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MULTIPLE EXTENSIONS OF GENERALIZED HEXAGONS RELATED TO THE SIMPLE GROUPS McL AND Co_3

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ABSTRACT

The groups McL, Co_3 and $2 \times Co_3$ are all contained in the automorphism group of a 2-, respectively, 3-fold extension of a generalized hexagon of order (4, 1). We give a geometric characterization of these multiple extensions of this generalized hexagon.

1. *Introduction*

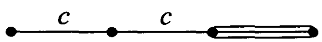
Several of the finite sporadic simple groups act as automorphism groups on extensions of generalized polygons. The large Mathieu groups act on extensions of the projective plane of order 4, the groups McL, Co_3 and HS are automorphism groups of extended generalized quadrangles, Ru and He are automorphism groups of extended octagons, and finally the Hall-Janko group HJ and the Suzuki group Suz, are known to act on extensions of generalized hexagons of order (2, 2), respectively, (4, 4). In general, extensions of generalized hexagons and octagons have infinite covers, see for example [**8**]. Thus it seems natural to impose extra conditions on extensions Γ of generalized hexagons or octagons to obtain characterizations of these geometries. The two extended generalized hexagons mentioned above satisfy the following condition:

(*) $\{x_1, x_2, x_3\}$ is a clique of the point graph not contained in a circle if and only if x_2 and x_3 are at distance 3 in the local generalized hexagon Γ_{x_1}.

It is just this condition that was imposed on extensions of generalized hexagons in [**5**, Theorem 1.1]. There the above mentioned extended generalized hexagons related to the sporadic groups HJ and Suz, as well as two other extensions of a generalized hexagon of order (2, 1), respectively, (4, 1), related to the groups $G_2(2)$, respectively, $PSU_4(3)$, were characterized. In particular, it was shown in [**5**], that these four extended generalized hexagons are the only extensions of finite, regular, line thick, generalized hexagons satisfying (*).

In this paper we are concerned with (possibly multiple) extensions of the four extended generalized hexagons related to the groups $G_2(2)$, HJ, $PSU_4(3)$ and Suz mentioned above. The interest in such geometries comes from the existence of two examples related to the sporadic simple groups McL, respectively, Co_3, that are 1-, respectively, 2-fold extensions of the extended generalized hexagon on 162 points related to $PSU_4(3)$. Their point graphs are the complement of the McLaughlin graph on 275 points, with automorphism group McL:2, and a graph on 552 points which is locally the complement of the McLaughlin graph and has as automorphism group the group $2 \times \mathrm{Co}_3$. This last graph is one of the two Taylor graphs for the third Conway group; it is the one with intersection array $\{275, 112, 1; 1, 112, 275\}$, see [**1**]. These graphs contain a unique class of 7-, respectively, 8-cliques inducing the multiple extended hexagon, which will be called the *multiple extended hexagon related to the group* McL, respectively, $2 \times \mathrm{Co}_3$.

The second example related to $2 \times \mathrm{Co}_3$ admits a quotient geometry on 276 points, which is a one point extension of the McL multiple extended hexagon, and has automorphism group Co_3. Obviously these multiple extended hexagons also carry the structure of a Buekenhout geometry with diagram



for the group McL, respectively,

for the groups $2 \times \mathrm{Co}_3$ and Co_3.

The groups McL:2, Co_3 and $2 \times \mathrm{Co}_3$, respectively, act flag-transitively on these Buekenhout geometries. As such, these groups and geometries were characterized by Weiss in [**10**], under some extra geometric condition closely related to (∗). In this paper we give a purely geometric characterization of these multiple extensions of the generalized hexagon of order $(4, 1)$.

THEOREM 1.1. *Let Γ be an r-fold extension of a finite, regular, line thick generalized hexagon, $r \geqslant 2$, in which the residue of any set of $r-1$ cocircular points is an extended hexagon satisfying* (∗). *Then Γ is isomorphic to the* McL *2-fold extended hexagon, or to one of the two 3-fold extended hexagons related to* Co_3, *respectively,* $2 \times \mathrm{Co}_3$.

We notice that the McLaughlin graph and the Taylor graph related to $2 \times \mathrm{Co}_3$ with intersection array $\{275, 162, 1; 1, 162, 275\}$ are extensions of the unique generalized quadrangle of order $(3, 9)$. These geometries are characterized in [**7**]. The point graph of the 3-fold extended generalized hexagon related to Co_3 is a regular 2-graph, as such it is characterized by Goethals and Seidel [**6**].

2. *Definitions and notation*

In this paper we use notation and definitions of [**3, 5**]. For convenience of the reader we recall some of these definitions and fix the notation.

Let Γ be an incidence structure $(\mathscr{P}, \mathscr{C})$ consisting of nonempty sets $\mathscr{P}$ of *points* and $\mathscr{C}$ of *circles*, where a circle is a subset of $\mathscr{P}$ of size at least 2. Then the *point graph* of Γ is the graph with vertex set $\mathscr{P}$ and as edges the pairs of distinct points that are *cocircular* (that is, are in some circle). That two points p and q are cocircular is denoted by $p \perp q$. For each subset X of $\mathscr{P}$, the set $X^{\perp}$ consists of all points q cocircular with all the points of X. We usually write $p^{\perp}$ instead of $\{p\}^{\perp}$ for a point p in $\mathscr{P}$.

Fix a point p of Γ. If all circles on p contain at least 3 points, then the residue $\Gamma_p := (\mathscr{P}_p, \mathscr{C}_p)$ of Γ at p, where $\mathscr{P}_p$ consists of all points of $\mathscr{P}$ distinct but cocircular with p in Γ, and $\mathscr{C}_p = \{C - \{p\} \mid C \in \mathscr{C}, p \in C\}$ is also an incidence structure. For any graph $\mathscr{G}$ and point p of the graph, the induced subgraph on the neighbours of p will be denoted by $\mathscr{G}_p$.

If $\mathscr{F}$, respectively, Δ, is a family of incidence structures, respectively, just an incidence structure, then Γ is called a (1-fold) *extension of* $\mathscr{F}$, respectively, Δ, if and only if all circles of Γ have at least 3 points, its point graph is connected and Γ_p is in $\mathscr{F}$, respectively, is isomorphic to Δ, for all points p of Γ. Inductively, for each integer $r > 1$, we define Γ to be an *r-fold extension of* $\mathscr{F}$ or Δ, if and only if all circles contain at least $r+2$ points, its point graph is connected, and Γ_p is an $(r-1)$-fold extension of $\mathscr{F}$, respectively, Δ for all points p of Γ. An *extended* (*generalized*) *hexagon* is an extension of a generalized hexagon.

In a generalized hexagon circles are usually referred to as *lines*.

An extension is called *triangular* if and only if any triangle of the point graph is contained in a circle.

3. *The triangular* 2-*fold extended hexagon related to* McL

Let $\Gamma = (\mathscr{P}, \mathscr{C})$ be a triangular 2-fold extension of a generalized hexagon satisfying the hypothesis of Theorem 1.1. For all points p of Γ, the residue Γ_p is an extended generalized hexagon satisfying (∗). In particular, the results of [5] apply and we find that Γ_p is isomorphic to one of the four extended generalized hexagons related to $G_2(2)$, HJ, $PSU_4(3)$ or Suz.

LEMMA 3.1. *For all points p and q of Γ we have $\Gamma_p \simeq \Gamma_q$.*

Proof. Suppose that p and q are adjacent points of the point graph of Γ. Then $(\Gamma_p)_q \simeq (\Gamma_q)_p$. But, by the results of [5], that implies that $\Gamma_p \simeq \Gamma_q$. Since the point graph of Γ is connected, we have proved the lemma.

Fix a point p of Γ. By Δ we denote the residue Γ_p at p. The point graph of Γ will be denoted by $\mathscr{G}$, that of Δ by $\mathscr{D}$. The above lemma implies that $\mathscr{G}$ is locally isomorphic to $\mathscr{D}$. Suppose that (p, q, r) is a path of length 2 in the point graph $\mathscr{G}$ of Γ. Then let $H = H_q$ be the complement of $p^\perp \cap q^\perp \cap r^\perp$ in $p^\perp \cap q^\perp - \{p, q\}$. The set H is a subset of the point set of the generalized hexagon $\Delta_q = \Gamma_{p,q}$. It plays an important role in [5] and will also be very useful in the situation considered here. The set H is the complement of a μ-graph in the point graph of Γ_q. The following properties of H will be useful later in this section.

LEMMA 3.2. *Suppose that Γ is an extension of a generalized hexagon of order (s, t). Then we have the following:*

(i) *each point of Δ_q not in H is on a unique line of the generalized hexagon Δ_q meeting H in $s-1$ points;*

(ii) *H contains $(s^2-1)(t^2+t+1)$ points;*

(iii) *each line of Δ_q is in H, or disjoint from H, or meets H in $s-1$ points.*

Proof. Inside Γ_q which is one of the four extended generalized hexagons the points p and r are at distance 2. Thus we can apply the results of [5, Section 3], in particular Lemma 3.4.

The possible embeddings of sets H in Δ_q or $\mathscr{D}_q$ having the properties (i) to (iii) of Lemma 3.2 are classified in [5]. To state that result we first have to fix some notation. Suppose that Δ is the $PSU_4(3)$ extended hexagon. Then the generalized hexagon Δ_q

of order (4, 1) will be identified with the generalized hexagon on the flags of PG(2, 4). A *hyperoval* of PG(2, 4) is a set of 6 points no three collinear, its dual is the set of 6 lines missing all the 6 points. This dual hyperoval is a hyperoval of the dual plane, that is, no three of the lines meet in a point.

LEMMA 3.3. *Let H be a set of points in Δ_q having the properties* (i) *to* (iii) *of Lemma* 3.2.

(i) *If Δ_q has order* (2, 1), (2, 2) *or* (4, 4), *then H is the complement of a μ-graph $q^{\perp} \cap x^{\perp}$ for some point x at distance two from q in the point graph $\mathscr{D}$ of Δ.*

(ii) *If Δ_q has order* (4, 1) *then H consists of the flags of* PG(2, 4) *missing a hyperoval and its dual hyperoval.*

Proof. This is a straightforward consequence of the results of [**5**, Section 4].

In the graph $\mathscr{G}$ the μ-graph $M = p^{\perp} \cap r^{\perp}$ is a subgraph of the graph $\mathscr{D}$, which is locally isomorphic to the complement of H inside the point graph of the local generalized hexagon of Δ.

PROPOSITION 3.4. *We have that Δ is isomorphic to the* $PSU_4(3)$ *extended hexagon.*

Proof. Suppose that Δ is not the $PSU_4(3)$ extended hexagon. Consider the subgraph M of $\mathscr{D}$. For each point q of M the vertex set of the local graph M_q is the complement of a set H in Δ_q satisfying (i) to (iii) of Lemma 3.2.

Fix a point q of M and inside M_q a point x. By Lemma 3.3 there is a point $y \in \Delta_q$ at distance 2 from x such that M_x is the induced subgraph on the set of common neighbours of x and y inside $\mathscr{D}$.

Inside the generalized hexagon Δ_q, we find that the points x and y have mutual distance 2, so that there is a unique point z collinear with both x and y. By the choice of y, the points x and z are the unique points on the line through x and z inside M_q. Thus, by 3.2(i), this line is the unique line on z containing some point outside M. In particular, $y \in M_q$.

Repeating the above argument with the role of x and y interchanged, we see that the line of Δ_q on y and z is the unique line on z containing a point outside M. A contradiction, and the proposition is proved.

Thus from now on we can assume that Γ is locally the extended generalized hexagon on 162 points related to the group $PSU_4(3)$. The generalized hexagon of order (4, 1) is isomorphic to the generalized hexagon on the 105 flags of the projective plane PG(2, 4). We shall identify $\Gamma_{p,q} = \Delta_q$ with this hexagon. By Lemma 3.3 we know that the set H consists of all the flags missing an hyperoval and its dual in PG(2, 4). The following can be obtained easily, especially using the information from the Atlas [**4**].

The projective plane PG(2, 4) admits 168 hyperovals, the group $P\Gamma L_3(4)$ being transitive on them. Thus, there are 168 subgraphs in $\mathscr{D}_q$ isomorphic to H. Of these subgraphs 56 are the complement of a μ-graph of $\mathscr{D}$. The group $PSL_3(4)$ has 3 orbits on the hyperovals, all of length 56. Two hyperovals are in the same orbit if and only if they meet in an even number of points. The stabilizer of a hyperoval O in $PSL_3(4)$ is isomorphic to A_6. It has two more orbits on the $PSL_3(4)$-orbit of O, one of length 45, consisting of all hyperovals meeting O in 2 points, and one of length 10 consisting of the hyperovals disjoint from O. This stabilizer has two orbits of length 36 and 20 on the two other $\mathrm{PSL}_3(4)$-orbits on the hyperovals, consisting of all hyperovals of that orbit meeting O in, respectively, 1 or 3 points of $\mathrm{PG}(2,4)$. The stabilizer of one of the orbits in $\mathrm{P\Gamma L}_3(4)$ is isomorphic to $\mathrm{P\Sigma L}_3(4)$, and permutes the two other orbits.

We shall use the following description of $\mathscr{D}$ (see [5]). First we fix the point q. The neighbours of q are the 105 flags of $\mathrm{PG}(2,4)$, where two flags are adjacent if and only if they are at distance 1 or 3 in the generalized hexagon on these flags. Now fix one of the 3 orbits of $\mathrm{PSL}_3(4)$ of length 56 on the hyperovals, say $\mathscr{O}$. The vertices at distance 2 from q are the elements of $\mathscr{O}$. Two hyperovals are adjacent if and only if they meet in 2 points. A flag and a hyperoval are adjacent if and only if the point (respectively, line) of the flag lies in the hyperoval (respectively, dual hyperoval). Fix a hyperoval O not in $\mathscr{O}$, and consider the subgraph $\mathscr{M}$ of $\mathscr{D}$ consisting of q, all flags on O or its dual and all elements of $\mathscr{O}$ that meet O in 3 points of the plane. Then $\mathscr{M}$ consists of $1+60+20=81$ points. This graph is the μ-graph appearing in the complement of the McLaughlin graph and thus locally the complement of H in the distance 1-or-3 graph of the generalized hexagon of order $(4,1)$. It is a strongly regular graph with parameters (v,k,λ,μ) equal to $(81,60,45,42)$. The complement of $\mathscr{M}$ in $\mathscr{D}$ is also isomorphic to $\mathscr{M}$. There are 112 elements in the $\mathrm{PSU}_4(3)$ orbit of $\mathscr{M}$, the complement of $\mathscr{M}$ not being one of them. Any subgraph of this orbit meets $\mathscr{M}$ in 81, 45 or 27 points, and hence meets the complement of $\mathscr{M}$ in 0, 36 or 54 points. (This can be checked within the McLaughlin graph.)

We shall show that in fact all subgraphs of $\mathscr{D}$ that are locally the complement of H in the distance 1-or-3 graph of the generalized hexagon of order $(4,1)$ are in the $\mathrm{PSU}_4(3)$ orbit of $\mathscr{M}$ or its complement in $\mathscr{D}$.

LEMMA 3.5. *The graph M is in the* $\mathrm{PSU}_4(3)$ *orbit of $\mathscr{M}$ or of its complement in $\mathscr{D}$.*

Proof. Fix the point q, and consider M_q and its complement in $\mathscr{D}_q$, the set H_q. The graph M_q is isomorphic to the graph whose vertex set consists of the 60 flags of $\mathrm{PG}(2,4)$ meeting a hyperoval O_q or its dual. Two such flags are adjacent if and only if they are at distance 1 or 3 in the generalized hexagon of order $(4,1)$ defined on the flags of $\mathrm{PG}(2,4)$. In particular, one can easily check that inside M_q two nonadjacent vertices have at least 32 common neighbours. Thus inside M the μ-graphs consist of at least 33 vertices. By the above, M_q is either a μ-graph in $\mathscr{D}$, or it consists of the 60 flags of $\mathrm{PG}(2,4)$ meeting one of the 112 hyperovals not in $\mathscr{O}$.

The same arguments as used in the proof of Proposition 3.4 rule out the case where M_q is the μ-graph for some point r in $\mathscr{D}$ at distance 2 from q, that is, $M_q = r^\perp \cap q^\perp$ in $\mathscr{D}$.

Thus assume that M_q is not the μ-graph of some point r at distance 2 from q, that is, $O_q \notin \mathscr{O}$. There are 36 hyperovals in $\mathscr{O}$ meeting the hyperoval O_q in a unique point of $\mathrm{PG}(2,4)$. Thus each of these 36 points of $\mathscr{D}$ is adjacent to 30 points in H_q and 30 points in M_q. Since μ-graphs in M contain at least 32 points, none of the 36 points is in M.

The remaining 20 points at distance 2 from q are hyperovals of $\mathscr{O}$ meeting the hyperoval O_q in 3 points of $\mathrm{PG}(2,4)$. Hence these 20 points are adjacent to 18 points of H_q and 42 points in M_q. So μ-graphs of M contain 42 points, and as M_q has valency 45, there are at least $60.(60-45-1)/42=20$ points at distance 2 from q. Thus, all 20 points in $\mathscr{O}$ meeting O_q in 3 points of $\mathrm{PG}(2,4)$ are in M. In particular, M consists of q, M_q and the 20 points in $\mathscr{O}$ meeting O_q in 3 points of the projective plane $\mathrm{PG}(2,4)$.

Since there are 112 hyperovals not in $\mathcal{O}$, there are $162.112/81 = 224$ subgraphs in $\mathcal{D}$ that are locally isomorphic to the complement of H in the distance 1-or-3 graph of the generalized hexagon of order $(4, 1)$. The lemma follows now easily.

PROPOSITION 3.6. *If Δ is the extended generalized hexagon on* 162 *points related to* $\mathrm{PSU}_4(3)$, *then Γ is isomorphic to the 2-fold extended hexagon related to* McL.

Proof. By the above lemma we find that for any two points at distance 2 in $\mathcal{G}$ the number of common neighbours is 81. Moreover, since each point of the complement of $\mathcal{M}$ in $\mathcal{D}$ is adjacent to some point in $\mathcal{M}$, the diameter of $\mathcal{G}$ is 2. Hence $\mathcal{G}$ is a strongly regular graph with parameters $(v, k, \lambda, \mu) = (275, 162, 105, 81)$. A strongly regular graph with these parameters has been shown to be isomorphic to the complement of the McLaughlin graph by Cameron, Goethals and Seidel [2]. Here however, we quickly obtain the uniqueness of Γ in the following way, without using [2].

There are 112 vertices at distance 2 from q. We can identify each of these points with a subgraph of $\mathcal{D}$ isomorphic to $\mathcal{M}$. Without loss of generality we may identify one of these points with $\mathcal{M}$.

The subgraph of $\mathcal{G}$ induced on the 112 points not adjacent to q is locally isomorphic with the complement of $\mathcal{M}$ in $\mathcal{D}$, and thus with $\mathcal{M}$. Since its valency is larger than 112/2, this subgraph is connected.

Two adjacent vertices x and y in this graph have 60 common neighbours, and thus 45 common neighbours in $\mathcal{G}_q$. But that implies that the two subgraphs $q^\perp \cap x^\perp$ and $q^\perp \cap y^\perp$ are in the same $\mathrm{PSU}_4(3)$ orbit. In particular, by connectivity of the subgraph, the points at distance 2 from q are in one orbit under the action of $\mathrm{PSU}_4(3)$. But then it is easy to see that the graph $\mathcal{G}$ is unique up to isomorphism. Moreover, Γ is unique: as there is only one set of 5-cliques in $\mathcal{D}$ inducing an extended generalized hexagon on $\mathcal{D}$, there is only one set of 6-cliques making Γ locally Δ. The McL 2-fold extended generalized hexagon does satisfy the hypothesis, and we find that it is isomorphic to Γ.

4. *The triangular 3-fold extended hexagon related to* $2 \times \mathrm{Co}_3$

The purpose of this section is to show that the $2 \times \mathrm{Co}_3$ 3-fold extended hexagon is the only triangular extension of the McL 2-fold extended hexagon satisfying the hypothesis of Theorem 1.1.

Assume that Δ is the 2-fold extension related to McL, $\mathcal{D}$ its point graph, and Γ a triangular extension of Δ with point graph $\mathcal{G}$. Then $\mathcal{G}$ is locally $\mathcal{D}$.

Let (p, q, r) be a path of length 2 in $\mathcal{G}$, and set M to be the subgraph induced by $p^\perp \cap r^\perp$ in $\mathcal{D}$. Then the graph M is locally isomorphic to the μ-graph of the complement of the McLaughlin graph, and thus to the graph $\mathcal{M}$ of the previous section. By Lemma 3.5 we find that M_q is either a μ-graph in Δ, or the complement of a μ-graph. For each point x of M denote by x' the unique point at distance 2 from x in $\mathcal{D}$ such that M_x is either $x^\perp \cap x'^\perp$ or its complement in $\mathcal{D}_x$.

Let x be a point in M, and $y \in M_x$. Then $M_x \cap M_y$ consists of 60 points. The points of $\mathcal{D}_y - M_y$ have 45 neighbours in M_y, so that we can conclude that x' is not adjacent to y, and M_x consists of the points of $\mathcal{D}_x$ not adjacent to x'. By the same argument we find that M_y consists of all the points in $\mathcal{D}_y$ not adjacent to y'.

The 81 points at distance 1 from x', but 2 from x have 36 neighbours in M_x, and thus also some neighbour in $M_x \cap M_y$. The 30 vertices at distance 2 from both x and

x' have 54 neighbours in M_x, and therefore also neighbours in $M_x \cap M_y$. Thus the point y' has to be equal to x'. Since each point at distance 2 from both x and x' is adjacent to some $y \in M_x$ and thus is M_y, we find that M contains and hence consists of the 112 points not adjacent to x'.

Now we can count the number of points at distance 2 from r in $\mathcal{G}$. There are exactly $275.112/112 = 275$ such points.

The point q' is at distance 3 from r. Any point of $\mathcal{G}_p$ different from q' is either adjacent to q' or adjacent to r. Thus all common neighbours of q' and p are at distance 2 from r. Since the graph $\mathcal{G}_{q'}$ is connected we find that all neighbours of q' are at distance 2 from r. Moreover, this implies that q' is the unique point at distance 3 from r.

But now uniqueness of the graph $\mathcal{G}$ is obvious, and, since there is a unique way to fix a set of 6-cliques in $\mathcal{D}$ making it into a 2-fold extended hexagon, we also obtain uniqueness of Γ. We have proved the following.

PROPOSITION 4.1. *If Γ is triangular and locally the* McL *2-fold extended hexagon, then it is isomorphic to the* $2 \times \mathrm{Co}_3$ *3-fold extended hexagon.*

5. *The nontriangular 3-fold extended hexagon related to* Co_3

In this final section we give a characterization of the 3-fold extended hexagon on 276 points related to the sporadic group Co_3. But first we consider 2-fold extensions of generalized hexagons.

PROPOSITION 5.1. *Let Γ be an extension of one of the four extended hexagons related to* $\mathrm{G}_2(2)$, $\mathrm{PSU}_4(3)$, HJ, *respectively,* Suz. *Then Γ is triangular and thus isomorphic to the* McL *2-fold extended hexagon.*

Proof. Let Δ be one of the four extended hexagons satisfying $(*)$ such that Γ is an extension of Δ. The point graph of Δ is strongly regular with parameters (v, k, λ, μ), say. Let x, p and q be 3 pairwise adjacent points of Γ. We want to show that they are in a common circle.

Assume to the contrary that p and q are not adjacent in Γ_x. Let

$$\mathcal{B} = \{U \cap V - \{x\} \mid x \in U \in \mathcal{C}_p, x \in V \in \mathcal{C}_q, |U \cap V| > 2\},$$

and set $\mathcal{N} = \bigcup_{U \in \mathcal{B}} U$. Notice that $|\mathcal{N}|$ equals μ. (See also [5].)

Fix a point $y \in \mathcal{N}$. Then there is no circle through p, q and y. Assume on the contrary that W is such a circle. There are circles $U \in \mathcal{C}_p$ and $V \in \mathcal{C}_q$ with $U \cap V$ containing x and y together with some third point z. Inside Γ_y we see that p, q and x form a triangle not in a circle. Thus the distance between the points p and q in the generalized hexagon $\Gamma_{y,x}$ is 3. However, it is at most 2, since z is collinear to both p and q inside $\Gamma_{y,x}$. Thus indeed, there is no circle through p, q and y.

This implies that no point of $\mathcal{N}$ is collinear to q inside Γ_p. Thus $\mathcal{N}$ contains at most $k - \mu$ points. Hence, $\mu + 1 \leqslant k - \mu$, which is a contradiction in all the four cases.

We can conclude that Γ is triangular. By Proposition 3.6, it is isomorphic to the McL 2-fold extended hexagon.

PROPOSITION 5.2. *Suppose that* Γ *is an extension of the* McL *2-fold extended hexagon. Then* Γ *is isomorphic to the 3-fold extended hexagon on 552 points related to* $2\times \mathrm{Co}_3$, *or to its quotient on 276 points related to* Co_3.

Proof. Let Δ be the McL 2-fold extended hexagon. If Γ is a triangular extension of Δ, then, by Proposition 4.1, it is isomorphic to the $2\times \mathrm{Co}_3$ 3-fold extended hexagon on 552 points. Suppose that Γ is not triangular. Let p, q and x be a triple of pairwise adjacent points of Γ, such that p and q are not adjacent in Γ_x. Let

$$\mathscr{B} = \{U\cap V-\{x\} \mid x\in U\in \mathscr{C}_p, x\in V\in \mathscr{C}_q, |U\cap V|>3\},$$

and set $\mathscr{N} = \bigcup_{U\in\mathscr{B}} U$. Note that $\mathscr{N}$ contains 81 points.

Let $y\in\mathscr{N}$. Since Γ_y is triangular, there is no circle on p, q and y. So q and y are not adjacent inside Γ_p. The valency of the subgraph of the point graph of Γ_p induced on the vertices not adjacent to q (inside Γ_p) is 81. Moreover, this subgraph is connected. Hence all points of Γ_p not adjacent to q inside Γ_p are adjacent to q inside Γ. Since the complement of the point graph of Δ is connected, we find that any two points of Γ_p are adjacent inside Γ. But that implies that all pairs of points of Γ are adjacent, and that Γ is a one point extension of Δ.

Since the point graph of Δ is strongly regular with parameters k and μ such that $k=2\mu$, we find that Γ carries the structure of a regular 2-graph. (See [**1, 6, 9**].) Now we can finish the proof of the proposition by referring to [**6**], or by considering the universal cover obtained by a standard construction for 2-graphs, see [**1, 6, 9**]. Its point graph is the Taylor graph of the regular 2-graph. This cover is a triangular extension of Δ and, by Proposition 4.1, it is isomorphic to the $2\times\mathrm{Co}_3$ extension of Δ. This proves the proposition.

We finish the proof of Theorem 1.1 with the following proposition.

PROPOSITION 5.3. *There exists no r-fold extended hexagon with* $r\geqslant 4$ *satisfying the hypothesis of Theorem* 1.1.

Proof. Suppose that Γ is a 4-fold extended hexagon satisfying the hypothesis of Theorem 1.1. Fix 4 cocircular points x_1, x_2, x_3 and x_4, and denote by X the set $\{x_1, x_2, x_3, x_4\}$.

By Proposition 3.4 we have that Γ_X is isomorphic to the unique generalized hexagon of order $(4, 1)$. Identify the points of Γ_X with the flags of the projective plane $\mathrm{PG}(2,4)$.

Consider a point y_i in $\Gamma_{X\setminus\{x_i\}}$ which is not cocircular inside this extended hexagon with x_i. As we have seen in Lemma 3.3, we can associate to y_i a unique subset H_{y_i} of the hexagon Γ_X consisting of the flags of $\mathrm{PG}(2,4)$ missing a hyperoval and its dual. Namely, the set $\{y_i\}\cup X\setminus\{x_i\}$ is cocircular with a point $z\in\Gamma_X$ if and only if the point z is not in H_{y_i}. For fixed i there are 56 different points y_i which are all associated to distinct sets H_{y_i}.

Let $\{i,j,k,l\}=\{1,2,3,4\}$. As, by Proposition 5.1, the residue of any two cocircular points of Γ is isomorphic to the 2-fold extended hexagon related to McL, we find in the residue Γ_{x_k,x_l} that the subsets H_{y_i} and H_{y_j} are distinct. Thus, we can find at least 4.56 different subsets of the form H_{y_i} inside the hexagon Γ_X. However, as $\mathrm{PG}(2,4)$

admits only 168 hyperovals, there are precisely 168 such subsets H_{y_i}. This contradiction implies that there is no 4-fold extended hexagon and hence also no r-fold extended hexagon with $r \geqslant 4$ satisfying the hypothesis of Theorem 1.1.

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