{2}$ or $\nu k \bar{w}_1 > 3$.

Since $\frac{1}{k} \leq \frac{3}{\nu k}$ when $2 < \nu \leq 3$, ξ_c does not exist for $\frac{2}{\nu k} < \bar{w}_1 \leq \frac{1}{k}$, and thus,

$-1 \leq q \leq 0$ is valid in $\tau_c \leq \tau < \tau_0$ or $\frac{2}{\nu k} < \bar{w}_1 \leq \frac{1}{k}$.

The above analysis is also applicable to $\nu > 3$, and therefore, $-1 \leq q \leq 0$ can be satisfied

in $\tau_c \leq \tau < \tau_a$ or $\frac{2}{\nu k} < \bar{w}_1 \leq \frac{3}{\nu k}$ when $\nu > 3$.

According to Eq.(1b), m is a monotonically increasing function of ξ . $m|_{\xi=0} = -1$

and $m|_{\xi=1} = 1$ bound the bending moment in the range of $-1 \leq m \leq 1$ for $0 \leq \xi \leq 1$.

A.3 $\nu > 3$

For $\bar{w}_1 < \frac{3}{\nu k}$, or $\tau < \tau_a$, the solution is the same as in Sections 3.2.1 and 3.2.2, and

therefore, the static admissibility has been proved in Section A.2.

For $\bar{w}_1 \geq \frac{1}{k}$, or $\tau \geq \tau_0$, the static admissibility of the generalized stress field has been

discussed by Li and Jones(1995a).

For $\frac{1}{k} > \bar{w}_1 \geq \frac{3}{\nu k}$, or $\tau_a \leq \tau < \tau_0$, the static admissibility is proved here.

Considering Eq. (28), Eq. (24) is rewritten as

$$\begin{aligned}
 q &= \frac{\xi - z}{\nu(1-z)^2} \left[(1-z)(\ddot{w}_1 - \ddot{w}_z)(\xi + z) + 2(1-z)(\ddot{w}_z - \ddot{w}_1 z) + \right. \\
 &\quad \left. - 2(\ddot{w}_z - p)(1-z) + (\ddot{w}_z - p)(1-z)(\xi + z) - 2p(1-z)^2 \right] \\
 &= \frac{\xi - z}{\nu(1-z)^2} \left[(1-z)(\ddot{w}_1 - p)(\xi + z) + 2(1-z)(p - \ddot{w}_1 z) - 2p(1-z)^2 \right] \\
 &= \frac{\xi - z}{\nu(1-z)} \left[(\ddot{w}_1 - p)(\xi + z) + 2(p - \ddot{w}_1 z) - 2p(1-z) \right] \\
 &= \frac{(\xi - z)^2}{\nu(1-z)} (\ddot{w}_1 - p) = -\frac{(\xi - z)^2}{\nu(1-z)} \frac{(k\nu\bar{w}_1)^2}{3}
 \end{aligned} \tag{A5}$$

by using Eq.(30). It can be seen from the above equation that q is a monotonically decreasing function of ξ .

Therefore, in $z \leq \xi \leq 1$, $0 \geq q \geq -\frac{(1-z)(k\bar{w}_1)^2}{3\nu} = -k\bar{w}_1 \geq -1$ when Eq.(31) is used.

The bending moment can be obtained as

$$m = -1 - 2\nu \int_z^\xi q d\xi = -1 + \frac{2(k\bar{w}_1)^2}{3(1-z)} \int_z^\xi (\xi - z)^2 d\xi = -1 + \frac{2(k\bar{w}_1)^2 (\xi - z)^3}{9(1-z)}, \quad (\text{A6})$$

which is a monotonically increasing function of ξ .

When $\xi=1$,

$$m|_{\xi=1} = -1 + \frac{2(k\bar{w}_1)^2 (1-z)^2}{9} = -1 + 2 = 1, \quad (\text{A7})$$

according to Eq.(31). Hence $-1 \leq m \leq 1$, for $z \leq \xi \leq 1$.

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(a)

(b)

(c)

(d)

Fig.1 Fully clamped beam, (a) The idealized problem, (b) Notations, (c) Blast loading, and (d) Relationship between shear force and shear displacement.

Fig.2 Independent yield condition.

(a)

(b)

(c)

Fig.3 Velocity field patterns, (a) Uniform velocity distribution (case I), (b) linear velocity distribution (case II), and (c) Curved and linear velocity distributions with travelling plastic bending hinges (case III).

Fig.4 Variation of non-dimensional transverse shear displacements with non-dimensional time for case-I ($\nu=1.5$).

Fig.5 Variation of non-dimensional transverse shear displacements with non-dimensional time for case-II ($\nu=2.5$).

Fig.6 Variation of non-dimensional transverse shear displacement with non-dimensional time for case-III ($\nu=4.0$).

Fig.7 Variation of non-dimensional maximum transverse shear displacement with non-dimensional elastic shear stiffness k .

Fig.8 Variation of non-dimensional maximum transverse plastic shear displacement with non-dimensional elastic shear stiffness k .

Fig.9 Variation of non-dimensional critical impulsive velocity with non-dimensional elastic shear stiffness $\bar{k}$ for a titanium-alloy beam ($\beta=0.2$).

Fig.10 Variation of non-dimensional critical impulsive velocity with non-dimensional elastic shear stiffness $\bar{k}$ for a mild steel beam ($\beta=0.4$).

Fig.11 Variation of maximum transverse displacements with non-dimensional elastic shear stiffness k predicted by numerical simulation and analytical model ($\nu=2.0$).