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Iterative nonlinear beam propagation using Hamiltonian ray tracing and Wigner distribution function

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We present an iterative method for simulating beam propagation in nonlinear media using Hamiltonian ray tracing. The Wigner distribution function of the input beam is computed at the entrance plane and is used as the initial condition for solving the Hamiltonian equations. Examples are given for the study of periodic self-focusing, spatial solitons, and Gaussian-Schell model in Kerr-effect media. Simulation results show good agreement with the split-step beam propagation method. The main advantage of ray tracing, even in the nonlinear case, is that ray diagrams are intuitive and easy to interpret in terms of traditional optical engineering terms, such as aberrations, ray-intercept plots, etc. © 2010 Optical Society of America

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Investigation of beam propagation in nonlinear media is a fast-developing research area with great potential applications in optical-fiber communication systems [1], integrated optical systems [2], nonlinear photonic crystals [3,4], and so on. The conventional treatment of this phenomenon is the split-step beam-propagation method (BPM) [1,5,6], a stepwise solution of the nonlinear Schrödinger equation in the coherent regime. With the discovery of incoherent solitons, methods in the partially coherent regime, such as mutual coherence function approach [7], self-consistent multimode theory [8] and coherent density method [9], have been developed. It has been shown that these methods are equivalent [10]. These methods yield excellent accuracy and are computationally efficient.

Ray diagrams, as opposed to fields, are physically intuitive and may give more insights into the radiance evolution in nonlinear media. The Wigner distribution function has been proposed to calculate the ray diagrams, and a set of differential equations governing the transport of radiance was derived [11,12]. However, because of the coupling between the optical intensity and the refractive index, solving these equations is not straightforward.

In this Letter, we present an iterative method for analyzing beam propagation in nonlinear media based on Hamiltonian ray tracing [13]. Our method provides nonlinear beam evolution as a ray trace, which is physically more intuitive. It is better than traditional geometrical ray tracing, because wave effects are taken into account by using the Wigner distribution function (WDF) [14–17] of the input field to initialize position and momentum of the rays. The coherence properties of the beam can also be easily included by the WDF [18]. Nonlinear effects are included iteratively by dynamically modifying the refractive index distribution at every iteration. Results of this method match well with those produced by the BPM or multimode BPM [19]. Also,

our method is applicable to structures such as nonlinear photonic crystals and metamaterials [20]. It has been shown that the Hamiltonian method can be applied to photonic crystals [21,22]; thus the extension of our method to such complex media should be straightforward.

Throughout this Letter, without loss of generality, bulk media with Kerr effect nonlinearity [23] are used to demonstrate the method. We use the common phenomenological Kerr index dependence on the optical intensity as $n = n_0 + n_2 I$, where n_0 is the linear component of the refractive index, n_2 is the Kerr coefficient and I is the optical intensity.

The iterative method is shown in the block diagram of Fig. 1. Each iteration consists of the following three steps: (1) define the initial condition for each ray emanating from the input plane; the value of the radiance of each ray being computed from the WDF of the input field; (2) solve the coupled ordinary differential Hamiltonian equations for each ray; (3) generate the intensity distribution from the ray distribution as a projection of its WDF along the momentum direction. The resulting intensity distribution is used to update the nonlinear refractive index, which is then used to solve the Hamiltonian equations in the next iteration. The convergence condition is the sum of absolute values of the refractive index differences between two consecutive iterations below a certain threshold.

In the first step, the initial ray input from the source with certain position and momentum is defined. To account for the wave effect, each position should have multiple rays emanating with different momentum, as suggested by Huygens' principle. The WDF [16,17] defines a generalized radiance function, which is a function of both position and momentum. Generalized radiance, as compared with classical radiance, includes wave effects into ray tracing [17]. It is also shown that WDF is conserved along the ray paths, and its transport equation is equivalent to the Hamiltonian equations [12,16].

Fig. 1. (Color online) Block diagram of the iterative method for nonlinear beam propagation.

To illustrate this point, Fig. 2(a) shows the WDF of a Gaussian source computed at the initial plane. In Fig. 2(b), the familiar Gaussian intensity profile along the propagation direction was generated by solving the Hamiltonian equations for each ray with the WDF in Fig. 2(a) as the initial conditions.

Our ray tracing consists of solving the three-dimensional (3D) Hamiltonian equation [13]

$$\frac{d\mathbf{q}}{d\sigma} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|}, \quad \frac{d\mathbf{p}}{d\sigma} = -\frac{\partial H}{\partial \mathbf{q}} = \frac{\partial n}{\partial \mathbf{q}}, \quad (1)$$

where $\mathbf{q}$ and $\mathbf{p}$ are position and momentum along the path, $H = |\mathbf{p}| - n$ is the 3D Hamiltonian, n is the refractive index, and σ parameterizes the ray trajectories. In this equation, for homogeneous nonlinear medium, $\partial n / \partial \mathbf{q}$ results from the nonlinearity [12]. Hamiltonian ray tracing is convenient to use in many cases, not only because it is computationally efficient as compared to the finite-difference time-domain (FDTD) method but also because it can be applied to complex media such as photonic crystals [21,22].

To validate this method, we applied it to study well-known self-focusing phenomena [24,25]. In a bulk Kerr-effect medium with Gaussian beam input, multiple foci occur in periodically spaced locations along the op-

Fig. 2. (Color online) (a) WDF of the Gaussian beam at the input plane in 1D and (b) the Gaussian beam generated from rays with initial condition specified by the WDF. Here x is position and p is momentum. Numbers are scaled according to the waist w_0 . Wavelength is $\lambda = w_0/20$. Nine points on the WDF plane [white dots in (a)] correspond to nine rays [arrows in (b)]. Three different ray directions are shown for each of the three positions (A–C). The lengths of arrows are proportional to the generalized radiance of the WDF.

Fig. 3. (Color online) Periodic self-focusing of Gaussian beam in Kerr medium produced by (a) the iterative method and (b) the BPM. Periodic focusing of two Gaussian beams in a Kerr medium produced by (c) the iterative method and (d) the BPM. The white curves are sampled from the set of all 10,100 rays used in the simulation. The color shading indicates intensity, computed as projection from the WDF. (see text).

tical axis owing to the interaction between the diffraction and the nonlinearity [23].

As the first example, periodic self-focusing of a Gaussian beam in a weak Kerr-effect medium is investigated. In this example, the original refractive index of medium is $n_0 = 1.5$ and Kerr-effect coefficient is $n_2 = 2 \times 10^{-13} \text{ (m/V)}^2$. The input Gaussian beam has a waist of 2 mm and a peak amplitude of 250 V/m, equivalent to intensity of 12.4 mW/cm^2 . The propagation length is 100 m. After 11 iterations, the intensity distribution estimates converge, and the results are shown in Fig. 3(a).

Fig. 4. (Color online) (a) Spatial soliton in a Kerr medium produced by the proposed iterative method and a subset of all 10,100 rays used in the simulation. (b) Comparison of intensity distribution along transverse direction between the results from the iterative method and the analytical solution. (c) Intensity difference in percentage of spatial soliton along the transverse direction between the iterative method and the analytical solution.

Fig. 5. (Color online) (a) Gaussian-Schell model beam propagation in a weak Kerr-effect medium. (b) FWHM at different Δn for a fixed propagation length.

For comparison, the result produced by BPM is shown in Fig. 3(b). Computation time for the iterative method is 2.1 min, while for BPM it is 33 s.

In the second case, two parallel Gaussian beams are incident on the Kerr-effect medium. The converged result after 19 iterations is shown in Fig. 3(c). In the simulation, the same medium is used, and each beam has a waist of 1 mm and a peak amplitude 282 V/m, equivalent to intensity of 15.8 mW/cm². The distance between the two waist centers is 1.6 mm and propagation distance is 80 m. Periodic focusing and divergence of the two beams are observed. The computation times for the iterative method and the BPM are 3.9 min and 43 s, respectively.

When self-focusing balances diffraction, spatial solitons can be generated. The soliton solution to the nonlinear Schrödinger equation at the input source plane is given by $A(x) = A \text{sech}(x/w_0)$ where w_0 is the beam width and A is the amplitude [26]. In this third example, a spatial soliton with $w_0 = 0.55$ mm and a peak amplitude $A = 281$ V/m, equivalent to intensity of 15.7 mW/cm², is investigated with the material parameters the same as in the previous example. The estimates of the intensity profile converge after 16 iterations, and result is shown in Fig. 4(a). In Figs. 4(b) and 4(c), the computed intensity profile is compared with the analytical result, which shows good agreement of the two results. Simulation time for iterative method is 3.1 min. A subset of rays used in calculation is shown in Fig. 4(a) to give a ray picture of the soliton. Interestingly, each ray follows an oscillating trajectory with a different period.

As the final example, the Gaussian-Schell model [19] as partially coherent illumination is examined. Figure 5(a) shows the beam propagation estimation of the Gaussian-Schell model in a weak Kerr-effect medium after five iterations. The input beam width is $a_0 = 25 \mu\text{m}$, and the correlation length is $l_c = 1.5 \mu\text{m}$. The beam width evolution is consistent with Eq. 4 of [19]. The FWHM change in the case of $l_c = 1 \mu\text{m}$ and $2 \mu\text{m}$ is illustrated in Fig. 5(b). The results show good agreement with Fig. 1 of [19].

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