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Lateral Shift Makes a Ground-Plane Cloak Detectable

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We examine the effectiveness of the ground-plane invisibility cloak generated from quasiconformal mapping of electromagnetic space. This cloak without anisotropy will generally lead to a lateral shift of the scattered wave, whose value is comparable to the height of the cloaked object, making the object detectable. This can be explained by the fact that the corresponding virtual space is thinner and wider than it should be. Ray tracing on a concrete model shows that, for a bump with a maximum height of 0.2 units to be hidden, the lateral shift of a ray with 45° incidence is around 0.15 units.

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The invisibility cloaks formed by the coordinate transformation method are rapidly developing from narrow band to broadband [1–10]. For a given coordinate transformation from "virtual space" into "physical space," the homogeneous Maxwell equations retain their form, with only the constitutive parameters changing together with the field values. This formal invariance gives rise to a powerful technique for designing optical devices, one of which is the invisibility cloak. Recently there have been reports on the applications of quasiconformal mapping to achieve broadband invisibility by utilizing a ground plane of the perfect electric conductor (PEC) [5,6,8–10]. This new strategy is based on the assumption that a sufficiently minimized anisotropy of transformed media can be dropped and consequently isotropic media in place of anisotropic media can be used to construct a broadband cloak [5]. Subsequent experiments have implemented this model in both microwave [6] and optical frequencies [8–10]. Most studies have considered this kind of ground-plane cloak as being undetectable. However, the physical consequences as well as the validity of this quasiconformal mapping technique, i.e., neglecting of the anisotropy and replacement of anisotropic materials with isotropic materials, have not been thoroughly discussed in previous studies.

In this Letter we examine the cloaking effectiveness of the quasiconformal-mapped ground-plane cloak which omits the minimized anisotropy. We demonstrate that the anisotropy of the transformed cloak medium is of fundamental importance in preserving both the phase and energy propagation of electromagnetic waves. Although the isotropic ground-plane cloak is able to largely preserve the phase propagation, the energy propagation is subject to a lateral shift which will render the cloak detectable. We show in a simplified case that by dropping the minimized anisotropy, although it can be arbitrarily small, a lateral shift on the order of the height of the object to be hidden will occur. In the model of the ground-plane cloak hiding a bump with the maximum height being 0.2 units, we show by ray tracing that the reflected ray of an incident ray with 45° incidence is laterally shifted on the incident side of the cloak by a distance of 0.15 units compared to the ideal case. Our theoretical analysis shows that this ground-plane cloak is equivalent to a free space thinner and wider than the ideal case, causing a lateral shift. This laterally shifting phenomena is similar to the effect of simply putting a ground plane above the object to be cloaked. The conclusion of a perceptible lateral shift, being inconsistent with previous verifications through ray tracing [6], electromagnetic numerical simulation [5,8–10], and experiments [6,8–10], provides a balanced view on the performance of a ground-plane cloak.

In order to understand the fundamental role of anisotropy in transformation-based invisibility cloaks, we first recall the required forms of the constitutive parameters of the transformation medium. In the transformation medium that is transformed from an empty space with permittivity ϵ_0 (or permeability μ_0), the transformed permittivity tensor is $\bar{\bar{\epsilon}} = \epsilon_0 \bar{\bar{J}} \cdot \bar{\bar{J}}^T/|J|$ (the permeability $\bar{\mu}$ has similar expression), where $\bar{\bar{J}}$ is the Jacobian of the transformation. If we require $\bar{\bar{\epsilon}}$ to be isotropic, the Jacobian must be a constant times an orthogonal matrix and the transformation corresponds to a rotation with scaling (here we do not consider inversion operation). However, for a freestanding cloak [1], the transformation does not correspond to a conformal mapping, and therefore orthogonality is not preserved by the transformation. This has a physical significance and we can quickly give an interpretation. Consider Fig. 1, where a transformation is applied to a
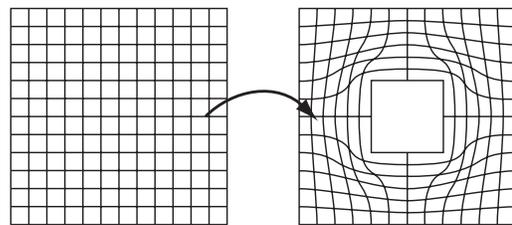

FIG. 1. An example of creating an invisibility cloak by coordinate transformation. In general the transformation is not a conformal mapping and the transformed cloak is anisotropic.

virtual empty space and a void is created in the center. In the virtual electromagnetic space, we can consider the horizontal lines as rays, and correspondingly the vertical lines as phase fronts. After transformation, the mesh is distorted. Take the first quadrant as an example. In order to fit the void in, the horizontal lines in this region are bent clockwise, while the vertical lines at the same location are bent counterclockwise. This means the energy propagation is not in the same direction as the phase propagation, which is not possible in an isotropic medium. It should be pointed out that a strict conformal mapping [2] does not suffer from this problem, but is restrictive in the choice of boundary conditions.

For the ground-plane cloak, the anisotropy is also important. Let us first consider a simplified case, as shown in Fig. 2. Consider a PEC ground plane in free space at the origin, as shown in Fig. 2(a), where the trajectory of a ray with incident angle $\theta_i = 45°$ is plotted. Using the transformation method, it is possible to construct an anisotropic but homogeneous layer to cloak the entire plane, as shown in Fig. 2(c). In this transformation, the virtual electromagnetic space between $z' = 0$ and $z' = d$ in Fig. 2(a) is squeezed into the physical space between $z = \delta d$ and $z =$

FIG. 2 (color online).   Left column: Ray tracing diagrams of a ray with incident angle of θ_i. Right column: A Gaussian beam with incident angle of θ_i. (a),(b) A PEC ground plane in free space with no cloak. (c),(d) A perfect anisotropic cloak squeezing space between $z = 0$ and $z = d$ into space between $z = \delta d$ and $z = d$. (e),(f) Replace the anisotropic cloak with isotropic medium. In the illustration, $\delta = 0.5$ and $\theta_i = 45°$.

d in Fig. 2(c) ($\delta = 0.5$ in the illustration). The effect of such a cloak is to have the ground plane appear to be at the original plane at $z = 0$ while in reality it has been moved to the plane at $z = \delta d$ so that there is room to hide objects between planes at $z = 0$ and $z = \delta d$. The transformation used to generate this cloak is a one-dimensional compression: $x = x'$, $y = y'$, and $z = (1 - \delta)z' + \delta d$, where the unprimed variables are the physical space and the primed variables are the space before transformation. The resulting relative permittivity and permeability tensors are $\bar{\bar{\epsilon}} = \bar{\bar{\mu}} = \hat{x}\hat{x}\frac{1}{1-\delta} + \hat{y}\hat{y}\frac{1}{1-\delta} + \hat{z}\hat{z}(1 - \delta)$. Note that we can also treat it as an extreme case of the previous ground-plane cloak model where a curved bump is changed to be a flat one here [5].

We will first discuss how the cloak in Fig. 2(c) works. For the cloak to be perfect, we require the wave vectors $k_z = k_z'/(1 - \delta)$ and $k_x = k_x'$, such that the total phase after a round-trip propagation in the cloak layer is the same as when the cloak is not there, i.e., as in Fig. 2(a). Because of increased k_z after the wave enters the cloak layer, the wave vector (as opposed to the ray) is refracted as if the wave has entered into an optically denser medium with a smaller refracted angle. Second, we also require the ray to come out at the same point C in both Figs. 2(c) and 2(a), and therefore the Poynting vector, or the ray, should follow the path A-B-C in Fig. 2(c). If we take the view of energy or ray propagation, the light seems to be refracted into an optically less dense medium with a larger refracted angle.

This discrepancy between the refracted angles required, respectively, by the phase and energy propagation is solved by the anisotropy of the cloak. In the k-surface diagram (Fig. 3), the incident medium, free space, gives a circle of radius ω/c and the cloak medium, a negative uniaxial medium, gives an ellipse with the major axis along the $-k_z$ axis. The phase matching condition is represented by the line at $k_x = k_{ix}$ and determines the transmitted wave vector k_t. Note that k_t undergoes a refraction as if the medium is optically denser, but since the medium is anisotropic, the normal of the k surface determines the direction of power propagation, which is denoted by the arrow, $\langle S_t \rangle$. Thus, the transmitted ray is refracted at an angle $\theta_t > \theta_i$, while maintaining the phase in the z direction, just as desired. Therefore, the original reflected beam in the region of interest is recovered, producing a perfect cloak.

We can use the exact solution of Maxwell's equations for an incident Gaussian beam [11] to illustrate this point. Figures 2(b) and 2(d) show a Gaussian beam incident at 45° to the PEC ground plane and the same beam incident onto the cloak, respectively, corresponding to the ray tracing diagrams in Figs. 2(a) and 2(c). The instantaneous E field is plotted. The parameters of the Gaussian beam are so chosen that the phase and energy propagation can be clearly seen. It can be seen that the phase and energy propagation directions are refracted at different angles in
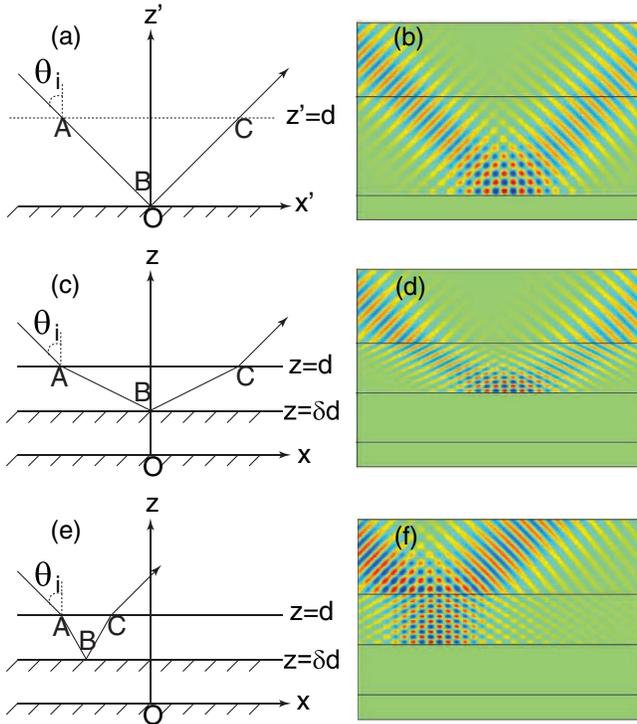
Fig. 2(d), resulting in the fields being identical to Fig. 2(b) above $z = d$.

It is interesting to investigate what will happen if we use an isotropic medium instead. The desired anisotropic medium has two principle refractive indices n_L and n_T [5], where in our case $n_L = 1$ and $n_T = 1/(1 - \delta)$. The refractive index of the candidate isotropic medium is chosen to be the geometrical average $n = \sqrt{n_L n_T}$ [5] which means an infinitesimal area should be preserved in the approximate implementation. Therefore, the radius of the circular k surface of this isotropic medium is between the shorter and longer axes of the elliptical k surface of the anisotropic medium, as the dotted circle shows in Fig. 3. With the same incident beam, the phase-matching condition determines the refracted $\tilde{k}_t$. Because of isotropy, the transmitted Poynting power $\langle \tilde{S}_t \rangle$ is in the same direction as $\tilde{k}_t$. It can be seen that $\tilde{k}_t$ is very close to k_t, meaning that the emitting ray will possess a phase close to the case of a perfect cloak, while $\langle \tilde{S}_t \rangle$ deviates from $\langle S_t \rangle$ with a large angle $\Delta \theta_t$, meaning that the emitting ray will be laterally shifted by a perceptible amount. Figures 2(e) and 2(f) clearly show the ray-tracing diagram and the Gaussian beam incident onto this isotropic medium. It can be seen that although the emitting ray as a whole has been shifted by a large amount, the phase of this emitting ray does not change much.

We have thus demonstrated the fundamental role of anisotropy using a relatively large anisotropy of 2. We now examine the effects of small anisotropy. In the case of Fig. 2(c), if we let δ be small, the anisotropy α will be small. In this case the anisotropy α is homogeneous and to further reduce α at one location will cause an increase of α at another location. In other words, the anisotropy is already minimal. According to Ref. [5], if α is very small, we can drop it and replace the anisotropic material by isotropic material with refractive index $n = \sqrt{n_L n_T}$. For a ray incident on the perfect cloak with incident angle θ_i, the distance between the incident point and the emitting point, i.e., the segment AC in Fig. 2(c) is expected to be $s = 2d \tan\theta_i$, since it is equivalent to a virtual space with height of d. However, in the isotropic case, this distance is $\tilde{s} = 2(1 - \delta)d \frac{\sin\theta_i/n}{\sqrt{1 - \sin^2\theta_i/n^2}} < 2d(1 - \delta)\tan\theta_i$. Therefore the reflected ray is laterally shifted toward the incident point by $s - \tilde{s} > 2\delta d \tan\theta_i = O(\delta d)$, which is proportional to the height δd of the object to be hidden, even if the thickness of the cloak d is greatly increased.

We now consider a curved bump similar to the case in Ref. [5], as shown in Fig. 4. The cloak is created from a quasiconformal mapping, which omits an anisotropy ratio of 1.042 [5]. The ray tracing exercise in Fig. 4 shows that there is indeed a lateral shift of the reflected ray comparable to the maximum height of the bump. Note that the incident point of the ray on the surface of the bump is not on the exact peak of the bump, but a little bit in front of the peak. This is also caused by the omission of anisotropy.
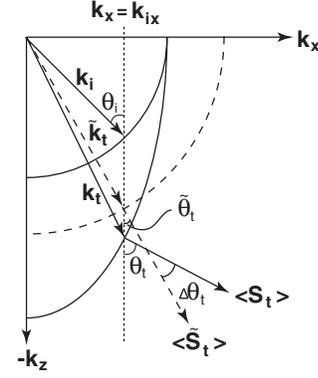

FIG. 3. k vectors used in phase matching between the incident and cloak media. The solid circle represents the incident medium and the solid ellipse represents the cloak medium. The dotted circle represents the isotropic medium that replaces the cloak medium.

However, if we examine the phase of the ray, due to a similar reason of two close wave vectors k_t and $\tilde{k}_t$ as shown in Fig. 3 (in the present case these two vectors are much closer due to smaller anisotropy), the phase propagation can be preserved to a large extent, whose distortion is difficult to perceive.

Now we discuss the origin of this shift in the quasiconformal mapping-based isotropic ground-plane cloak. In Fig. 5, we first show the procedure of such quasiconformal mapping. The virtual free space that appears as a rectangle with dimensions $1 \times m$ [Fig. 5(a)] needs to be compressed vertically to fit the rectangle of dimensions $1 \times M$ in the intermediate space [Fig. 5(b)], where M is the conformal module of the profile of the cloak with one curved boundary [5] [Fig. 5(c)]. From the intermediate space to the cloak is a typical conformal mapping with slipping boundary conditions. The anisotropy of $\alpha = m/M$ in the anisotropic cloak [5] is inherited from the homogeneous anisotropy of
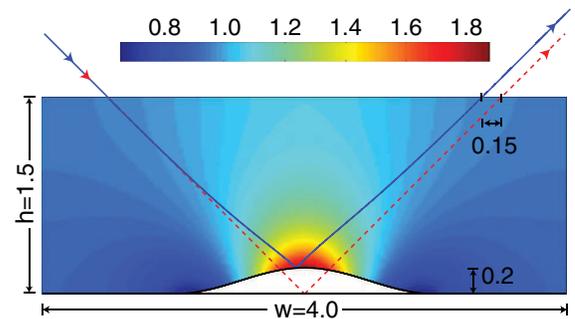

FIG. 4 (color online). A ray (blue solid line) incident at 45° to the normal onto an isotropic ground-plane cloak. The bump has a maximum height of 0.2 units [5]. The red dotted line represents the trajectory of an undistorted ray in the absence of the cloak and the bump. The background represents n^2. Notice the distorted ray has been shifted on top of the cloak toward the incident point by 0.15 units.
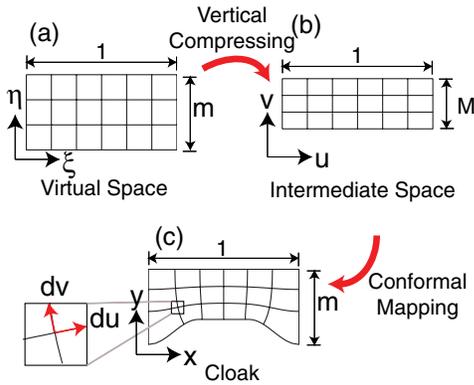
FIG. 5 (color online). Procedure of quasiconformal mapping. (a) A rectangle of $1 \times m$ in virtual free space. (b) Compressed rectangle of $1 \times M$ in the intermediate space. (c) An anisotropic cloak created from quasiconformal mapping.

$\alpha = m/M$ in the intermediate space in Fig. 5(b) that is similar to the case we discussed in Fig. 2(c). The assumption of the quasiconformally mapped isotropic cloak is that the minimized anisotropy can be dropped by letting $\alpha = 1$ and taking the refractive index $n = \sqrt{n_L n_T}$. We now study the consequence of this operation and consider the inverse procedure of Fig. 5. As shown in Fig. 5(c), every mesh point of the anisotropic cloak has two orthogonal directions, corresponding to principal directions in the compressed rectangle [Fig. 5(b)]. In the ground-plane cloak the anisotropy is $\frac{m}{M}$, because in Fig. 5(c), the metric in dv direction $\sqrt{g_{vv}}$ and the metric in du direction $\sqrt{g_{uu}}$ satisfy the relation $\sqrt{g_{vv}} = \frac{m}{M}\sqrt{g_{uu}}$. When we force the two different principle refractive indexes n_L and n_T to be $n = \sqrt{n_L n_T}$, it is equivalent to decreasing the metric along dv direction and increasing the metric along du direction by the same factor of $\sqrt{\alpha}$. With the inverse conformal mapping, in the intermediate space, the metric along du direction is increased from 1 to $\sqrt{\alpha}$, while the metric along dv direction is decreased from α to $\sqrt{\alpha}$, meaning now the intermediate space is an isotropic space. What happens in the virtual free space is that its height becomes thinner and its width becomes wider. It should be pointed out that due to the slipping boundary condition in the quasiconformal mapping, the interface of the top of the cloak is not perfectly matched to the free space above the cloak. Therefore there is some additional distortion caused by this mismatch. However, for the current case we neglect this distortion.

Now we consider the virtual space of the example in Fig. 4. Because of the smaller height, the distance between the impinging point and the emitting point of the incident and reflected rays in the virtual space (similar to AC in Fig. 2) is $2(h/\sqrt{\alpha})\tan 45°$. In order to fit the wider width into the physical space, we need to squeeze this segment AC by $\sqrt{\alpha}$ to $2(h/\alpha)\tan 45°$. We also need to consider that due to the slipping boundary condition, the compression of mesh in the middle is denser that other places. So the squeezed segment AC will be a little shorter and the final lateral shift caused by dropping anisotropy is larger than $2h\tan 45° - 2(h/\alpha)\tan 45° \approx 0.12$ units. In our ray tracing exercise, the final lateral shift is about 0.15 units. Note that this shift at 45° incidence can be reduced by increasing the width of the cloak. However, if the cloak is very wide, the incident wave with the largest incident angle can have a very long trajectory inside the cloak, which will accumulate a perceptible lateral shift.

In conclusion, the invisibility performance of the quasiconformal-mapped isotropic ground-plane cloak is examined. A lateral shift of scattered waves, being comparable to the height of the hidden object, will result when anisotropy is neglected. This is because the corresponding virtual space is thinner and wider than the ideal case. Ray tracing and theoretical analysis confirm our point on a concrete example.

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