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Extended Generalized Octagons and the Group *He*

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Abstract. Let Γ be an extended generalized octagon such that the points of a triple $\{u, v, w\}$ not on a block are pairwise adjacent if and only if the distance between v and w in the local generalized octagon Γ_u equals 3 and there is a thick line through any point of Γ_u. Then Γ is one of the two examples related to the groups $2{\cdot}L_3(4).2^2$ and *He*. It is also shown that Γ does not admit further extensions.

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1. Introduction and the Results

A number of sporadic simple groups arise as automorphism groups of extensions of classical geometries or buildings. The Mathieu groups act on (multiple) extensions of PG(2, 4) or AG(2, 3), and the sporadic Fischer groups act on extensions of the $U_6(2)$-polar space. The Suzuki chain groups act on extensions of a certain subgeometry of the $O_8^+(2)$.3-building. The latter groups also, along with McL, Co_2, HS, He and Ru, act on extensions of generalized polygons. The group Co_2 acts on an extension on the $U_6(2)$-dual polar space, the group BM acts on an extension of ${}^2E_6(2)$-building. These geometries were characterized under the assumption that they admit a flag-transitive automorphism group (see, e.g., [3], [9], [12], [10], [24], [23], [18]).

Purely combinatorial characterizations of the extensions of the projective plane of order 4 and the affine plane of order 3, that is, without any assumption on group actions, were obtained by Witt in the 1930's. Recently, similar characterizations of some other sporadic geometries were obtained. The author gave such characterizations of the geometries for the Suzuki chain groups related to the $O_8^+(2)$.3-building ([13]), of those related to the sporadic Fischer groups ([16], [15]), and of the extensions of the generalized quadrangle of order (3, 9) related to the groups McL and Co_3 ([14]). Cuypers characterized the extended hexagons related to the Suzuki chain groups and the geometry for Co_2([4], [5], [6]). Cuypers, Kasikova and the author characterized the multiple extensions of a generalized hexagon related to McL and Co_3 ([7]).

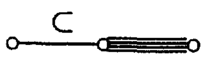
Here we consider a class of extended generalized octagons. In general, they have infinite universal covers, see [17], thus additional conditions are needed to characterize finite examples. The example Γ related to the group He satisfies the following condition.

(*) The points of a triple $\{u, v, w\}$ of points not on a block of Γ are pairwise adjacent if and only if v and w are at distance 3 in the local generalized octagon Γ_u.

There is one more example of an extended generalized octagon satisfying (*), it relates to the group $2 \cdot L_3(4).2^2$. It turns out that (*), along with a natural assumption on point residues, characterizes those two examples.

THEOREM 1.1. *Let* Γ *be an extension of a nondegenerate generalized octagon satisfying* (*). *Assume that for any point* u *the local generalized octagon* Γ_u *has at least one thick line through each point. Then* Γ *is isomorphic either to the extended generalized octagon on* 2048 *points related to the group He, or to the extended generalized octagon on* 112 *points related to* $2 \cdot L_3(4).2^2$.

R. Weiss ([24]) characterized these extended generalized octagons as flag-transitive geometries satisfying a property related to (*) and having diagram



The example related to $2 \cdot L_3(4).2^2$ admits a 2-fold quotient satisfying the property $(*)_{3,4}$ (cf. [5]) obtained from (*) by replacing the words ‘distance 3’ to the words ‘distance 3 or 4’. This geometry appears in the list in [24], as well. Cuypers mentions in [5] that the methods he uses there could be applied to the extended generalized octagons satisfying $(*)_{3,4}$.

We also settle the question about further extensions of the geometries under consideration.

THEOREM 1.2. *Let* Γ *be as in Theorem 1.1. Then* Γ *does not admit any further extensions.*

Note that R. Weiss shows in [24] that there are no flag-transitive geometries with the diagram

and with residues of the left-hand side type of elements as in the remark following Theorem 1.1.

2. Definitions and Notation

Let Γ be an *incidence system* $(\mathcal{P}, \mathcal{B})$ of *points* and *blocks*, where the latter are subsets of $\mathcal{P}$ is size at least 2. The *point graph* of Γ is the graph with vertex set $\mathcal{P}$ such that two vertices p and q are *adjacent* (notation $p \perp q$) if there is a block containing both of them. The distance between subsets X, Y of points of Γ (notation $d(X,Y)$ or $d_\Gamma(X,Y)$) is the minimal distance in the point graph between a point of X and a point of Y. We denote by $\Gamma_n(X)$ the set of points at distance n from $X \subseteq \mathcal{P}$, we also use $\Gamma(X)$ instead of $\Gamma_1(X)$. If $X = \{x\}$ we often use x instead of $\{x\}$. For any $X \subseteq \mathcal{P}$, we denote $X^\perp = \{p \in \mathcal{P} | p \perp x$ for any $x \in X\}$. The *subsystem* of Γ induced by the set X is the incidence system $\Gamma(X \cup \mathcal{P}, \{B \cap X | B \in \mathcal{B}, |B \cap X| > 1\})$. We call Γ *connected* if its point graph is connected.

An incidence system is called a *generalized* $2d$*-gon* if its point graph has diameter d, for any $p \in \mathcal{P}$ and $B \in \mathcal{B}$ there exists a unique point on B closest to p, and for any $p, q \in \mathcal{P}$ such that $d(p,q) = i < d$ there is a unique block on p containing a point at distance $i-1$ from q. Since it follows that there is at most one block on any pair of points, the blocks of Γ are usually called *lines*, and the word *collinearity* is used instead of the word *adjacency*. Given Γ, we define the dual system Γ^*, in fact also a $2d$-gon, whose points (respectively lines) are lines (respectively points) on Γ, incidence is by the inverse inclusion. We say that Γ is *nondegenerate* if for each point there exists a point at distance d from it, and the same holds in Γ^*. A line of Γ is *thick* if it contains more than two points, otherwise it is *thin*. We say that Γ is *regular* of order (s,t) if each point is on exactly $t+1$ lines and each line contains exactly $s+1$ points. The numbers s and t are called *parameters* in this case. The famous Feit–Higman theorem ([8]) says that a line-thick finite regular $2d$-gon satisfies $d \in \{1,2,3,4,6\}$. An incident point–line pair of Γ is called a *flag*. Given a generalized $2d$-gon Γ, one can always construct a $4d$-gon Γ^F, whose points are flags of Γ and whose lines are the points and the lines of Γ, incidence being the natural one. Note that it is customary to refer to a generalized 4-gon as a *generalized quadrangle* (GQ, for short, or GQ(s,t)). Similarly, we refer to generalized 8-gons as *generalized octagons* (respectively GO and GO(s,t)).

In what follows a well-known family of generalized quadrangles with parameters (s,s) usually called $W(s)$ (see [19], a standard reference on GQ's), will play a significant role. For a prime power, s, the points and the lines of $W(s)$ are the points and the lines of the projective space PG(3, s) which are totally isotropic with respect to a nondegenerate alternating form f. Without loss of generality, f can be chosen as follows:

$$f(x,y) = x_1y_4 - x_2y_3 + x_3y_2 - x_4y_1. \tag{1}$$

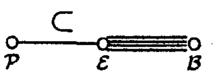
Note that $W(s)^* \cong W(s)$ if and only if $s = 2^k$.

Let Γ be an incidence system. Given $X \subset \mathcal{P}(\Gamma)$ satisfying $|B - X| > 1$ for any $B \in \mathcal{B}(\Gamma)$, we refer to the incidence system

$$\Gamma_X = (\Gamma(X), \{B - X | X \subset B \in \mathcal{B}(\Gamma)\})$$

as the *residue* of X, or as the *local* system in X. Let $\mathcal{D}$ be a class of incidence systems. We say that Γ is an *extension* of $\mathcal{D}$ (or *extended* $\mathcal{D}$) if, for any $x \in \mathcal{P}(\Gamma)$, Γ_x is isomorphic to a member of $\mathcal{D}$. If $\mathcal{D} = \{\Delta\}$, it is customary to refer to extensions of $\mathcal{D}$ as extensions of Δ (or extended Δ). If Γ is an extension of $\mathcal{D}$ then the connected components of Γ are extensions of $\mathcal{D}$, also. Hence, unless otherwise stated, we assume our extensions to be connected.

In particular, if $\mathcal{D}$ is a class of generalized octagons, we say that Γ is an *extended generalized octagon*. Note that Γ can be considered as a geometry with the following diagram (for this notion see, e.g., [2]).



Here $\mathcal{E}$ is the set of edges of the point graph of Γ. Note that the subgraph Ξ of the point graph of Γ induced by $x^\perp - \{x\}$ is not necessarily isomorphic to the point graph Υ and Γ_x. In particular, Ξ and Υ are not isomorphic if Γ is an extended generalized octagon satisfying $(*)$, namely two vertices of Ξ are adjacent if and only if the distance between them in Υ is 1 or 3.

3. The Type of Point Residues

In what follows $\Gamma = \Gamma(\mathcal{P}, \mathcal{B})$ denotes an extended GO satisfying the conditions of Theorem 1.1. The main result of this section is as follows.

PROPOSITION 3.1. Γ *is an extension of the generalized octagon* $W(s)^F$ *of flags of the generalized quadrangle* $W(s)$, *where* $s = 2^k$.

In the remainder of this section we prove Proposition 3.1. Then in Lemma 3.8 we summarise a few facts arising mainly as by-products of the proof.

The following statement can be easily deduced from the main result of Yanushka [25].

RESULT 3.2 (cf. [25]). *Let* Δ *be a nondegenerate* GO *such that for any point* $p \in \mathcal{P}(\Delta)$ *there exists a thick line on* p. *Then exactly one of the following holds:*

(i) Δ *is regular of order* (s, t) *for some* $s > 1, t \geq 1$.
(ii) Δ *is a GQ* $(s_1, s_2)^F$ *for some* $s_1 < s_2, s_1 > 1$ *and* $s_2 > 1$.
(iii) Δ *is a* G^F, *where G is the* $(a+1) \times (b+1)$-*grid for some* $a > 1$ *and* $b > 1$. □

Below we say that a GO is of *type* (x) if it belongs to the family (x) of Result 3.2.

Let $u \in \mathcal{P}$. Pick $x, z \in u^{\perp} - u, x \neq z, \{x, u, z\} \subseteq B \in \mathcal{B}$. If Γ_u is of type (ii) (respectively, (iii)) we make our choice to satisfy $|B| = s_1 + 2$ (respectively, $|B| = 3$ and the size of the other block on $\{u, x\}$ to be $a+2$). Choose $v \in x^{\perp} - u^{\perp}$ satisfying $v \perp_{\Gamma_x} z$.

LEMMA 3.3. *Let Ω be the subsystem of the residue Γ_u induced by $v^{\perp}$, and Ξ the connected component of Ω containing x. Then*

(a) $d_{\Gamma_y}(u, v) = 2$ *for any $y \in \Xi$;*
(b) *there is exactly one thin line of Ξ on y, and the remaining lines of Ξ on y coincide with those of Γ_u, for any $y \in \Xi$.*

Proof. Note that $u \perp_{\Gamma_x} z \perp_{\Gamma_x} v$. In Γ_x we find that for any line L on u but not on z each point $w \in L$ distinct from U is at distance 3 from v, so $w \perp v$ by $(*)$. The remaining line on u has all its points, except z, at distance 2 from v, so z is the only point on it adjacent to v. Thus we have that $x^{\perp} \cap \Xi$ contains each line L of Γ_u on x such that $z \notin L$, and xz (as a line of Ξ) is thin. This proves part (b) for $y = x$.

By the choice of z, there is a thick line of Ξ on x, namely any line of Ξ on x missing z is thick. On the other hand, let us choose $v' \in x^{\perp}$ satisfying $d_{\Gamma_x}(u, v') = 4$. Let Ξ' be the connected component containing x of the subsystem of Γ_u induced by $v'^{\perp}$. Then $x^{\perp} \cap \Xi'$ intersects each line of Γ_u on x in exactly two points, that is, all the lines of Ξ' on x are thin. This gives us a criterion for checking whether $d_{\Gamma_y}(u, v)$ is 2 or 4.

Clearly $d_{\Gamma_z}(u, v) = 2$. If $y \in x^{\perp} \cap \Xi - \{x, z\}$ then the line xy of Γ_u lies within Ξ. By the choice of z, xy is thick. Using the observation above in the case $d_{\Gamma_x}(u, v') = 4$ we see that $d_{\Gamma_u}(u, v) = 2$. Hence $d_{\Gamma_y}(u, v) = 2$ for any $y \in \Xi$. Thus part (a) is proved.

To complete the proof of (b) it suffices to repeat the first paragraph of the proof with y in place of x and with $z' \in y^{\perp}$ satisfying $u \perp_{\Gamma_y} z' \perp_{\Gamma_y} v$ in place of z. □

LEMMA 3.4. *Let Ξ be as defined in Lemma* 3.3, *and set* $X' = \Xi(x) \cup \Xi_3(x)$. *Then*

(i) $|X'| = 1 + st + s^2t^2 + st(1 + st - s)^2$,
(ii) $|X'| = 1 + 2s_2 + s_2^2$,
(iii) $|X'| = 1 + a + b + ab$,

according as Γ_x is of type (i), (ii) *or* (iii).

Proof. By Lemma 3.3 (b), we have that $|\Xi(x)| = 1+st, 1+s_2$, or $1+a$ according as the type of Γ_u is (i), (ii), or (iii). Since the point graph of Ξ behaves as a 'cactus' in the first three layers, it is straightforward to count $|\Xi_3(x)|$, and hence $|X'|$. □

LEMMA 3.5. *Let* $\Delta = \Gamma_x$, *and set*

$$X = \bigcup_{\substack{i=1,3 \\ j=1,3}} \Delta_i(u) \cap \Delta_j(v),$$

where u, x *and* v *are as defined above. Then*

(i) $|X| = 1 + 2st + s^2t^3 + s^2t(t-1)$,
(ii) $|X| = 1 + s_1 + s_2 + s_1s_2$,
(iii) $|X| = 2 + 2a$,

according as Γ_u *is of type* (i), (ii) *or* (iii). *Moreover,* $X = x^\perp \cap \Omega - \{x\}$, *where* Ω *is as defined in Lemma* 3.3.

Proof. We begin by proving the first part of the lemma. Observe that the type of Δ can be easily determined, using the blocks on $\{u, x\}$. Namely, it is the same as the type of Γ_u, with the exception that if Γ_u is of type (iii) then the corresponding parameters b for Γ_u and Δ are not necessary equal.

For Δ of type (i), we determine $|X|$ using standard calculations with parameters of distance-regular graphs [1], as follows. The point graph of Δ is distance regular with intersection array

$$\{s(t+1), st, st, st; 1, 1, 1, t+1\}.$$

Given any two vertices x, y satisfying $d_\Delta(x,y) = k$, the number of vertices w such that $d_\Delta(x,w) = i$ and $d_\Delta(y,w) = j$ is a constant p_{ij}^k. Note that $|X| = p_{11}^2 + 2p_{13}^2 + p_{33}^2$ and that $p_{11}^2 = 1, p_{13}^2 = st$. The constant $p_{33}^2 = s^2t(t^2+t-1)$ is computed using the recurrence formulae in [1, Lemma 4.1.7]. The consideration of this case is complete.

For Δ of type (ii), it is clear that

$$|\Delta(u) \cap \Delta(v)| + |\Delta(u) \cap \Delta_3(v)| + |\Delta_3(u) \cap \Delta(v)| = 1 + s_1 + s_2.$$

Now let us consider $\Phi = \Delta_3(u) \cap \Delta_3(v)$. Let $vpqw$ be a path from v to Φ. Since there are exactly two lines on each point, it follows that $p \in \Delta_3(u)$ and $q \in \Delta_4(u)$. By the same reason, for any $q \in \Delta_2(v) \cap \Delta_4(u)$ there is a unique point $w_q \in q^\perp \cap \Phi$. Moreover, $w_q = w_{q'}$ implies $q = q'$, for otherwise Δ possesses a circuit on q, q' and w_q of length less than 8. Hence

$$|\Phi| = \Delta_2(v) \cap \Delta_4(u)| = s_1s_2.$$

The consideration of this case is complete.

The remaining case of Δ of type (iii) is dealt with by arguments similar to those used for type (ii).

The last part of the lemma immediately follows from (*). □

LEMMA 3.6. *Γ_p is isomorphic to a* GO $(s, 1)$,*for some constant $s > 1$ independent from particular choice of $p \in \mathcal{P}$.*

Proof. Note that $X' \subseteq X$, where X and X' are as defined in Lemmas 3.5 and 3.4. It follows that $t = 1$ if Γ_u is of type (i). If Γ_u is of type (ii) then $X' \subseteq X$ implies that $s_1 \geq s_2$, a contradiction. If Γ_u is of type (iii) then $X' \subseteq X$ implies that $b = 1$, which is also a contradiction.

Thus Γ_u is a GO$(s, 1)$. Let $w \in u^\perp - \{u\}$. In Γ_w we see that there are exactly two lines on u, both of size $s + 1$. Hence, by Result 3.2, Γ_w is a GO$(s, 1)$, as well, and by the connectivity of Γ the result follows. □

Our next task is to investigate Ξ more closely.

LEMMA 3.7. *$\Omega = \Xi$, that is, Ω is connected. Moreover, Ω, is a sub*GO *of Γ_u. It is of type* (iii) *such that $a = b = s$.*

Proof. Using Lemmas 3.3 and 3.6, we observe that each point of Ξ has one thin line and one line of size $s + 1$ through it. Combining Lemmas 3.4 and 3.5 with $t = 1$, we have $X = X'$. Hence $x^{\perp_\Xi} \cup \Xi_3(x) = x^\perp \cap \Xi = x^\perp \cap \Omega$. It follows that every thick line of Ω intersects $x^\perp$, since each line of Γ_u intersects $x^\perp$. Hence the first part of the lemma holds, and the diameter of Ω is at most 4.

Since Ω has exactly s thin lines ww' satisfying $d_\Omega(x, w) = d_\Omega(x, w') - 1 = 2$, it has (exactly) s thick lines L at distance 3 from x, since for each w with $d_\Omega(x, w) = 2$ and such L there exists $w' \in L$ such that ww' is one of the s thin lines just mentioned. Hence $|\Omega_4(x)| = s^2$. Also, there are s^2 points $p \in \Omega_3(x)$ such that the thick line on p is at distance 2 from x. If p is such a point then there exists a thin line pp' satisfying $p' \in \Omega_4(x)$. It follows that for any $y \in \Omega_4(x)$ the thin line on y is at distance 3 from x. Hence Ω is a GO. It is clear that it is of type (iii) with $a = b = s$. □

Let $\Delta = \Gamma_u, x, z \in u^\perp - \{u\}$ and $x \perp_\Delta z$. Pick $L, M \in \mathcal{B}(\Delta) - \{xz\}$ such that $z \in L$ and $x \in M$. We claim that there are s points p at distance 2 from u in Γ_x satisfying $L, M \subset u^\perp \cap p^\perp$ and such that $u^\perp \cap p^\perp \neq u^\perp \cap p'^\perp$ for any pair p, p' of those s points.

Pick $v \in x^\perp - u^\perp$ such that $v \perp_{\Gamma_x} z \perp_{\Gamma_x} u$. We check that the set of s points on the line vz of Γ_x not containing z satisfies the requirements above. Let $p \in vz - \{v, z\}$. Clearly $L, M \subset u^\perp \cap p^\perp$. On the other hand, we see within Γ_v that there is a point $q \in u^\perp \cap v^\perp$ such that $p \not\perp q$. Hence $u^\perp \cap v^\perp \neq u^\perp \cap p^\perp$, as required.

Thus Δ admits at least s distinct subGO's isomorphic to Ω which contain L and M, where Ω is the subsystem of Δ induced by $v^\perp$. Let Θ be the GQ(s, s) such that $\Delta = \Theta^F$, and L, M are (collinear) Points of Θ (we use capital 'P' to distinguish Points of Θ from points of Δ). It is clear that Ω corresponds to a subGQ$(1, s)$ of Θ, among the s subGO's of Δ just constructed. So we have distinct subGQ$(1, s)$'s $\mathcal{S}_1, \ldots, \mathcal{S}_s$ containing the Points L and M of Θ. Pick a Point $P \in L^{\perp_\Theta} - M^{\perp_\Theta}$.

It is easy to see that there is at most one subGQ$(1, s)$ through P and M. Denoting $\mathcal{S}_i \cap LP = \{L, P_i\}$, we see that $P_i = P_j$ implies $i = j$, for $i, j, \in 1, \ldots, s$.

We have established that for any $Q \in \mathcal{P}(\Theta) - M^{\perp_\Theta}$ there is a subGQ$(1, s)$ on M and Q. In other words, $|\{M, Q\}^{\perp_\theta \perp_\theta}| = s + 1$, that is, M is *regular*, cf. [19]. Therefore each Point of Θ is regular, for we are free to choose M to be any line of Δ. Hence $\Theta \cong W(s)$ by [19, 5.2.1].

Let N be a line of Δ intersecting $M - \{x\}$. The lines N and xz of Δ correspond to the collinear Points of Θ^*, the dual of Θ. Repeating the argument above with N and xz in place of L and M, we see that $\Theta^* \cong W(s)$ also. Hence $s = 2^k$, cf. [19]. The proof of Proposition 3.1 is complete. □

Several facts arising in the proof above will be required later. We summarize them and some other facts in the following lemma.

LEMMA 3.8. *Let Ξ be a sub*GO *of $\Delta = \Gamma_u = \Theta^F$ isomorphic to Ω. Then there exists $q \in \Gamma_2(u)$ such that $\Xi = u^\perp \cap q^\perp$. There are two classes $\mathcal{O}, \mathcal{O}^*$ of such sub*GO, *where $\mathcal{O}$ (respectively $\mathcal{O}^*$) corresponds to the sub*GQ$(1, s)$*'s of Θ (respectively Θ^*), $|\mathcal{O}| = |\mathcal{O}^*| = s^2(s^2+1)/2$, and $\Omega \in \mathcal{O}$. There are $(s-1)(s+1)^2$ elements $\Xi \in \mathcal{O}$ such that $|\Omega \cap \Xi| = 2s + 2$ and Ω and Ξ have two thick lines in common. For all other $\Xi \in \mathcal{O} - \{\Omega\}$ we have $|\Omega \cap \Xi| = 0$. For $\Xi \in \mathcal{O}^*, \Omega \cap \Xi$ is either an ordinary octagon or empty.*

Proof. Without loss of generality we can assume that the lines L, M of Δ defined in the final part of the proof of Proposition 3.1 lie in Ξ. We saw that there are exactly s subGO's containing L and M, and all of them are of the form $q^\perp \cap u^\perp$ for some $q \in \Gamma_2(u)$. Thus Ξ is among them, and the first statement is proved.

The second statement just repeats an observation made at the end of the proof of Proposition 3.1, and the formula for $|\mathcal{O}| = |\mathcal{O}^*|$ is well known.

Let $\Xi \in \mathcal{O} - \{\Omega\}$. If $\bar{\Omega} \cap \bar{\Xi}$ contains a Line then there are exactly 2 Points in $\bar{\Omega} \cap \bar{\Xi}$. There are $(s + 1)^2$ Lines in $\bar{\Omega}$ and there are s subGQ$(1, s)$'s of Θ through each Line. Thus there are $(s - 1)(s + 1)^2$ elements of $\mathcal{O}$ intersecting Ω in two thick lines. If $\Omega \cap \Xi$ is nonempty then $\bar{\Omega} \cap \bar{\Xi}$ contains a flag. Hence $\Omega \cap \Xi$ contains a thin line. Thus it also contains two thick lines, and we are in the already considered situation.

It remains to consider the case $\Xi \in \mathcal{O}^*$. Here $\bar{\Xi}$ is a subGQ$(s, 1)$ of Θ. Assume that $\bar{\Omega}$ and $\bar{\Xi}$ have a common Point. It is easy to deduce that they intersect in an ordinary quadrangle of Θ, so $\Omega \cap \Xi$ is an 8-gon. Otherwise (that is, if $\bar{\Omega}$ and $\bar{\Xi}$ do not have common Points), $\Omega \cap \Xi = \emptyset$. □

4. More on Point Residues

In this section we show that Γ is an extension of either $W(2)^F$ or $W(4)^F$, and as a by-product we obtain more information about the set $\Gamma_2(u)$, where $u \in \mathcal{P}$. Let $v \in \Gamma_2(u)$.

LEMMA 4.1. *Let* $x \in u^\perp \cap v^\perp$. *Then, for any* $y \in u^\perp \cap v^\perp$, $d_{\Gamma_y}(u,v) = d_{\Gamma_x}(u,v) = d \in \{2,4\}$.

Proof. If $d = 2$ then the statement follows from Lemma 3.7. The only other possibility is $d = 4$. □

LEMMA 4.2. *Let* $a, b \in u^\perp \cap v^\perp$ *satisfy* $a \perp_{\Gamma_u} c \perp_{\Gamma_u} b$ *for* $c \in u^\perp - \{u\}$. *Then* $c \perp v$.

Proof. By Lemma 3.7, if $d_{\Gamma_a}(u,v) = 2$ then the subsystem of Γ_u induced by $v^\perp$ is geodetically closed, so the statement holds in this case. By Lemma 4.1 we may assume that $d_{\Gamma_a}(u,v) = d_{\Gamma_b}(u,v) = 4$. Observe that the subsystem Ω of Γ_b induced by $a^\perp$ is a subGO, and that uc is a thin line of Ω. Since $d_{\Gamma_b}(u,v) = 4$, v is at distance 3 from any line on u of Ω. Hence $d_\Omega(v,c) = 3$, so $v \perp c$. □

Let $d_{\Gamma_x}(u,v) = 4$ for some (and so, by Lemma 4.1, for any) $x \in u^\perp \cap v^\perp$, and let Ω be the subsystem of Γ_u induced by $u^\perp$. It is easy to see that the connected components of Ω are ordinary polygons. The following is an immediate consequence of Lemma 4.2.

LEMMA 4.3. *The connected components of* Ω *are ordinary polygons. The distance between any two of them is at least 3.* □

It turns out that the components are the polygons of the least possible girth.

LEMMA 4.4. *The connected components of* Ω *are ordinary* 8*-gons.*

Proof. Let axb be a 2-path contained in a component of Ω. Let ubb_1dvca_1a be an 8-gon inside Γ_x containing u and v (points are listed in the natural cyclic order). This 8-gon is unique, since Γ is an extended $\mathrm{GO}(s,1)$. Since $d_{\Gamma_x}(u,c) = d_{\Gamma_x}(u,d) = 3$, we have $c \perp u \perp d$. Since $x, u \in a^\perp \cap c^\perp$ and $d_{\Gamma_x}(a,c) = 2$, by Lemma 4.1 we have that $d_{\Gamma_u}(a,c) = 2$. So there exists $p = p_{ac} \in u^\perp$ satisfying $a \perp_{\Gamma_u} p \perp_{\Gamma_u} c$. By Lemma 4.2, $p \in v^\perp$.

Similarly we find $p_{bd}, p_{cd} \in u^\perp \cap v^\perp$. So $xap_{ac}cp_{cd}dp_{bd}b$ is an 8-gon within Ω containing axb. By Lemma 4.3, it is a full connected component of Ω. □

Denote by Ξ the connected component of Ω containing axb, $\{c,d\} = \Xi_3(x)$. Considering Γ_x, we easily find that $X = \{u,v,x\}^\perp - \Xi$ satisfies $|X| = 4(s-1)^2$ and that $X \subseteq c^\perp \cup d^\perp$. Moreover, for any $y \in X$ there exists the unique $y' \in X - \{y\}$ such that $y \perp_{\Gamma_u} y'$. Denote by Υ the connected component of Ω containing y. Let y_1, y_2 be the two points of Υ at distance 3 from y. As $x \in \{u,v,y\}^\perp - \Upsilon$, we see that $x \in y_1^\perp \cup u_2^\perp$. So Υ has at least four, and so exactly four, of its points within X. Summarizing, we have the following.

LEMMA 4.5. *The set* X *is the disjoint union of* $(s-1)^2$ *sets of size* 4 *of the form* $X \cap \Upsilon$, *where* Υ *is a connected component of* Ω. □

Let $q \in \{x,v\}^\perp - u^\perp$ satisfy $d_{\Gamma_x}(u,q) = 2$. We shall establish a correspondence between the subsystem Π of Γ_u induced by $q^\perp$ and the subsystem Π' of Γ_q induced by $u^\perp$.

LEMMA 4.6. *Let xz be a thin line of Π. Then it is a thin line of Π' as well. Let $\{z, a_1, \dots, a_s\}$ be the thick line on z of Π and for $i = 1, \dots, s$ let $a_i a_i'$ be the thin line on a_i of Π. Then the thick line of Π' on x is $\{x, a_1', \dots, a_s'\}$.*

Proof. The first statement follows immediately from the observation that for any $x \in u^\perp \cap q^\perp$ the thin line xz of Π is defined by $q \perp_{\Gamma_x} z \perp_{\Gamma_x} u$. Let L be the thick line of Π' on x, and let $y \in L - \{x\}$. Note that $d_{\Pi'}(z, y) = d_{\Gamma_q}(z, y) = 2$. Hence by Lemma 4.1 we have that $d_{\Gamma_u}(z, y) = 2$. So either y is one of the a_i' or $y \perp_{\Gamma_u} x$. But the latter is clearly impossible. Hence the lemma. □

LEMMA 4.7. *The subsystem of Π induced by $v^\perp$ is the disjoint union of s copies of the 3-path. Moreover, if $\alpha\beta\gamma\delta$ and $\alpha'\beta'\gamma'\delta'$ are any two of them written in the natural order, then, without loss in generality, $\alpha, \gamma, \delta \in \Pi_3(\alpha')$ and $\alpha, \beta, \delta \in \Pi_3(\delta')$.*

Proof. Let Π' be the subsystem of Γ_q induced by $u^\perp$. Let $\Theta \cong W(s)$ such that $\Gamma_q = \Theta^F$ and Π' corresponds to a subGQ$(1, s)$ $\bar{\Pi}'$ of Θ. Denote by ab the thin line of Π' at distance 2 from v. Let A, B be the thick lines of Π' satisfying $a \in A, b \in B$. Let v correspond to the flag (P, l) of Θ. Then $P \in AB \in \mathcal{B}(\bar{\Pi}')$.

For any $W \in \mathcal{P}(\bar{\Pi}') - \{A, B\}$ there exists a Line w through W intersecting l. There are $2s$ such Points W. As they correspond to thick lines of Π', they give us $2s$ thick lines (and so $2s$ points) of Π' at distance 3 from v. Also, there are $2s$ points at distance 3 from v lying on the thick lines A, B of Π'. Clearly, there are no more points at distance 3 from v within Π'. It shows that the subsystem of Π' induced by $v^\perp$ consists of $4s$ points.

Next, for w as above let $w \cap \bar{\Pi}' = \{W, W'\}$. This defines an equivalence relation with classes of size 2 on $\mathcal{P}(\bar{\Pi}') - \{A, B\}$, and so gives s thin lines of Π' at distance 3 from v. The translation of this situation back into Π' is presented diagrammatically in Figure 1. Now we use Lemma 4.6 to look at the pointset we are interested in within Π. By Lemma 4.6 we have that $\alpha \perp_{\Gamma_u} \beta \perp_{\Gamma_u} \gamma \perp_{\Gamma_u} \delta$, and none of the points just listed is collinear within Γ_u to any other point at distance 3 from v within Π. The same holds for α', β', γ' and δ', and it is straightforward to check the remaining claimed distances. □

Now we return to the connected components Ξ and Υ of Ω, that we considered in the text preceding Lemma 4.5. We shall improve the result of Lemma 4.5. As above, denote $X = \{u, v, x\}^\perp - \Xi$. By Lemma 4.5, the subsystem $X \cap \Upsilon$ has two connected components with sets of points $\{y_1, y_1'\}$ and $\{y_2, y_2'\}$ respectively. Let $\{y, y'\} = \Xi_3(x)$. As already noted, each point of X is at distance 3 within Γ_u from either y or y' (but not both of them).

LEMMA 4.8. *Let $d_{\Gamma_u}(y, y_1) = 3$. Then $d_{\Gamma_u}(y', y_1') = 3$.*
Also, $d_\Upsilon(\{y_1, y_1'\}\{y_2, y_2'\}) = 2$.

Proof. First, we claim that there exists $q \in \{x, y, v, y_1\}^\perp - u^\perp$ such that $u \perp_{\Gamma_y} w \perp_{\Gamma_y} q$ for $w \in u^\perp \cap q^\perp$. Note that $d_\Delta(y, y_1) = d_\Delta(v, y_1) = 3$, where $\Delta = \Gamma_x$. Let L be the unique line of Δ at distance 1 from v, y and y_1, and let z be

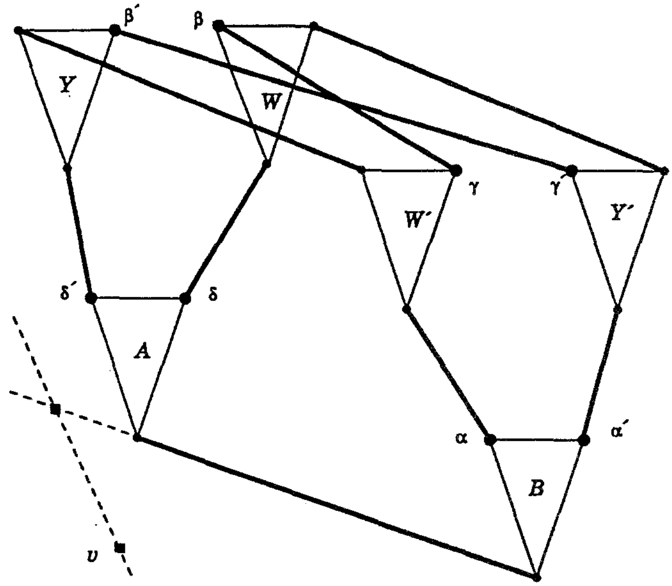


Fig. 1. The subsystem Π' of Γ_q, see the proof of Lemma 4.7. (Unless $s = 2$, only a part of Π' is actually shown.) Points inside Π' are shown as circles whereas those outside are shown as squares. Thick lines of Π' are shown as triangles, thin lines of Π' are shown as bold lines. Points of Π' at distance 3 from v are shown by bigger circles than the rest of them.

the point on L satisfying $\{z\} = L \cap \Delta_3(u)$. Denote $\{q\} = z^{\perp_\Delta} \cap \Delta_2(u)$. Clearly $d_\Delta(v, q) = d_\Delta(y, q) = 3$ and $d_\Delta(y_1, q) = 1$ or 3. We are done.

Now we apply Lemma 4.7. Since $x, y, y_1 \in q^\perp$, the subsystem Q of Ω induced by $q^\perp$ contains the 3-path joining x and y and a 3-path of Υ containing y_1. In the notation of Lemma 4.7, without loss in generality let us take $\alpha' = x, \delta' = y$. Since y_1 is at distance 3 from both x and y, either $\alpha = y_1$ or $\delta = y_1$.

If $\alpha = y_1$ then, by Lemma 4.7, $y_1' \notin Q$ and, by the same lemma, $\{\beta\} = \Upsilon \cap \{y, y_1\}^\perp \neq \{y_1'\}$. The line $y_1\beta$ of Γ_u is at distance 2 from y, so the line y_1y_1' is at distance 3 from y. Hence $y \not\perp y_1'$. Therefore $y' \perp y_1'$.

If $\delta = y_1$ the $y_1' = \gamma$ and so $y \not\perp y_1'$. Hence again $y' \perp y_1'$. This completes the proof of the first part of the lemma.

To prove the second part, observe that by Lemma 4.7 the system $X \cap \Upsilon \cap Q$ has two components and that the distance between them equals 2. □

At this point we are able to analyse Ω yet more precisely. We shall establish that $s = 2$ or 4 and that Ω is as in the known examples. We translate the situation with the 8-gons of Ω into the corresponding one with the 4-gons in Θ, where $\Delta = \Gamma_u = \Theta^F$.

It is obvious that a (nondegenerate) 8-gon of Δ corresponds to a quadrangle $ABCD$ of Θ, such that the points of the 8-gon are the flags $(A, AB), (A, AC), \ldots$, (C, AC) of Θ. Denote by $\bar{\Omega}$ the set of 4-gons of Θ corresponding to the set of 8-gons of Ω. We write the Points of 4-gons in the natural cyclic order. We call the

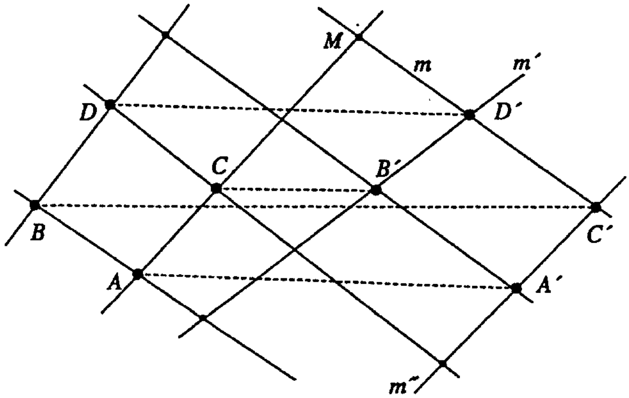


Fig. 2. Points of Θ are shown as circles. Points inside $\bar{\Omega}$ are shown as bigger circles than Points outside. Sides of $\bar{\Omega}$ are shown by continuous lines. Other Lines of Θ are shown by discontinuous lines.

Lines AB, AC, BD and CD of Θ the *Sides* of $ABCD$. The next statement follows immediately from Lemma 4.3.

LEMMA 4.9. *Let P, P' (respectively L, L') be the set of Points (respectively, of Sides) of two distinct 4-gons of $\bar{\Omega}$ corresponding to the 8-gons Υ, Υ' of Ω. Then $P \cap P' = L \cap L' = \emptyset$ and $Y \notin s'$ for any $Y \in P$ and $s' \in L'$.* □

LEMMA 4.10. *With the notation of Lemma* 4.9, let $d_\Delta(\Upsilon, \Upsilon') = 3$. Then for any $Y \in P$ there exists a unique $Y' \in P'$ such that $Y \perp_\Theta Y'$, and for any $s \in L$ there exists a unique $s' \in L'$ intersecting s.

Proof. In this proof we shall write $\perp$ instead of $\perp_\Theta$. Figure 2 illustrates the argument. Let $P = ABDC, P' = A'B'C'D'$. Without loss of generality $A \perp A'$. So the lines $l_1 = (A', A'B')(A', A'C')$ of Υ' and $(A, AB)(A, AC)$ of Υ are at distance 3 in Δ. There is one more line l_2 of Υ' at distance 3 from (A, AB) such that l_2 and $(A, AB)(B, AB)$ are at distance 3 in Δ. By Lemma 4.8, $d_{\Upsilon'}(l_1, l_2) = 2$. Without loss of generality, $l_2 = (B', B'D')(D', B'D')$. It follows that the Lines AB and $B'D'$ of Θ intersect. Since there must be two lines of Υ' at distance 3 from (B, AB), and one of them is l_2, we see that $B \perp C'$. Continuing in the same vein we find that AC intersects $C'D', C \perp B', BD$ intersects $A'B'$. Finally, we see that $D \perp D'$ and CD intersects $A'C'$.

Thus the existence of Y' and s' mentioned in the statement is established. The uniqueness easily follows from Lemma 4.5. □

Our next goal is to reconstruct the whole of Ω starting from a single 8-gon $\Upsilon \subset \Omega$.

Let $\bar{\Upsilon}$ and $\bar{\Upsilon}'$ be the 4-gons of Θ corresponding to Υ and Υ' as in Lemma 4.10. We shall show that the choice of the point $M = AC \cap C'D'$ and the line $m = C'D'$

(see Figure 2) determine Υ' uniquely. That is, if there exists $\Upsilon' \subset \Omega$ having the Side m then it is determined uniquely.

Let M and m be chosen. Note that the points of $\bar{\Upsilon}'$ collinear to B or to D lie on m. By Lemma 4.10, , $m \cap BD = \emptyset$. So the two points D' and C' of $\bar{\Upsilon}'$ satisfying $D' \perp_\Theta D$, $C' \perp_\Theta B$ are distinct and uniquely determined. Next we see that the Side m' of $\bar{\Upsilon}'$ on D' distinct from m must intersect AB. It determines m'. Similarly we reconstruct the remaining Side on C'. Finally we are forced to set $A' = A^{\perp_\Theta} \cap m''$, and $B' = C^{\perp_\Theta} \cap m'$. Thus $\bar{\Upsilon}'$, and so Υ' is determined.

There are $s-1$ possible choices of M and $s-1$ possible choices of m through M. These gives a total of at most $(s-1)^2$ 8-gons of Ω at distance 3 from $x \in \Upsilon$. Since by Lemma 4.5 this number must be exactly $(s-1)^2$, and hence the 8-gons of Ω at distance 3 from x are determined once Υ is chosen.

Since $\Theta \cong W(s), s$ even, the 8-gons of Γ_u all lie in one orbit of $\mathrm{Aut}(\Gamma_u)$. Thus there is no loss of generality in choosing $\bar{\Upsilon}$ to have points $A = (1000), B = (0010), D = (0001), C = (0100)$. (Here we present Θ as it was explained in Section 2. Collinearity is determined by (1).)

Choose $M = (1100)$ and $m = MD'$, where $D' = (0111)$. We see that $\bar{\Upsilon}' = \bar{\Upsilon}(m, M)$ has the following points, where the notation is chosen to match those of Figure 2: $A' = (1110), B' = (1101), C' = (1011), D'$ as above.

Let $M_1 = (1a00)$, where $a \in \mathrm{GF}(s) - \mathrm{GF}(2), m_1 = M_1D_1$, where $D_1 = (011a)$. Let us find $\bar{\Upsilon}'' = \bar{\Upsilon}(m_1, M_1)$ with points A'', B'', etc. First, observe that $D \perp D_1$, so $D'' = D_1$, and $B'' = B^\perp \cap m_1 = (a^{-1}, 0, 1, a)$. Since $D' \perp D''$, it follows that $d_{\Gamma_u}(\bar{\Upsilon}', \bar{\Upsilon}'') = 3$. Therefore there exists a point Y of $\bar{\Upsilon}'$ satisfying $Y \perp B''$.

If $Y = C'$ then $f(C', B'') = a + a^{-1} = 0$, so $a^2 = 1$, contradicting the choice of a. (Here f is the form in (1).)

If $Y = B'$ then $1 + a + a^{-1} = 0$, so $a^2 + a = 1$. The later implies $a^3 + a^2 = a$, so $a^3 = 1$. Hence in this case $s \leq 4$.

Finally, $Y = A'$ implies $1 + a = 0$, again contradicting the choice of a, and $Y = D'$ is impossible. So $s \leq 4$.

To summarize, we state the following.

PROPOSITION 4.11. *Γ is either an extension of $W(2)^F$ or an extension of $W(4)^F$.* □

5. The Remaining Cases

We shall reconstruct the point graph of Γ layer by layer. Let $u \in \mathcal{P}(\Gamma)$. The set $\Gamma_2(u)$ is naturally divided into two parts $\Gamma_2'(u)$ and $\Gamma_2''(u)$ according as $d_{\Gamma_x}(u, v) = 2$ or 4, respectively, where $v \in \Gamma_2(u)$ and $x \in u^\perp \cap v^\perp$, cf. Lemma 4.1. We proceed by showing that Γ is isomorphic to one of the known examples. Table I is self-explanatory. It shows certain infromation on the subsystems Ω of $\Delta = \Gamma_u$ induced by $v^\perp, v \in \Gamma_2(u)$, for the examples related to the groups $2 \cdot L_3(4).2^2$ (here $s = 2$)

TABLE I. Sets $x^{\perp} \cap \Omega$ in the examples, where $\Omega = u^{\perp} \cap v^{\perp} \subset \Gamma_u$.

s	Location	Location of $x \in \Delta - \Omega$				
	of v	$d_{\Delta}(x, \Omega)$	1	2	3	4
2	$\Gamma_2'(u)$	$\lvert x^{\perp} \cap \Omega \rvert$	6	8		
		$\#x$	9	18	0	0
	$\Gamma_2''(u)$	$\lvert x^{\perp} \cap \Omega \rvert$	6	8	8	0
		$\#x$	16	8	4	1
4	$\Gamma_2'(u)$	$\lvert x^{\perp} \cap \Omega \rvert$	10	16		
		$\#x$	75	300	0	0
	$\Gamma_2''(u)$	$\lvert x^{\perp} \cap \Omega \rvert$	22	16	32	
		$\#x$	240	60	45	0

and He (here $s = 4$). By Lemma 3.8, for any $v \in \Gamma_2'(u)$ the subsystem Ω of $\Delta = \Gamma_u$ induced by $v^{\perp}$ is a subGO, and each subGO of Δ isomorphic to Ω appears in this form for a point of $\Gamma_2'(u)$. We shall prove similar statement for $\Gamma_2''(u)$.

LEMMA 5.1. *Let $s = 2, \Phi = \mathcal{P}(\Delta)$ and $m = 4$, respecively let $s = 4, \Phi$ be the set of sub*GO(2, 1)*'s of Δ forming an orbit of* Aut(Δ) *of length* 1360 *and $m = 3$.*

Then for any $v \in \Gamma_2''(u)$ there exist unique $\phi \in \Phi$ such that

$$u^{\perp} \cap v^{\perp} = \{x \in \mathcal{P}(\Delta) | d_{\Delta}(x, \phi) = m\}.$$

Moreover, ϕ is at maximal distance in Δ from $u^{\perp} \cap v^{\perp}$.

Proof. The statement of the lemma merely describes the situation with $\Gamma_2''(u)$ and Φ in the known examples. All we need to show is that once Δ is given, the set $\Gamma_2''(u)$ and the edges joining $u^{\perp}$ and $\Gamma_2''(u)$ are uniquely determined.

Let $v \in \Gamma_2''(u)$. Pick an 8-gon Υ of $\Omega = u^{\perp} \cap v^{\perp} \subset \Delta$. We know that Aut$(\Delta)$ acts transitively on the 8-gons of Δ, so there is no loss in generality in choosing $\Omega = u^{\perp} \cap v^{\perp}$. We saw in the final part of the proof of Proposition 4.11 that once Υ is chosen, the set Ω^x of the 8-gons of Ω at distance 3 from a point x of Υ is determined. It is easy to check that Ω^x does not depend upon the particular choice of the point x, so we denote $\Omega^{\Upsilon} = \Omega^x$. It is straightforward to check that $\Pi = \{\Upsilon\} \cup \Omega^{\Upsilon} = \{\Xi\} \cup \Omega^{\Xi}$ for any 8-gon Ξ of Π. It follows that Ω is the disjoint union of several Components (to distinguish form 8-gons of Ω) of the form $\{\Xi\} \cup \Omega^{\Xi}$, where Ξ is an 8-gon of Ω. Each Component is one of the sets arising in the known examples, so it remains to show that there is only one Component.

Clearly the Components are at distance 4 from each other. But in both cases the size of the set of points of Δ at distance 4 from a Component is less than the size of a Component. We are done. □

TABLE II. The intersections of sets $u^\perp \cap v^\perp$ in the examples.

<table>
<tr><th rowspan="2">s</th><th rowspan="2">Location of v</th><th rowspan="2"></th><th colspan="9">Location of x</th></tr>
<tr><th colspan="3">$\Gamma_2'(u) - \{x\}$</th><th colspan="6">$\Gamma_2''(u) - \{v\}$</th></tr>
<tr><td rowspan="4">2</td><td rowspan="2">$\Gamma_2'(u)$</td><td>$\lvert\{x,u,v\}^\perp\rvert$</td><td>8</td><td>6</td><td>0</td><td colspan="2">8*</td><td colspan="2">8</td><td colspan="2">4</td></tr>
<tr><td>$\#x$</td><td>9</td><td>9</td><td>1</td><td colspan="2">9</td><td colspan="2">18</td><td colspan="2">18</td></tr>
<tr><td rowspan="2">$\Gamma_2''(u)$</td><td>$\lvert\{x,u,v\}^\perp\rvert$</td><td>8*</td><td>8</td><td>4</td><td>8</td><td>6</td><td colspan="2">5</td><td colspan="2">4</td></tr>
<tr><td>$\#x$</td><td>4</td><td>8</td><td>8</td><td>4</td><td>16</td><td colspan="2">16</td><td colspan="2">8</td></tr>
<tr><td rowspan="4">4</td><td rowspan="2">$\Gamma_2'(u)$</td><td>$\lvert\{x,u,v\}^\perp\rvert$</td><td>10</td><td>8</td><td>0</td><td colspan="2">16</td><td colspan="2">8</td><td colspan="2">0</td></tr>
<tr><td>$\#x$</td><td>75</td><td>100</td><td>$60+36$</td><td colspan="2">300</td><td colspan="2">$100+900$</td><td colspan="2">60</td></tr>
<tr><td rowspan="2">$\Gamma_2''(u)$</td><td>$\lvert\{x,u,v\}^\perp\rvert$</td><td>16</td><td>8</td><td>0</td><td>32</td><td>25</td><td>22</td><td>12</td><td>8</td><td>0</td></tr>
<tr><td>$\#x$</td><td>60</td><td>$20+180$</td><td>12</td><td>45</td><td>144</td><td>240</td><td>720</td><td>180</td><td>30</td></tr>
</table>

It immediately follows from Lemma 5.1 and the observation preceding it that the partial subgraph of Γ consisting of the points $u^\perp \cup \Gamma_2(u)$ and the edges at distance at most 1 from u is isomorphic to the same subgraph in the example. Next task is to show that the edges within $\Gamma_2(u)$ are uniquely determined, as well.

In Table II we present information about the sets $\{x, u, v\}^\perp$, where $x, v \in \Gamma_2(u)$. We denote there by 8^* the entries corresponding to the subsystems of Γ_u not isomorphic to those mentioned in Lemma 4.7, that is the disjoint union of two copies of the 3-path.

Comparing Tables I and II, we see that the edges within $\Gamma_2(u)$ are uniquely determined, as well. As an example, let as consider the case $s = 4, v \in \Gamma_2'(u)$. In Table 1 we see that the set $v^\perp - u^\perp - \{v\}$ consists of 75 points x such that $|\{u, v, x\}^\perp| = 10$ and 300 points x such that $|\{u, v, x\}^\perp| = 16$. In Table 2 we look at the corresponding entries and see that in $\Gamma_2(u)$ there are exactly 75 points satisfying $|\{u, v, x\}^\perp| = 10$ and exactly 300 points satisfying $|\{u, v, x\}^\perp| = 16$. Hence the $v^\perp$ is determined.

Thus the consideration of the case $s = 4$ as complete, since by Table 1 the diameter of the point graph is 2.

Let us turn to the case $s = 2$. We see that for any $v \in \Gamma_2''(u)$ there is a unique $z_v = v^\perp \cap \Gamma_3(u)$. Each $w \in v^\perp \cap \Gamma_2''(u)$ lies in a block on v and z_v. Hence $z_w = z_v$. The subgraph of the point graph induced on $\Gamma_2''(u)$ is isomorphic to those induced on $\Gamma(u)$, in particular it is connected. Hence $\Gamma_3(u) = \{z_v\}$ and $z_v^\perp = \{z_v\} \cup \Gamma_2''(u)$.

The proof of Theorem 1.1 is complete.

6. Proof of Theorem 1.2

Let Γ be an extension of $\mathcal{D}$, where $\mathcal{D}$ is the class of extended generalized octagons satisfying the conditions of Theorem 1.1.

Pick $u \in \mathcal{P}(\Gamma), x \in u^\perp - \{u\}$. Note that Γ_{ux} is a GO$(s, 1)$, for some $s \in \{2, 4\}$. By the connectivity argument, the value of s does not depend on the particular choice of u and x. There exists a point $v \in \mathcal{P}(\Gamma_x)$ such that for some $y, z \in \{u, v, x\}^{\perp}$ one has $u \perp_{\Gamma_{xy}} z \perp_{\Gamma_{xy}} v$. Define the incidence system Ω as follows. Let

$$\mathcal{B}_0(\Omega) = \{B \cap B' | v \notin B \in \mathcal{B}(\Gamma_u), u \notin B' \in \mathcal{B}(\Gamma_u), |B \cap B'| = 3\}$$

and let $\mathcal{P}(\Omega) = \bigcup_{B \in \mathcal{B}_0(\Omega)} B$. Then let

$$\mathcal{B}(\Omega) = \mathcal{B}_0(\Omega) \cup \{B - \{u\} | v \notin B \in \mathcal{B}(\Gamma_u), B - \{u\} \subset \mathcal{P}(\Omega)\}.$$

Note the Ω_x is a subGO of Γ_{ux} of type (iii), where $a = b = s$. Since there exist blocks of Γ containing, respectively $\{u, x, y, z\}$ and $\{v, x, y, z\}$, one has that Ω_y is a subGO of Γ_{uy} and $\Omega_y \cong \Omega_x$ for any $y \in \mathcal{P}(\Omega_x)$ and an appropriate choice of z.

The idea of the following argument is adapted from [7]. By Lemma 3.8, there exists $r \in \mathcal{P}(\Gamma_{ux})$ such that Ω_y is the subsystem of Γ_{uy} induced by $r^{\perp_{\Gamma_u}}$. In particular, $y \perp_{\Gamma_{ux}} w \perp_{\Gamma_{ux}} r$ for $w \in \mathcal{P}(\Gamma_{ux})$.

By the choice of r, we have $wy \cap \mathcal{B}(\Omega_x) = \{w, y\}$, where $wy \in \mathcal{B}(\Gamma_{ux})$. Hence the other line wr of Ω_x on w is thick. In particular, $r \in \mathcal{P}(\Omega_x)$. Repeating the argument above with the roles of y and r interchanged, we have that the line wr of Ω_x is thin. This is the contradiction.

The proof of Theorem 1.2 is complete.

Acknowledgement

The author used the computer algebra system GAP ([11]) and its shared package GRAPE ([20]) to investigate combinatorial properties of the extended generalized octagons for the groups He and $2 \cdot L_3(4)$ and compute contents of Tables I and II.

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