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2019

Victorin, N., Haug, T., Kwek, L.-C., Amico, L., & Minguzzi, A. (2019). Nonclassical states in strongly correlated bosonic ring ladders. *Physical Review A*, 99(3), 033616-.
doi:10.1103/PhysRevA.99.033616

<https://hdl.handle.net/10356/96041>

<https://doi.org/10.1103/PhysRevA.99.033616>

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Nonclassical states in strongly correlated bosonic ring ladders

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(Received 10 October 2018; published 21 March 2019)

We study the ground state of a bosonic ring ladder under a gauge flux in the vortex phase, corresponding to the case where the single-particle dispersion relation has two degenerate minima. By combining exact diagonalization and an approximate fermionization approach we show that the ground state of the system evolves from a fragmented state of two single-particle states at weak interparticle interactions to a fragmented state of two Fermi seas at large interactions. Fragmentation is inferred from the study of the eigenvalues of the reduced single-particle density matrix as well as from the calculation of the fidelity of the states. We characterize these nonclassical states by the momentum distribution, the chiral currents, and the current-current correlations.

DOI: [10.1103/PhysRevA.99.033616](https://doi.org/10.1103/PhysRevA.99.033616)

I. INTRODUCTION

Recent progress in ultracold lattice bosons allows us to study strongly correlated many-body systems with enhanced tunability [1]. One of the problems that have attracted interest in the community is a system of ladders pierced by an external magnetic flux. Indeed, such systems provide an exciting playground to address relevant aspects of many-body correlated matter such as the Meissner effect in type-II superconductors and quantum Hall effect in semiconductors [2–4]. Flux ladders have been already experimentally realized in a linear geometry with state-of-the-art ultracold atoms quantum technology [5–9]. Depending on the system parameters, interaction, and effective magnetic field, the system enters different physical regimes characterized by distinctive current patterns [10–16].

Here we focus on the avenue of research studying ultracold atoms loaded in closed geometries [17–32]. Atomtronics, in particular, promises to substantially enlarge the scope of cold atom quantum simulators, and establish the basis for new quantum devices and sensors [33–36]. Here, we consider a specific atomtronic network made of two coupled rings, which provide an analogical quantum simulator of bosonic flux ladders (see also [37]). The ring geometry allows us to naturally study the pattern of current flows through the ladder, as well as to minimize the effects of boundary conditions in the quantum simulator. In coupled ring lattices, currents, time-of-flight images, and spiral interferograms have been studied in various regimes of interactions, from mean-field approaches for large fillings and weak interactions [38] to exact diagonalization methods for small fillings and arbitrary interactions [39].

A bosonic system is in a single Bose-Einstein condensate if its single-particle density matrix has one macroscopic

eigenvalue (i.e., order of the number of particles) [40]. If the single-particle density matrix has more than one macroscopically occupied eigenvalue, then the state is called fragmented [41,42]. Nozières and Saint James [41] demonstrated that no fragmentation can take place in a homogeneous Bose gas with repulsive interactions. For dispersion relations with degenerate minima, instead, fragmented states may emerge [43–45].

Such a type of dispersion relation occurs in the vortex phase of double ring lattices, displaying in particular a two-minima structure. Here, we investigate the nature of the system's ground state at arbitrary interactions. The mean-field approach assumes a coherent state, made of a superposition of single-particle occupancies of each minimum. However, it has been shown that the ground state at small lattice fillings and weak interactions is indeed a fragmented state constructed with single-particle momentum states [46]. This result can be related to the studies of spin-orbit coupled systems, which share the same type of Hamiltonian as bosonic flux ladders, and where the fragmentation was also observed with an *ab initio* numerical study [45]. In this work, we explore the fate of the fragmented state at increasing interactions. In particular, we show that strong repulsive interactions destroy the fragmented single-particle state, and give rise to a different type of nonclassical state, which can be described as a fragmentation of two Fermi spheres. The ground-state crossover is analyzed by studying the correlation functions as implied in the one-body density matrix and the specific changes in the configuration of the currents flowing in the ladder.

The paper is outlined as follows. In Sec. II, we introduce the model Hamiltonian and discuss the different entanglement properties of the ground states in the different physical regimes of the ladder. In addition we sketch the analytical

methods (an approximate fermionization scheme) that we employ to establish regime-appropriate observables. In Sec. III, we present the results obtained with the analytical methods and compare them with the exact diagonalization; our findings are corroborated by the study of the configuration of currents flowing in the ring ladder. Finally Sec. IV is devoted to a summary and concluding remarks.

II. MODEL SYSTEM AND METHODS

We consider two ring lattices A and B , occupied by bosons at zero temperature, with the same number of sites in each ring and tunnel coupled via the rungs. The total Hamiltonian of the system reads $\mathcal{H} = \mathcal{H}_A + \mathcal{H}_B + \mathcal{H}_I$, where

$$\begin{aligned}\hat{\mathcal{H}}_A &= \sum_{m=1}^L (-J e^{i2\pi\phi_A} \hat{a}_m^\dagger \hat{a}_{m+1} + \text{H.c.}) + \frac{U}{2} \hat{n}_m^A (\hat{n}_m^A - 1), \\ \hat{\mathcal{H}}_B &= \sum_{m=1}^L (-J e^{i2\pi\phi_B} \hat{b}_m^\dagger \hat{b}_{m+1} + \text{H.c.}) + \frac{U}{2} \hat{n}_m^B (\hat{n}_m^B - 1), \\ \hat{\mathcal{H}}_I &= -K \sum_{m=1}^L (\hat{a}_m^\dagger \hat{b}_m + \text{H.c.}),\end{aligned}\quad (1)$$

where J is the intraring tunnel energy, K is the inter-ring tunnel energy, U is the on-site interaction energy, and ϕ_A, ϕ_B are the artificial gauge fields applied to each ring, which we assume to be separately tunable. In the following we will restrict for simplicity [47] to the case $\phi_A = \phi$, $\phi_B = -\phi$, corresponding to the case where counterpropagating currents are driven by the gauge fields on the two rings.

In order to characterize the ground state of the system we use various observables: the momentum distribution in ring A is defined as $n_k^A = \langle \hat{a}_k^\dagger \hat{a}_k \rangle$, with $\hat{a}_k = \frac{1}{\sqrt{L}} \sum_{m=1}^L \hat{a}_m e^{imk}$, where k is the wave vector in units of inverse lattice spacing. Similarly, we have $n_k^B = \langle \hat{b}_k^\dagger \hat{b}_k \rangle$ for ring B. The current operator at position m along the same ring is defined as $\hat{j}_{a,m}^\parallel = -iJ(\hat{a}_m^\dagger \hat{a}_{m+1} - \text{H.c.})$. We define the chiral current in the ladder as $\hat{j}_c = \frac{1}{L} \sum_m (\hat{j}_{a,m}^\parallel - \hat{j}_{b,m}^\parallel)$. The inter-ring current operator at site m is $\hat{j}_m^\perp = -iK(\hat{a}_m^\dagger \hat{b}_m - \text{H.c.})$. Correspondingly, we define the current-current correlation of the current between the rings $\langle \hat{j}_0^\perp \hat{j}_m^\perp \rangle$ and the current inside the ring $\langle \hat{j}_0^\parallel \hat{j}_m^\parallel \rangle$. We also investigate the density-density correlations in the leg $\Delta n_m = \langle \hat{n}_0 \hat{n}_m \rangle - \langle \hat{n}_0 \rangle \langle \hat{n}_m \rangle$ and between legs $\Delta n_m^\perp = \langle \hat{n}_0^A \hat{n}_m^B \rangle - \langle \hat{n}_0^A \rangle \langle \hat{n}_m^B \rangle$.

By Fourier transforming, the Hamiltonian Eq. (1) is readily expressed in momentum space according to

$$\begin{aligned}\hat{\mathcal{H}}_A &= \sum_k -2J \cos(k + 2\pi\phi) \hat{n}_k^A + \frac{U}{2L} \sum_{k,k',q} \hat{a}_{k+q}^\dagger \hat{a}_{k'-q}^\dagger \hat{a}_k \hat{a}_k, \\ \hat{\mathcal{H}}_B &= \sum_k -2J \cos(k - 2\pi\phi) \hat{n}_k^B + \frac{U}{2L} \sum_{k,k',q} \hat{b}_{k+q}^\dagger \hat{b}_{k'-q}^\dagger \hat{b}_k \hat{b}_k, \\ \hat{\mathcal{H}}_I &= -K \sum_k (\hat{a}_k^\dagger \hat{b}_k + \text{H.c.}).\end{aligned}\quad (2)$$

It is useful to transform the above Hamiltonian to a new basis, which is diagonal in the absence of interactions. We call this

the “diagonal” basis. By using the transformation

$$\begin{pmatrix} \hat{a}_k \\ \hat{b}_k \end{pmatrix} = \begin{pmatrix} v_k & u_k \\ -u_k & v_k \end{pmatrix} \begin{pmatrix} \hat{\alpha}_k \\ \hat{\beta}_k \end{pmatrix}, \quad (3)$$

where u_k and v_k are given by

$$v_k = \sqrt{\frac{1}{2} \left(1 + \frac{\sin(2\pi\phi) \sin(k)}{\sqrt{(K/2J)^2 + \sin^2(2\pi\phi) \sin^2(k)}} \right)}, \quad (4)$$

$$u_k = \sqrt{\frac{1}{2} \left(1 - \frac{\sin(2\pi\phi) \sin(k)}{\sqrt{(K/2J)^2 + \sin^2(2\pi\phi) \sin^2(k)}} \right)}, \quad (5)$$

the noninteracting part of the Hamiltonian becomes

$$\hat{H}_0 = \sum_k E_+(k) \hat{\alpha}_k^\dagger \hat{\alpha}_k + E_-(k) \hat{\beta}_k^\dagger \hat{\beta}_k, \quad (6)$$

with $E_\pm = -2J \cos(k) \cos(2\pi\phi) \pm \sqrt{K^2 + 4J^2 \sin^2(k) \sin^2(2\pi\phi)}$. In the vortex phase the lowest branch $E_-(k)$ of the dispersion relation has two degenerate minima at $k_{1,2} = \pm \arcsin \sqrt{\sin^2(2\pi\phi) - \frac{K^2}{[2J \tan(2\pi\phi)]^2}}$. It will be useful to study the momentum distributions in the diagonal basis, i.e., the one of the lower branch $n_\beta(k) = \langle \hat{\beta}_k^\dagger \hat{\beta}_k \rangle$ and the one of the upper branch $n_\alpha(k) = \langle \hat{\alpha}_k^\dagger \hat{\alpha}_k \rangle$.

In the vortex phase, for very small ring-ring coupling K and small, nonzero interaction strength U , the ground state is fragmented, i.e., it displays macroscopic occupation of the two single-particle momentum states $k_{1,2}$ as described by the ansatz [46]

$$|\Psi_0^{(sp)}\rangle = \hat{\beta}_{k=k_1}^\dagger{}^{N/2} \hat{\beta}_{k=k_2}^\dagger{}^{N/2} |0\rangle. \quad (7)$$

Notice that the ground state is built only with the field operators β associated to the lowest branch of the dispersion relation, and the problem has been mapped to an effectively one-dimensional one. In the presence of interactions, the effective one-dimensional Hamiltonian restricted to the lowest branch reads

$$\begin{aligned}\hat{H} &= \sum_k E_-(k) \hat{\beta}_k^\dagger \hat{\beta}_k \\ &+ \frac{U}{2N_s} \sum_{q,k,r} \kappa(k-q, r+q, k, r) \hat{\beta}_{k-q}^\dagger \hat{\beta}_{r+q}^\dagger \hat{\beta}_k \hat{\beta}_r,\end{aligned}\quad (8)$$

where the kernel $\kappa(k-q, r+q, k, r) = u_{k-q} u_{r+q} u_k u_r + v_{k-q} v_{r+q} v_k v_r$ is an effective interaction potential in momentum space, which has some involved momentum structure. However, if the ratio K/J is small, the parameters u_k and v_k can be approximated as constants for wave vector k close to k_1 and k_2 . In this case, for the sake of finding the ground state, the Hamiltonian is equivalent to the one of a one-dimensional Bose gas with contact interactions with a single-particle dispersion $E_-(k)$.

At increasing interaction strength, clearly the fragmented single-particle state ansatz (7) is not expected to describe the ground state of the system well, since repulsive interactions give rise to a spread in momentum occupancy. In the regime $U \rightarrow \infty$ of very strong repulsions, we predict an effective fermionization of the ground state, i.e., two particles cannot

occupy the same single-particle state and we propose the following fragmented Fermi-sea ansatz:

$$|\Psi_0^{(Fs)}\rangle = \prod_{-k_F+k_1 < k < k_F+k_1} \hat{c}_k^\dagger \prod_{-k_F+k_2 < k < k_F+k_2} \hat{c}_k^\dagger |0\rangle, \quad (9)$$

where k_F is the Fermi wave vector corresponding to $N/2$ particles and $\hat{c}_k^\dagger$ is the fermionic creation operator for the lower band of the noninteracting Hamiltonian (e.g., $\{\hat{c}_k^\dagger, \hat{c}_{k'}^\dagger\} = \delta_{k,k'}$, where $\{\cdot\}$ is the anticommutator). This ansatz is only valid as long as the Fermi energy is smaller than the energy of the upper band, i.e., $E_{\text{Fermi}} < E_+(k)$.

Fermionization approach

Using the fragmented Fermi sea ground state, in the $U = \infty$ limit we calculate the one-body density matrix and the momentum distribution of the gas using a mapping onto noninteracting fermions. In detail, we Fourier transform the ansatz into real space, then apply the Jordan-Wigner transformation [48]

$$\hat{b}_i^\dagger = \hat{c}_i^\dagger \prod_{j=1}^{i-1} e^{-i\pi \hat{c}_j^\dagger \hat{c}_j}, \quad \hat{b}_i = \prod_{j=1}^{i-1} e^{i\pi \hat{c}_j^\dagger \hat{c}_j} \hat{c}_i. \quad (10)$$

Hard-core bosons and noninteracting fermions have the same spectrum in one dimension. Differences between the two appear in off-diagonal correlation functions, e.g., in the one-body density matrix. Using the relation between the one-body density matrix ρ_{ij} and the one-particle Green function $G_{ij} = \langle \hat{b}_i \hat{b}_j^\dagger \rangle$,

$$\rho_{ij} = \langle \hat{b}_i^\dagger \hat{b}_j \rangle = G_{ij} + \delta_{ij}(1 - G_{ii}), \quad (11)$$

we calculate the one-body density matrix using a method developed by Rigol and Muramatsu [49]. In particular the one-particle Green function can be expressed in terms of our ansatz and fermionic operators according to

$$\begin{aligned} G_{ij} &= \langle \Psi_0^{(\text{HCB})} | \hat{b}_i \hat{b}_j^\dagger | \Psi_0^{(\text{HCB})} \rangle \\ &= \langle \Psi_0^{(Fs)} | \prod_{p=1}^{i-1} e^{i\pi \hat{c}_p^\dagger \hat{c}_p} \hat{c}_i \hat{c}_j^\dagger \prod_{s=1}^{j-1} e^{-i\pi \hat{c}_s^\dagger \hat{c}_s} | \Psi_0^{(Fs)} \rangle, \end{aligned} \quad (12)$$

where $|\Psi_0^{(\text{HCB})}\rangle$ is the ground state for hard-core bosons.

III. RESULTS

A. Fidelity with respect to fragmented states

In order to infer the nature of the ground state of the system we calculate the fidelity of the ground state obtained by exact diagonalization of the many-body Hamiltonian with the Lanczos algorithm and we project it onto the two ansatz states discussed in Sec. II, i.e., we take $F = |\langle \Psi_0 | \Psi_{\text{GS}} \rangle|^2$, where $|\Psi_0\rangle$ is either the single-particle fragmented state Eq. (7) or the Fermi-sea fragmented state Eq. (9). Notice that since we use a real-space basis for the numerical diagonalization we perform first a Fourier transform of the ansatz state onto real space.

Our results are shown in Fig. 1. In the left panel we show that the fidelity with respect to the single-particle fragmented state decreases at increasing interactions. In the right panel,

FIG. 1. Fidelity (dimensionless) of the ground state obtained from numerical diagonalization as a function of respectively interaction strength in units of tunnel energy J and flux ϕ (dimensionless): (a) and (c) with respect to the single-particle fragmented state $|\Psi_0^{(sp)}\rangle$ and (b) and (d) with respect to the fragmented Fermi sea $|\Psi_0^{(FS)}\rangle$. The parameters used are $L = 12$, $N = 6$. The values of K/J and ϕ are indicated in the figure legend for (a) and (b). For (c) and (d) interaction strength is specified in the figure legend and $K/J = 0.1$. A third dip occurs in (c) and (d), in correspondence with the third jump in the chiral current, but is not resolved in this figure.

we show that the fidelity with respect to the fragmented Fermi sea increases with interaction. For weak inter-ring coupling, we reach nearly unity fidelity for $U/J > 10$ for any flux, thus confirming the validity of our ansatz.

For strong inter-ring coupling, where the description of hard-core bosons breaks down, we find that the fidelity stays below 1 for any interaction and its value depends strongly on the choice of flux values. In this case there is no simple analytical description since bosons belonging to the lower band interact with a long-range interacting potential.

Fidelity at varying flux [see Figs. 1(c) and 1(d)] exhibits discontinuities associated to change of quantized momenta and jumps in the chiral current [see Fig. 2(a)]. At those transitions the eigenvalues of the single-particle density matrix are

FIG. 2. Chiral current j_c (in units of J) (a) as a function of flux ϕ (dimensionless) for $K/J = 1$ for different ring lengths L , (b) as a function of inter-ring coupling K (in units of J) for $\phi = 1/8$. In both panels we have taken $U = 0$ and half filling of the lattice.

FIG. 3. (a) Chiral current (in units of J) as a function of flux ϕ (dimensionless) for $K/J = 1$ and on-site interaction U/J as indicated in the legend. (b) Chiral current as function of inter-ring coupling K/J for $\phi = 1/8$ and interaction strength as indicated in the legend. The other parameters are $L = 12$ and $N = 6$ particles.

fourfold degenerate implying that the description in terms of the single-particle fragmented state (7) and fragmented Fermi sea (9) ansatz breaks down.

In the following, we provide an analysis of the observables characterizing the ground state of the system and identify the ones needed to infer the fragmented nature of the state in the vortex phase.

B. Currents

First, we show that the study of chiral currents can be used to identify univocally the vortex phase in parameter space, both in the interacting and noninteracting regimes.

The Hamiltonian (1) in the absence of interactions features the Meissner to vortex transition. At weak interactions, an additional biased ladder phase is found. At stronger interactions, the chiral Mott insulating phase with Meissner-like current and the vortex Mott insulating phase are predicted [10,16].

In Fig. 2(a) we show the chiral current as a function of the gauge field ϕ in the noninteracting case. The Meissner phase has an increasing chiral current, whereas the current decreases in the vortex phase. For finite-sized rings, the current acquires a step structure, each jump being associated to an integer change of the phase winding. This signals the formation of a vortex pair in the rings [38]. In Fig. 2(b) we show the chiral current at increasing inter-ring tunneling K/J . We see that a change of behavior occurs in the chiral currents in correspondence to the transition from the vortex phase at low values of K/J to the Meissner phase at large K/J : a jump in the chiral current is found in the finite-size system while the chiral current is continuous with the discontinuous derivative in the infinite-size limit. We expect the transition to be of first order in analogy to the case of spin-orbit-coupled bosons [50].

The chiral current for interacting systems is shown in Fig. 3. We see that even though interactions smooth out the steps of the current and reduce the positions of the steps to lower values of K/J , overall it is still possible to infer the vortex phase as the regime where the chiral current has a decreasing and oscillating behavior as a function of the flux ϕ . Similarly to the noninteracting case, the transition from vortex to Meissner phase is visible by studying the dependence of chiral current on inter-ring coupling K/J .

FIG. 4. Inter-ring current-current correlation function $\langle j_0^\perp j_m^\perp \rangle$ in units of J^2 at varying flux ϕ (dimensionless) and lattice position m for $L = 60$, $N_p = 2$, $U/J = 1$, and $K/J = 1$.

C. Current-current correlations

Another way to identify the vortex phase is the study of current-current correlations [51].

For linear ladders the ground state in the vortex phase is characterized by a vortex structure along the ladder, a modulation of the density along the legs, and a modulation of the current between rungs. In the case of coupled lattice rings, corresponding to periodic boundary conditions, these features are not visible since the ground state displays rotational invariance. In this case as the vortex phase sets in, the mentioned features are encoded in the correlation functions. In particular, the current-current correlation $\langle j_0^\perp j_x^\perp \rangle$, where $j_x^\perp$ is the current operator for the current between the rings, shows a clear vortex structure in Fig. 4.

D. Density-density correlations

Next, we show that density-density correlations of the ground state may be used to infer the onset of strong correlations and the fermionized regime.

A hallmark of fermionization is the presence of Friedel-like oscillations, characterized by a wave vector $2k_F$, with k_F the Fermi wave vector. These are found, e.g., in the density-density correlation function Δn_m , shown in Fig. 5 along one chosen ring (the same results are found for the other ring). For small inter-ring coupling, we observe the buildup of Friedel-like oscillations at increasing interaction, with wavelength corresponding to four times the lattice spacing, corresponding to the chosen average lattice filling in each ring. For large values of K/J , the system is not any more quasi-one-dimensional. In this case we find that the wavelength of Friedel oscillations changes with the applied flux.

E. One-body density matrix and momentum distribution

Finally, in this section we show that the study of the first-order correlations and momentum distribution allows us to obtain information about fragmentation.

The momentum distribution in each ring A and B is plotted in Fig. 6. We see that at weak inter-ring coupling the momentum distribution is centered in each of the two minima

FIG. 5. Density-density correlation function Δn_m (dimensionless) as a function of the position m along the lattice. (a) $K/J = 0.1, \phi = 1/4$; (b) $K/J = 0.1, \phi = 1/6$; (c) $K/J = 1, \phi = 1/4$; (d) $K/J = 1, \phi = 1/6$. The other parameters are $N = 6, L = 12$, and the flux ϕ is dimensionless.

$k = k_1$ or k_2 , corresponding to the applied gauge flux on each ring.

The momentum distribution in the diagonal basis, corresponding to occupation of lower and upper excitation branch in momentum space, is plotted in Figs. 7 and 8 for two choices of the inter-ring coupling. It displays a two-peak structure, corresponding to the two minima of the single-particle dispersion relation $k = k_1, k_2$. We notice first that most of the population occupies the lower branch, while the upper branch population is two orders of magnitude smaller. This validates the reduction to an effective one-dimensional system corresponding to the lower branch discussed in Sec. II. Focusing on the lower-branch momentum distribution, we notice that at strong interactions the momentum distribution broadens due to interaction effects and develops large-momentum tails characteristic of a strongly interacting regime. The peaks remain well defined even in the fermionized limit, as is typical of bosonic statistics, though their width coincides with the kinetic energy of the corresponding Fermi gas. The width of

FIG. 6. Momentum distribution (dimensionless) as a function of the wave vector k (in units of inverse of lattice spacing) in the leg basis for leg A with (a) $K/J = 1, \phi = 1/4$; (b) $\phi = 1/6$. The momentum distribution in ring B is the same graph reflected at zero momentum. The calculations are performed with $N = 6$ particles in total in the two rings and $L = 12$ sites per ring.

FIG. 7. Momentum distribution (dimensionless) as a function of the wave vector k (in units of inverse of lattice spacing) in the diagonal basis for weak inter-ring coupling $K/J = 0.1$ with (a), (b) $\phi = 1/4$; (c), (d) $\phi = 1/6$. Panels (a) and (c) show the lowest branch, (b) and (d) show the upper branch. We take $N = 6$ particles in total in the two rings and $L = 12$ sites per ring.

the momentum distribution for bosons and fermions is shown in Appendix A.

We finally analyze the single-particle density matrix (SPDM), which is independent of the chosen basis and whose Fourier transform yields the momentum distribution. By analyzing its eigenvalues, in the case of weak inter-ring coupling we find that for all values of interactions, it displays a double degeneracy of the two largest eigenvalues, as shown in Figs. 9 and 11. The degeneracy of the two largest eigenvalues of

FIG. 8. Momentum distribution (dimensionless) as a function of the wave vector k (in units of inverse of lattice spacing) in the diagonal basis for strong inter-ring coupling $K/J = 1$ with (a), (b) $\phi = 1/4$; (c), (d) $\phi = 1/6$. Panels (a) and (c) show the lowest branch, (b) and (d) show the upper branch. We take $N = 6$ particles in total in the two rings and $L = 12$ sites per ring.

FIG. 9. Two largest degenerate eigenvalues λ (superimposed, indicated by a single line) of the single-particle density matrix as a function of interaction strength U/J for different values of flux ϕ and inter-ring coupling K/J as indicated in the legend. The parameters used in the calculation are $L = 12$, $N = 6$.

the one-body density matrix provides a strong indication of fragmentation.

As already noticed in the study of other observables, for strong inter-ring coupling the quasi-one-dimensional description does not apply. We see in particular that the eigenvalues decay faster with increasing interactions, in a flux-dependent way, indicating the important role of the transverse direction in this case.

IV. SUMMARY AND CONCLUDING REMARKS

In this work we have studied the ground-state properties of two tunnel-coupled lattice rings in the quantum regime. In particular, we have shown that the ground state of the system is always fragmented at any interaction strength, and the nature of the fragmented state depends on the interaction strength: for weak interactions it consists of fragmentation among two single-particle states, while for strong interactions it corresponds to two fragmented Fermi seas. This Fermi seas description holds provided that the tunnel coupling between the two rings is sufficiently weak and the flux threading the system relatively strong. This allows for an analytical ansatz which well describes the limits of very weak or very strong interactions.

The information of the nature of the state can be inferred by combining the knowledge of various observables: the study of chiral currents and current-current correlation functions allows us to identify the vortex phase. By increasing interactions, the flux dependence of the currents across the transitions between states with different winding numbers is smoothed out; see Fig. 3. The density-density correlation function shows the onset to fermionization via the appearance of Friedel-like oscillations at large interactions, and a double-peak structure in the momentum distribution together with the demonstration of degenerate eigenvalues of the one-body density matrix establishes the fragmented nature of the state. In outlook, it would be interesting to explore the crossover from the quantum regime at very weak filling considered in this work and the mean-field Gross-Pitaevskii description used in the case of a very large number of bosons per lattice site.

FIG. 10. Width of the momentum distribution $\kappa = \sum_k k^2 n_k$ (in units of the inverse of the square of the lattice spacing) as a function of interaction strength U/J for various values of ϕ and K . The red dashed line is the corresponding value for noninteraction fermions. The calculations are performed for $N = 6$ particles, $L = 12$ sites.

ACKNOWLEDGMENTS

We acknowledge funding from the SuperRing ANR Project (Grant No. ANR-15-CE30-0012-02). The Grenoble LANEF framework (Grant No. ANR-10-LABX-51-01) is acknowledged for its support with mutualized infrastructure. We thank National Research Foundation Singapore and the Ministry of Education Singapore Academic Research Fund Tier 2 (Grant No. MOE2015-T2-1-101) for support.

APPENDIX A: MOMENTUM DISTRIBUTION WIDTH

We provide here some complementary material.

The width of the momentum distribution $\kappa = \sum_k k^2 n_k$ is shown in Fig. 10. At increasing interactions and small-inter-ring coupling it tends to the fermionic value, thereby further confirming the fermionized nature of the state.

FIG. 11. Single-particle density matrix eigenvalues λ_n (dimensionless) ordered in descending order n for (a) $K/J = 0.1$, $\phi = 1/4$; (b) $K/J = 0.1$, $\phi = 1/6$; (c) $K/J = 1$, $\phi = 1/4$; (d) $K/J = 1$, $\phi = 1/6$. First two eigenvalues are degenerate. Results for $N = 6$ particles, $L = 12$ sites.

APPENDIX B: DEGENERACY OF EIGENVALUES

In this appendix we provide further information about the eigenvalues of the reduced one-body density matrix of the double ring. In particular, Fig. 11 shows the eigenvalues in

decreasing order, at various values of the system parameters. One sees that the two largest eigenvalues are degenerate both at small and larger values of both K/J and ϕ , thus indicating fragmentation of the state for all choices of parameters investigated in this work.

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- [1] I. Bloch, J. Dalibard, and W. Zwerger, *Rev. Mod. Phys.* **80**, 885 (2008).
 - [2] M. Kardar, *Phys. Rev. B* **33**, 3125 (1986).
 - [3] E. Orignac and T. Giamarchi, *Phys. Rev. B* **64**, 144515 (2001).
 - [4] A. Petrescu and K. Le Hur, *Phys. Rev. B* **91**, 054520 (2015).
 - [5] M. Atala, M. Aidelsburger, M. Lohse, J. T. Barreiro, B. Paredes, and I. Bloch, *Nat. Phys.* **10**, 588 (2014).
 - [6] M. Mancini *et al.*, *Science* **349**, 1510 (2015).
 - [7] B. Stuhl *et al.*, *Science* **349**, 1514 (2015).
 - [8] F. A. An, E. J. Meier, and B. Gadway, *Sci. Adv.* **3**, e1602685 (2017).
 - [9] L. F. Livi, G. Cappellini, M. Diem, L. Franchi, C. Clivati, M. Frittelli, F. Levi, D. Calonico, J. Catani, M. Inguscio, and L. Fallani, *Phys. Rev. Lett.* **117**, 220401 (2016).
 - [10] A. Petrescu and K. Le Hur, *Phys. Rev. Lett.* **111**, 150601 (2013).
 - [11] A. Tokuno and A. Georges, *New J. Phys.* **16**, 073005 (2014).
 - [12] R. Wei and E. J. Mueller, *Phys. Rev. A* **89**, 063617 (2014).
 - [13] M. Di Dio *et al.*, *Eur. Phys. J.: Spec. Top.* **224**, 525 (2015).
 - [14] E. Orignac *et al.*, *New J. Phys.* **18**, 055017 (2016).
 - [15] S. Greschner, M. Piraud, F. Heidrich-Meisner, I. P. McCulloch, U. Schollwöck, and T. Vekua, *Phys. Rev. Lett.* **115**, 190402 (2015).
 - [16] M. Piraud, F. Heidrich-Meisner, I. P. McCulloch, S. Greschner, T. Vekua, and U. Schollwöck, *Phys. Rev. B* **91**, 140406 (2015).
 - [17] J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, *Rev. Mod. Phys.* **83**, 1523 (2011).
 - [18] K. C. Wright, R. B. Blakestad, C. J. Lobb, W. D. Phillips, and G. K. Campbell, *Phys. Rev. Lett.* **110**, 025302 (2013).
 - [19] A. Ramanathan, K. C. Wright, S. R. Muniz, M. Zelan, W. T. Hill, III, C. J. Lobb, K. Helmerson, W. D. Phillips, and G. K. Campbell, *Phys. Rev. Lett.* **106**, 130401 (2011).
 - [20] C. Ryu, P. W. Blackburn, A. A. Blinova, and M. G. Boshier, *Phys. Rev. Lett.* **111**, 205301 (2013).
 - [21] S. Eckel *et al.*, *Nature (London)* **506**, 200 (2014).
 - [22] A. I. Yakimenko, Y. M. Bidasyuk, M. Weyrauch, Y. I. Kuriatnikov, and S. I. Vilchinskii, *Phys. Rev. A* **91**, 033607 (2015).
 - [23] D. W. Hallwood, K. Burnett, and J. Dunningham, *New J. Phys.* **8**, 180 (2006).
 - [24] D. Solenov and D. Mozyrsky, *Phys. Rev. Lett.* **104**, 150405 (2010).
 - [25] C. Schenke, A. Minguzzi, and F. W. J. Hekking, *Phys. Rev. A* **84**, 053636 (2011).
 - [26] L. Amico, D. Aghamalyan, F. Auksztol, H. Crepaz, R. Dumke, and L. C. Kwek, *Sci. Rep.* **4**, 4298 (2014).
 - [27] D. Aghamalyan *et al.*, *New J. Phys.* **17**, 045023 (2015).
 - [28] D. Aghamalyan *et al.*, *New J. Phys.* **18**, 075013 (2016).
 - [29] A. C. Mathey and L. Mathey, *New J. Phys.* **18**, 055016 (2016).
 - [30] T. Haug, H. Heimonen, R. Dumke, L.-C. Kwek, and L. Amico, *arXiv:1706.05180*.
 - [31] T. Haug, J. Tan, M. Theng, R. Dumke, L.-C. Kwek, and L. Amico, *Phys. Rev. A* **97**, 013633 (2018).
 - [32] T. Haug, R. Dumke, L.-C. Kwek, and L. Amico, *arXiv:1807.03616*.
 - [33] B. T. Seaman, M. Krämer, D. Z. Anderson, and M. J. Holland, *Phys. Rev. A* **75**, 023615 (2007).
 - [34] L. Amico, A. Osterloh, and F. Cataliotti, *Phys. Rev. Lett.* **95**, 063201 (2005).
 - [35] R. Dumke *et al.*, *J. Opt.* **18**, 093001 (2016).
 - [36] L. Amico, G. Birkel, M. Boshier, and L.-C. Kwek, *New J. Phys.* **19**, 020201 (2017).
 - [37] D. Aghamalyan, L. Amico, and L. C. Kwek, *Phys. Rev. A* **88**, 063627 (2013).
 - [38] N. Victorin, F. Hekking, and A. Minguzzi, *Phys. Rev. A* **98**, 053626 (2018).
 - [39] T. Haug, L. Amico, R. Dumke, and L.-C. Kwek, *Quantum Sci. Technol.* **3**, 035006 (2018).
 - [40] O. Penrose and L. Onsager, *Phys. Rev.* **104**, 576 (1956).
 - [41] P. Nozières and D. Saint James, *J. Phys. (France)* **43**, 1133 (1982).
 - [42] E. J. Mueller, T.-L. Ho, M. Ueda, and G. Baym, *Phys. Rev. A* **74**, 033612 (2006).
 - [43] T.-L. Ho and S. K. Yip, *Phys. Rev. Lett.* **84**, 4031 (2000).
 - [44] N. K. Wilkin, J. M. F. Gunn, and R. A. Smith, *Phys. Rev. Lett.* **80**, 2265 (1998).
 - [45] E. Kawasaki and M. Holzmann, *Phys. Rev. A* **95**, 051601 (2017).
 - [46] A. R. Kolovsky, *Phys. Rev. A* **95**, 033622 (2017).
 - [47] The consequences of other choices for ϕ_A and ϕ_B are discussed in [38].
 - [48] P. Jordan and E. Wigner, *Z. Phys.* **47**, 631 (1928).
 - [49] M. Rigol and A. Muramatsu, *Phys. Rev. A* **72**, 013604 (2005).
 - [50] Y. Li, L. P. Pitaevskii, and S. Stringari, *Phys. Rev. Lett.* **108**, 225301 (2012).
 - [51] M.-C. Cha and J.-G. Shin, *Phys. Rev. A* **83**, 055602 (2011).