Shimura Subgroups and Degeneracy Maps

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For $M \geqslant 1$ an integer and M' a positive divisor of M, let $\phi: J_0(M')^\tau \to J_0(M)$ be the map defined by all the degeneracy maps, where τ is the number of positive divisors of M/M'. We determine the kernel of ϕ for certain M and M', as well as relate the pre-image of the Shimura subgroup $\Sigma(M)$ under ϕ to the group $\Sigma(M')^\tau$. We also study the restriction of degeneracy maps to Shimura subgroups.

Let $M \geqslant 1$ be an integer. The congruence subgroups $\Gamma_0(M)$ and $\Gamma_1(M)$ act on the Poincaré upper half-plane $\mathcal{H}$ and $\bar{\mathcal{H}} = \mathcal{H} \cup \mathbf{P}^1(\mathbf{Q})$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) \mapsto \frac{a\tau + b}{c\tau + d}.$$

The quotients $\Gamma_i(M) \backslash \bar{\mathcal{H}}$ $(i = 0, 1)$ are the classical modular curves $X_i(M)$. The inclusion $\Gamma_1(M) \subset \Gamma_0(M)$ induces a natural morphism $u: X_1(M) \to X_0(M)$. By Pic functoriality, u induces a morphism $u^*: J_0(M) \to J_1(M)$ between the Jacobian varieties. The kernel of u^*, which is a finite abelian group, is the Shimura subgroup $\Sigma(M)$. This group and its properties are studied in [9]. We recall in Section 2 some of the facts we need.

If M' is a divisor of M, and D is a divisor of M/M', then one can define the degeneracy map $v_D: X_0(M) \to X_0(M')$. The modular description of v_D on $\Gamma_0(M) \backslash \mathcal{H}$ is given by

$$v_D([E, C]) = [E/C_D, C_{M'D}/C_D],$$

where E is an elliptic curve over $\mathbf{C}$, C a cyclic subgroup of E of order M, $[E, C]$ denotes the isomorphism class of such a pair, and C_D and $C_{M'D}$ are subgroups of C of orders D and $M'D$, respectively. The map v_D induces, via

Pic functoriality, the map $v_D^*: J_0(M') \to J_0(M)$ between the Jacobian varieties. It also induces, via Albanese functoriality, the map $(v_D)_*$: $J_0(M) \to J_0(M')$.

The objective of this paper is to study some relationships between the Shimura subgroup $\Sigma(M)$ and the degeneracy maps v_D. Our discussion may be divided into three parts.

Let N be a positive integer and let $p \geqslant 5$ be a prime not dividing N. Let γ be the map

$$\gamma = v_1^* \times \cdots \times v_{p^{r-1}}^*: J_0(Np)^r \to J_0(Np^r), \qquad r \geqslant 2.$$

Let B be the p-new subvariety of $J_0(Np)$. By definition, B is the identity component of the intersection of the kernels of

$$(v_1)_*: J_0(Np) \to J_0(N) \qquad \text{and} \qquad (v_p)_*: J_0(Np) \to J_0(N).$$

Let $\Phi_{Np,p}$, $\Phi_{Np^r,p}$ and $\Phi(B)_p$ denote the component groups of the special fibre of the Néron model of $J_0(Np)$, $J_0(Np^r)$ and B, respectively, over $\mathbf{Z}_p$.

In Section 1, we relate the kernel of the restriction of γ to B^r to the component group $\Phi(B)_p^r$. More precisely, we prove

THEOREM 1. *There is a natural inclusion of the group $K_B \overset{\text{def}}{=} (\ker \gamma) \cap B^r$ into the kernel of $\bar{\gamma}: \Phi(B)_p^r \to \Phi_{Np^r,p}$, where $\bar{\gamma}$ is the map induced from the restriction of γ to B^r. This inclusion is compatible with the Hecke operators T_n for $n \neq p$.*

While Theorem 1 provides an upper bound for K_B, we shall also give in subsection 1.2 a lower bound for K_B in terms of $\Sigma(Np)$.

After recalling some facts on the Shimura subgroups and component groups in subsection 2.1, we use Theorem 1 to obtain in subsection 2.2 a generalisation of a result in [7].

THEOREM 2. *If $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$ and $p \geqslant 5$ is a prime not dividing N, then the kernel K_γ of the map*

$$\gamma = v_1^* \times \cdots \times v_{p^{r-1}}^*: J_0(Np)^r \to J_0(Np^r), \qquad r \geqslant 2,$$

is exactly

$$K_0 \overset{\text{def}}{=} \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \,\middle|\, x_i \in \Sigma(Np) \quad \text{for all} \quad i, \sum x_i = 0 \right\}.$$

The special case where $N = 1$ is Theorem 2 of [7]. Theorem 2 is then applied in Section 3 to establish congruence relations between cusp forms on $\Gamma_0(Np)$ and $\Gamma_0(Np^2)$, for $1 \leq N \leq 9$, $N \neq 7$.

In the second part of the paper, we begin by proving in Section 4

THEOREM 3. *Let $L \geq 1$ be an integer and let q be a prime not dividing L. Let η be the map*

$$\eta = v_1^* \times v_q^* : J_0(L) \times J_0(L) \to J_0(Lq).$$

(i) *If L is odd, then $\Sigma(L) \times \Sigma(L) = \eta^{-1}(\Sigma(Lq))$.*

(ii) *If L is even, then the index of $\Sigma(L) \times \Sigma(L)$ in $\eta^{-1}(\Sigma(Lq))$ is at most two.*

In Section 5, some consequences of Theorem 3 are discussed. We obtain, in particular, the following

THEOREM 4. *Let L be a positive integer, and let $M = q_1 \cdots q_t$ (q_i distinct primes) be such that $(L, M) = 1$. Let*

$$\Sigma(L)_0^{2^t} \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_{2^t} \end{pmatrix} \;\middle|\; x_i \in \Sigma(L) \quad \text{for all} \quad i, \sum x_i = 0 \right\}. \tag{1}$$

Let K_ϕ be the kernel of $\phi : J_0(L)^{2^t} \to J_0(LM)$.

(i) *If L is odd or $M = q_1$ is a prime, then $K_\phi = \Sigma(L)_0^{2^t}$.*

(ii) *If L is even and M is not a prime, then K_ϕ and $\Sigma(L)_0^{2^t}$ are equal up to a 2-group.*

Theorem 4 is essentially a generalisation of Theorem 4.3 of [12], which deals with the case where M is a prime. It should be noted that this theorem is used in the proof of Theorem 4.

Finally, we turn our attention to study the restriction of degeneracy maps to Shimura subgroups. For example, we show that, if $M_1, \ldots, M_r$ are pairwise relatively prime positive integers, then the map

$$\alpha : \Sigma(M_1) \times \cdots \times \Sigma(M_r) \to \Sigma(M_1 \cdots M_r N), \qquad N \geq 1,$$

where $\alpha \mid_{\Sigma(M_i)}$ is any degeneracy map from $\Sigma(M_i)$ to $\Sigma(M_1 \cdots M_r N)$, is injective. We also determine (up to a group of exponent dividing six) the kernel of α when $r = 2$ and the assumption $(M_1, M_2) = 1$ is dropped. A few corollaries of these results are also discussed.

1. The Kernel K_B

1.1. *Proof of Theorem* 1

We first note that γ restricted to B^r has a finite kernel. Indeed, the map $\gamma \mid_{B^r} \colon B^r \to J_0(Np^r)$ has a finite kernel if and only if the dual map is surjective, and the latter map is surjective if and only if its pullback on the differentials is injective. This last map is the map

$$(p\text{-new part of } S_2(Np))^r \to S_2(Np^r), \tag{2}$$

induced by the degeneracy maps, where $S_2(Np)$ (resp. $S_2(Np^r)$) denotes the weight-2 cusp forms on $\Gamma_0(Np)$ (resp. $\Gamma_0(Np^r)$). The injectivity of (2) is well-known (cf. [1], for example).

PROPOSITION 1. *The p-new subvariety B of $J_0(Np)$ has purely toric reduction at p; it is the maximal torus in the reduction of $J_0(Np)$ modulo p.*

Proof. Let C be the p-new quotient of $J_0(Np)$ (i.e., the quotient $J_0(Np)/((v_1^* \times v_p^*)(J_0(N)^2)))$. Then B and C are isogenous. Therefore, to show that B has purely toric reduction mod p, it suffices to show the same for C.

Results of Raynaud [11] and Deligne–Rapoport [3] yield an exact sequence

$$0 \to T(Np) \to J^o \to J_0(N)_{/\mathbf{F}_p} \times J_0(N)_{/\mathbf{F}_p} \to 0, \tag{3}$$

where J^o is the connected component of identity in the reduction mod p of $J_0(Np)$, $T(Np)$ is the maximal torus in J^o, and $J_0(N)_{/\mathbf{F}_p}$ is the special fibre of the Néron model of $J_0(N)$ over $\mathbf{Z}_p$.

The two degeneracy maps $v_1, v_p \colon X_0(Np) \to X_0(N)$ also give the standard degeneracy map $v_1^* \times v_p^* \colon J_0(N) \times J_0(N) \to J_0(Np)$. Passing to characteristic p, we obtain a map

$$v_1^* \times v_p^* \colon J_0(N)_{/\mathbf{F}_p} \times J_0(N)_{/\mathbf{F}_p} \to J^o. \tag{4}$$

Combining (3) and (4), we obtain the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & J_0(N)_{/\mathbf{F}_p} \times J_0(N)_{/\mathbf{F}_p} & \overset{\alpha}{\longrightarrow} & J_0(N)_{/\mathbf{F}_p} \times J_0(N)_{/\mathbf{F}_p} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle v_1^* \times v_p^*} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & T(Np) & \longrightarrow & J^o & \longrightarrow & J_0(N)_{/\mathbf{F}_p} \times J_0(N)_{/\mathbf{F}_p} & \longrightarrow & 0,
\end{array}
\tag{5}
$$

where the map α is the endomorphism of $J_0(N)_{/\mathbf{F}_p} \times J_0(N)_{/\mathbf{F}_p}$ given by the matrix $\left(\begin{smallmatrix} 1 & V \\ V & 1 \end{smallmatrix}\right)$, where V is the Verschiebung endomorphism of $J_0(N)_{/\mathbf{F}_p}$

(cf. [13]), and F is the kernel of α. Since α is an isogeny, F is a finite group.

Applying the Snake Lemma to (5), we see immediately that the cokernel of the map $v_1^* \times v_p^*\colon J_0(N)_{/\mathbf{F}_p} \times J_0(N)_{/\mathbf{F}_p} \to J^o$ is isomorphic to the torus $T(Np)/F$. Since this cokernel is isogenous to the reduction at p of the p-new quotient C ([2], §7.3), it follows that C and hence B have purely toric reduction $T(B)$ at p.

Passing to the Néron model, the inclusion $B \hookrightarrow J_0(Np)$ (in characteristic 0) induces a map $T(B) \to J^o$. Since B has semi-stable reduction, results of Raynaud [11] imply that the map $T(B) \to J^o$ is an injection, so $T(B)$ is a sub-torus of J^o. However, since $J^o/T(B)$ is an abelian variety, we conclude that $T(B)$ must be the maximal torus in the reduction modulo p of $J_0(Np)$. ∎

The intersection of the kernels of the degeneracy maps $(v_1)_*, \dots, (v_{p^r})_*\colon$ $J_0(Np^r) \to J_0(N)$ cuts out a subvariety of $J_0(Np^r)$, called the p-new subvariety of $J_0(Np^r)$. Since B has purely toric reduction at p, while $J_0(N)$ has good reduction at p, it follows that the image of B^r under γ lies in the p-new subvariety of $J_0(Np^r)$.

Since $\gamma\colon J_0(Np)^r \to J_0(Np^r)$ induces an injection $T(Np)^r \hookrightarrow T(Np^r)$ on the tori ([7], Theorem 1), the restriction $\gamma\mid_{B^r}\colon B^r \to J_0(Np^r)$ also induces an injection $T(B)^r \hookrightarrow T(Np^r)$ in view of Proposition 1. It follows that there is an injection $\underline{T}(B)^r \hookrightarrow \underline{T}(Np^r)$, where $\underline{T}(B)$ (resp. $\underline{T}(Np^r)$) denotes the lift of $T(B)$ (resp. $T(Np^r)$) to B (resp. $J_0(Np^r)$) in the same sense as in [7]. Since K_B has trivial intersection with $\underline{T}(B)^r$, the argument in [7] implies that K_B extends to a constant (finite, flat) group scheme $\mathbf{K}_B$ over $\mathbf{Z}_p$, which embeds in the Néron model of B^r, and that $\mathbf{K}_B$ has trivial intersection with $T(B)^r$. Therefore, there is a natural inclusion of K_B into $\Phi(B)_p^r$. Since K_B is the kernel of $\gamma\mid_{B^r}$ by definition, the image of K_B in $\Phi(B)_p^r$ actually lies in the kernel of the map $\bar{\gamma}\colon \Phi(B)_p^r \to \Phi_{Np^r, p}$ induced from $\gamma\mid_{B^r}$.

It is clear that this inclusion is compatible with the action of the Hecke operators T_n for $n \neq p$.

This completes the proof of Theorem 1.

1.2. A Lower Bound for K_B

While Theorem 1 provides an upper bound for K_B, the following proposition gives a lower bound for K_B.

PROPOSITION 2. *The kernel K_B contains the group*

$$(\Sigma(Np) \cap B)_0^r \overset{\text{def}}{=} \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \;\middle|\; x_i \in \Sigma(Np) \cap B \quad \text{for all} \quad i, \sum x_i = 0 \right\}.$$

Up to a group of exponent two, the group $\Sigma(Np) \cap B$ is canonically isomorphic to the group of homomorphisms $g: (\mathbf{Z}/N\mathbf{Z})^\times \times (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{U}$ (where $\mathbf{U}$ is the group of complex numbers of modulus 1) such that $g(d) = 1$ if $d = -1$, $d^2 + 1 = 0$, $d^2 + d + 1 = 0$, $(d-1)^2 = 0$ or

$$d \equiv \begin{cases} e^{p+1} \mod N & \textit{for some} \quad e \in (\mathbf{Z}/N\mathbf{Z})^\times \\ 1 & \mod p \end{cases}$$

Proof. Since the degeneracy maps coincide on $\Sigma(Np)$ ([9], Theorem 4), the first statement of the proposition follows immediately.

For the desciption of $\Sigma(Np) \cap B$, we observe first that $\Sigma(Np)$ is canonically isomorphic to the group of homomorphisms $g: (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathbf{U}$ such that $g(d) = 1$ if $d = -1$, $d^2 + 1 = 0$, $d^2 + d + 1 = 0$ or $(d-1)^2 = 0$ ([9], Theorem 1).

Recall that B is the identity component of the kernel of

$$\eta^\wedge : J_0(Np) \to J_0(N) \times J_0(N),$$

where $\eta^\wedge(x) = ((v_1)_* x, (v_p)_* x)$. In fact, this kernel is the extension of a finite group, canonically isomorphic to the Cartier dual $\Sigma(N)^\wedge$ of $\Sigma(N)$, by the abelian variety B (cf. [12]). By considering the Galois action on $\Sigma(Np) \cap \ker \eta^\wedge$ and that on $\Sigma(N)^\wedge$, we conclude that the image of any map from $\Sigma(Np) \cap \ker \eta^\wedge$ to $\Sigma(N)^\wedge$ has exponent at most two. Consequently,

$$2(\Sigma(Np) \cap \ker \eta^\wedge) \subseteq \Sigma(Np) \cap B \subseteq \Sigma(Np) \cap \ker \eta^\wedge.$$

Finally, using [9] Theorem 5, it follows that $\Sigma(Np) \cap \ker \eta^\wedge$ consists of the elements of $\Sigma(Np)$ (after identifying $\Sigma(Np)$ with a subgroup of $\mathrm{Hom}((\mathbf{Z}/Np\mathbf{Z})^\times, \mathbf{U})$) that satisfy the extra condition that

$$g(d) = 1 \quad \text{if} \quad d \equiv \begin{cases} e^{p+1} \mod N & \text{for some} \quad e \in (\mathbf{Z}/N\mathbf{Z})^\times \\ 1 & \mod p \end{cases} \quad \blacksquare$$

2. The Kernel K_y

2.1. *Some Facts on Shimura Subgroups and Component Groups*

For an integer $M \geqslant 1$, with prime power decomposition $M = \prod p^{r_p}$, we introduce the following notations:

(i)   let $m(M)$ be the largest integer such that $m(M)^2$ divides M;

(ii)   let $k(M)$ be the number of prime divisors of M distinct from 2 and 3;

(iii)   let $m_1(M)$ be 2 if M is the product of 1, 2 or 4 by a power of an odd prime, and let it be 1 otherwise;

(iv)   let $m_2(M)$ be 2 if -1 is a square mod M, and let it be 1 otherwise;

(v)   let $m_3(M)$ be 3 if $X^2 + X + 1$ has a root mod M, and let it be 1 otherwise;

(vi)   let $r'_p = r_p - 1 - [r_p/2]$ if $p \neq 2$;

(vii)   let $r'_2 = \max(0, r_2 - 2 - [r_2/2])$;

(viii)   let $e_0(M) = \mathrm{lcm}_{p \mid M} ((p-1) p^{r'_p})$.

We write, for example, m for $m(M)$, when no ambiguity is involved.
Then the order of the Shimura subgroup $\Sigma(M)$ is given by [9]

$$\mathrm{card}(\Sigma(M)) = \begin{cases} \phi(M)/2mm_2^k m_3^k & \text{if } M \geqslant 5 \\ 1 & \text{if } M \leqslant 4, \end{cases} \qquad (6)$$

and the exponent of $\Sigma(M)$ is given by (*loc. cit.*)

$$e(\Sigma(M)) = \begin{cases} e_0/(m_1 m_2 m_3) & \text{if } M \geqslant 5 \\ 1 & \text{if } M \leqslant 4. \end{cases} \qquad (7)$$

When $M = Np$, where $p > 3$ is a prime not dividing N, the description of $\Phi_{Np, p}$, the component group of the special fibre of the Néron model of $J_0(Np)$ over $\mathbf{Z}_p$, is given in [6, Section 4.4]. Using this description in [6] and the facts about Shimura subgroups above, we may obtain

PROPOSITION 3.   *If* $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$ *and* $p \geqslant 5$ *is a prime not dividing* N, *both the Shimura subgroup* $\Sigma(Np)$ *and the component group* $\Phi_{Np, p}$ *at* p *are cyclic, and they have the same order.*

Proof.   From [10], it follows that $\Sigma(p)$ and $\Phi_{p, p}$ are both cyclic of order $(p-1)/(p-1, 12)$.

For $N \neq 1$, using (6), (7) and the description in [6], we obtain the following table:

N	$p \bmod 12$	$\operatorname{card}(\Sigma(Np))$	$e(\Sigma(Np))$	$\operatorname{card}(\Phi_{Np,p})$
2	1,5	$(p-1)/4$	$(p-1)/4$	$(p-1)/4$
	7, 11	$(p-1)/2$	$(p-1)/2$	$(p-1)/2$
3	1, 7	$(p-1)/3$	$(p-1)/3$	$(p-1)/3$
	5, 11	$p-1$	$p-1$	$p-1$
4	1, 5, 7, 11	$(p-1)/2$	$(p-1)/2$	$(p-1)/2$
5	1, 5	$(p-1)/2$	$(p-1)/2$	$(p-1)/2$
	7, 11	$2(p-1)$	$2(p-1)$	$2(p-1)$
6	1, 5, 7, 11	$p-1$	$p-1$	$p-1$
8	1, 5, 7, 11	$p-1$	$p-1$	$p-1$
9	1, 5, 7, 11	$p-1$	$p-1$	$p-1$

The description in [6] also implies that $\Phi_{Np,p}$ is cyclic in all these cases. ∎

2.2. *Proof of Theorem* 2

Now as a corollary of Theorem 1, we deduce Theorem 2.

For the rest of this section we assume $1 \leqslant N \leqslant 9$, $N \neq 7$, and $p \geqslant 5$ is a prime not dividing N.

First, we note that under these assumptions, we have $J_0(N) = 0$, and so the p-new subvariety B is $J_0(Np)$. Therefore, K_y coincides with the group K_B.

Since the degeneracy maps $v_D^*: J_0(Np) \to J_0(Np^r)$ all coincide on $\Sigma(Np)$ ([9], Theorem 4), it follows that $K_0 \subseteq K_y$.

By Theorem 1, there is a natural inclusion of K_y into the kernel of $\tilde{\gamma}: \Phi_{Np,p}^r \to \Phi_{Np^r,p}$ (since $B = J_0(Np)$). Let

$$
\begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \in \ker \tilde{\gamma}.
$$

Composing $\tilde{\gamma}$ with the map $(v_1)_*: \Phi_{Np^r,p} \to \Phi_{Np,p}$ induced by v_1 via Albanese functoriality, we have $(v_1)_* v_1^* x_1 + \cdots (v_1)_* v_{p^{r-1}}^* x_r = 0$. Since $v_1: X_0(Np^r) \to X_0(Np)$ has degree p^{r-1}, the map $(v_1)_* v_1^*$ is multiplication by p^{r-1}. Since p^{r-1} is relatively prime to $\operatorname{card}(\Phi_{Np,p})$ (Proposition 3), $x_2, \ldots, x_r$ completely determine x_1. Therefore, K_y is contained in a subgroup

of $\Phi^r_{Np, p}$ of order $(\operatorname{card}(\Phi_{Np, p}))^{r-1}$. Since $\Sigma(Np)$ and $\Phi_{Np, p}$ have the same cardinality (cf. Proposition 3), we have that $\operatorname{card}(K_0) = (\operatorname{card}(\Phi_{Np, p}))^{r-1}$, so $K_p = K_0$.

This completes the proof of Theorem 2.

3. Congruence Relations between Cusp Forms on $\Gamma_0(Np)$ and $\Gamma_0(Np^2)$ $(p \nmid N)$

Let $f = \sum a_n q^n$ be a normalised weight-2 Hecke eigenform of level N. Let l be a prime not dividing N. The coefficients a_n are in a finite extension E of $\mathbf{Q}$ contained in $\mathbf{C}$. Let $\mathcal{O}_E$ be the ring of integers in E. Choose a prime ideal λ in $\mathcal{O}_E$ of residue characteristic l, and fix a homomorphism $\pi: \mathcal{O}_E \to \bar{\mathbf{F}}_l$ with kernel λ. By a theorem of Deligne ([4], Théorème 6.7), there is a semi-simple continuous representation

$$\rho_f: \operatorname{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\bar{\mathbf{F}}_l), \tag{8}$$

with the following property: the representation ρ_f is unramified at all primes q not dividing lN and, for each such q, let Frob_q denote the corresponding Frobenius element, then we have

$$\operatorname{Tr} \rho_f(\operatorname{Frob}_q) = \pi(a_q)$$

and

$$\det \rho_f(\operatorname{Frob}_q) \equiv q \bmod l.$$

In this section we apply Theorem 2 to prove

THEOREM 5. *Let* $f = \sum a_n q^n$ *be a normalised weight-2 Hecke eigenform of level Np, where $1 \leqslant N \leqslant 9$, $N \neq 7$ and $p \geqslant 5$ is a prime not dividing N. Let $l > 3$ be a prime such that $p^2 \equiv 1 \bmod l$ and $l \nmid N$. Assume that the associated Galois representation ρ_f is irreducible. Then there exists a weight-2 newform g of level Mp^2; where M divides N, such that the associated Galois representation ρ_g is isomorphic to ρ_f.*

The case where $N = 1$ has been studied carefully in [7, Section 3]. The argument there can be generalised to our present situation. Therefore we shall only present a sketch of the proof of Theorem 5. We shall also assume that $1 \leqslant N \leqslant 9$, $N \neq 7$ and $p \geqslant 5$ is a prime not dividing N for the rest of this section.

However, we should remark here that the congruences established in this section have recently been proved independently by Diamond and Taylor in [5] in greater generality. The method outlined below is different from

that in [5] and provides an additional perspective on the problem of establishing congruences between cusp forms in certain cases.

Let X be the image of

$$\gamma\colon J_0(Np)^2 \to J_0(Np^2).$$

Consider the map

$$J_0(Np^2) \xrightarrow{\ \phi_\Theta\ } J_0(Np^2)^\vee \xrightarrow{\ \iota^\vee\ } X^\vee, \tag{9}$$

where $\iota^\vee$ is dual to the inclusion $\iota\colon X \hookrightarrow J_0(Np^2)$, and ϕ_Θ denotes the canonical polarisation of $J_0(Np^2)$. Let Y be the kernel of this composition of maps in (9). It is known that Y is a subvariety of $J_0(Np^2)$, that $X + Y = J_0(Np^2)$ and that $X \cap Y$ is finite.

Let l be as in the statement of Theorem 5, and let $\mathfrak{m}$ be a maximal ideal of $\mathbf{T}$ of residue characteristic l. (Here, $\mathbf{T}$ denotes the ring generated by the Hecke operators on the space of weight-2 cusp forms on $\Gamma_0(Np^2)$.) From Theorem 5.2 of [15], the representation ρ_f can be regarded as arising from the subspace

$$V = J_0(Np)[\mathfrak{m}] = \bigcap_{\alpha \in \mathfrak{m}} J_0(Np)[\alpha]$$

of $J_0(Np)[l]$. Borrowing an argument from [14], to prove Theorem 5, it suffices to show that there is an inclusion $V \hookrightarrow Y$ over $\mathbf{Q}$ that is compatible with the action of the Hecke operators T_n for $n \neq p$.

The canonical Θ polarisation on $J_0(Np^2)$ induces a line bundle $\mathscr{L}$ on X, which in turn induces $\delta^*\mathscr{L}$ on $J_0(Np)^2$, where $\delta\colon J_0(Np)^2 \to X$ is the isogeny defined by γ. We take Δ to be $\ker(\phi_{\delta^*\mathscr{L}})$, where $\phi_{\delta^*\mathscr{L}}\colon J_0(Np)^2 \to J_0(Np)^2$ is the isogeny induced by $\delta^*\mathscr{L}$. Then $K_\gamma \subseteq \Delta$, and there is a canonical pairing on $\Delta \times \Delta$, trivial on $K_\gamma \times K_\gamma$. If $K_\gamma^\perp$ denotes the orthogonal complement of K_γ under this pairing, then $K_\gamma \subseteq K_\gamma^\perp$. If Ω is the kernel of $\phi_{\mathscr{L}}$, then we have further $K_\gamma^\perp/K_\gamma = \Omega$, and the support of Ω consists of primes of fusion.

Let w_p be the Atkin–Lehner involution on $X_0(Np)$ (and hence on $J_0(Np)$), defined in modular terms by

$$w_p([E, C]) = [E/C_p, (E[p] + C)/C_p],$$

where C_p is the unique subgroup of order p in C. It is easy to see that $w_p + T_p$ (acting on $J_0(Np)$) factors through the map $(v_p)_*\colon J_0(Np) \to J_0(N)$ (see, for example, proof of [15, Prop. 3.7]). Since $J_0(N) = 0$ in our situation, we have $w_p = -T_p$ on $J_0(Np)$.

The argument in [7] then shows that

$$\Delta = \left\{ \begin{pmatrix} w_p py \\ y \end{pmatrix} \;\middle|\; y \in J_0(Np)[p^2 - 1] \right\}.$$

Since $V = J_0(Np)[m] \subseteq J_0(Np)[l] \subseteq J_0(Np)[p^2 - 1]$, it follows that there is an inclusion $V \subsetneq \Delta$.

Lemma 1. *The inclusion $V \subsetneq \Delta$ above induces an inclusion $V \subsetneq \Delta/K_y$.*

Proof. Let $y \in V$. It suffices to show that if $\binom{w_p py}{y} \in K_y$, then $y = 0$.

From Theorem 2, $\binom{w_p py}{y} \in K_y$ implies $y \in \Sigma(Np)$, so $y \in \Sigma(Np)[m]$.

Let I be the ideal in T generated by the elements $T_r - (1 + r)$, with r prime and $r \nmid Np$. Then Theorem 6 of [9] shows that $\Sigma(Np)$ is annihilated by I. However, Theorem 5.2(c) of [15] shows that m and I are relatively prime, so $\Sigma(Np)[m] = 0$, i.e., $y = 0$. ∎

Lemma 2. *The image of the inclusion $V \subsetneq \Delta/K_y$ lies in the subgroup $K_y^{\perp}/K_y$.*

Proof. We need to prove that the image of V in $\Delta/K_y^{\perp}$ is trivial.

The group $\Delta/K_y^{\perp}$ is the G_m-dual of K_y, which may be identified Gal($\bar{Q}/Q$)-equivariantly with $\Sigma(Np)$. Since the action of Gal($\bar{Q}/Q$) on $\Sigma(Np)$ is given by the cyclotomic character Gal($\bar{Q}/Q$) $\to \hat{Z}^*$, the Galois action on $\Delta/K_y^{\perp}$ is trivial. Therefore, if V maps non-trivially to $\Delta/K_y^{\perp}$, then the semisimplification of V (as an $F_l[$Gal($\bar{Q}/Q$)$]$-module) contains the trivial representation, which contradicts Theorem 5.2 of [15]. ∎

Lemma 2 shows that there is a natural inclusion $V \subsetneq K_y^{\perp}/K_y = \Omega = X \cap Y$. This completes the proof of Theorem 5.

4. Proof of Theorem 3

We now proceed to prove Theorem 3. We first observe that $\eta(\Sigma(L) \times \Sigma(L)) \subseteq \Sigma(Lq)$ ([9], Theorem 4).

Let A denote the q-old part of $J_0(Lq)$, i.e., the image of $J_0(L)^2$ under $\eta = v_1^* \times v_q^*$. Let q' be a prime not dividing Lq. Consider the following (commutative) diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma(L)^2 & \longrightarrow & J_0(L)^4 & \xrightarrow{\eta \times \eta} & A^2 & \longrightarrow & 0. \\
& & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & \Sigma(Lq') & \longrightarrow & J_0(Lq')^2 & \xrightarrow{\eta'} & J_0(Lqq'). & &
\end{array}
\qquad (10)
$$

The map $\Sigma(L)^2 \to J_0(L)^4$ is defined by $(x, y) \mapsto (x, -x, y, -y)$ and the map $\Sigma(Lq') \to J_0(Lq')^2$ is given by $x \mapsto (x, -x)$. The map η' is $v_1^* \times v_{q'}^*$. The map α is defined by $\alpha(x, y, z, w) = (v_1^* x + v_{q'}^* z, \ v_1^* y + v_{q'}^* w)$, and β is obtained by restricting the map $v_1^* \times v_{q'}^* : J_0(Lq)^2 \to J_0(Lqq')$ to A^2. The map $\Sigma(L)^2 \to \Sigma(Lq')$ is induced by α.

The exactness of each row follows from [12, Theorem 4.3]. The commutativity is easy to see.

By applying the Snake Lemma to (10), we get the exact sequence

$$0 \longrightarrow \Sigma(L) \longrightarrow \Sigma(L)^2 \xrightarrow{\ \eta\ } (\Sigma(Lq) \cap A) \xrightarrow{\ \delta\ } \Sigma(Lq')/\Sigma(L).$$

Using (6), we see that, for $q' \neq 2, 3$,

$$\mathrm{card}(\Sigma(Lq')/\Sigma(L)) = \frac{(q'-1)\, m_2^k(L)\, m_3^k(L)}{m_2^{k+1}(Lq')\, m_3^{k+1}(Lq')}, \tag{11}$$

where $k = k(L)$.

If $l \neq 2, 3$ is a prime, we may choose q' such that $q' \not\equiv 1 \bmod l$. Then l does not divide $\mathrm{card}(\Sigma(Lq')/\Sigma(L))$, and so $\delta(\Sigma(Lq) \cap A)_l = 0$, where the subscript l denotes the l-primary part. Hence, $(\Sigma(Lq) \cap A)_l$ lies entirely in the image of $\Sigma(L)^2$ (under η), for $l \neq 2, 3$.

For $l = 3$, we split into two cases.

(I)   When $m_3(L) = 1$. In this case, $m_3(Lq') = 1$ also. We choose $q' \neq 2$ such that $q' \equiv 2 \bmod 3$, then 3 does not divide $\mathrm{card}\,(\Sigma(Lq')/\Sigma(L))$. Hence $(\Sigma(Lq) \cap A)_3$ lies entirely in $\eta(\Sigma(L)^2)$.

(II)   When $m_3(L) = 3$. In this case, we pick q' such that $q' \equiv 1 \bmod 3$, but $q' \not\equiv 1 \bmod 9$. Then 3 exactly divides $q' - 1$, and $m_3(Lq') = 3$. Hence, $3 \nmid \mathrm{card}(\Sigma(Lq')/\Sigma(L))$ again. Therefore, $(\Sigma(Lq) \cap A)_3$ also lies in $\eta(\Sigma(L)^2)$.

Now assume L to be odd.
If $q \neq 2$, then choose $q' = 2$. Then (6) implies

$$\mathrm{card}(\Sigma(2L)/\Sigma(L)) = \frac{(2-1)\, m_2^k(L)\, m_3^k(L)}{m_2^k(2L)\, m_3^k(2L)} = m_3^k(L).$$

Then $2 \nmid \mathrm{card}(\Sigma(2L)/\Sigma(L))$, implying that $(\Sigma(Lq) \cap A)_2$ is in the image of $\Sigma(L)^2$ under η.
If $q = 2$, then

$$\mathrm{card}(\Sigma(2L)) = m_3^k(L)\, \mathrm{card}(\Sigma(L)),$$

so clearly $(\Sigma(2L) \cap A)_2$ lies in $\eta(\Sigma(L)^2)$.

Therefore, $\Sigma(Lq) \cap A = \eta(\Sigma(L)^2)$, when L is odd. If $x \in J_0(L)^2$ and $y \in \Sigma(L)^2$ have the same image in $J_0(Lq)$, then $x - y \in \ker \eta$. By Theorem 4.3 of [12], $x - y \in \Sigma(L)^2$, so $x \in \Sigma(L)^2$. Therefore, $\Sigma(L) \times \Sigma(L) = \eta^{-1}(\Sigma(Lq))$.

When L is even, we split into two cases to consider.

(I) When $m_2(L) = 1$. It follows that $m_2(Lq') = 1$. We choose $q' \neq 3$ such that $q' \equiv 3 \bmod 4$. Then from (11), we see that 2 exactly divides $\text{card}(\Sigma(Lq')/\Sigma(L))$, which means 2 annihilates $(\Sigma(Lq) \cap A)/\eta(\Sigma(L)^2)$.

(II) When $m_2(L) = 2$. In this case, we choose q' such that $q' \equiv 1 \bmod 4$ but $q' \not\equiv 1 \bmod 8$, then $m_2(Lq') = 2$, and (11) again shows that 2 exactly divides card $(\Sigma(Lq')/\Sigma(L))$. Hence $2 \cdot (\Sigma(Lq) \cap A) \subseteq \eta(\Sigma(L)^2)$.

Therefore, as above, we conclude that the index of $\Sigma(L) \times \Sigma(L)$ in $\eta^{-1}(\Sigma(Lq))$ is at most two.

This completes the proof of Theorem 3.

5. KERNELS OF DEGENERACY MAPS

Given $M \geqslant 1$ and M' such that $M' \mid M$, let D denote a divisor of M/M'. Consider the map $\phi = \prod v_D^*: J_0(M')^\tau \to J_0(M)$, where τ is the number of positive divisors of M/M', and the product is taken over all the positive divisors D of M/M'.

In this section we apply Theorem 3 to determine the kernel of ϕ in some cases. Theorem 4 characterises the kernel of ϕ when M' is a positive integer, and $M = M'q_1 \cdots q_t$, where the q_i are distinct primes such that $(M', q_1 \cdots q_t) = 1$. Theorem 8 describes the kernel of ϕ when $M' = Np$, where $1 \leqslant N \leqslant 9$, $N \neq 7$ and $p \geqslant 5$ is a prime not dividing N, and $M = Np^rq_1 \cdots q_t$, where the q_i are distinct primes such that $(Np, q_1 \cdots q_t) = 1$.

Before we prove these theorems, we need some preparatory results.

THEOREM 6. *Let L be a positive integer, and let $M = q_1 \cdots q_t$ (q_i distinct primes) be relatively prime to L. Let ϕ be the map*

$$\phi: J_0(L)^{2^t} \to J_0(LM).$$

(i) *If L is odd, then $\phi^{-1}(\Sigma(LM)) = \Sigma(L)^{2^t}$.*

(ii) *If L is even, then $\phi^{-1}(\Sigma(LM))$ and $\Sigma(L)^{2^t}$ are equal up to a 2-group.*

Proof. First assume $M = q$ is a prime. The theorem is then Theorem 3.

Write $M = q_1 \cdots q_t$. If M is even (which only occurs if L is odd), we order the q_i such that $q_t = 2$.

By the inductive hypothesis, we assume that the theorem is true when M is replaced by $M' = q_1 \cdots q_{t-1}$.

Let $d_1, \ldots, d_{2^{t-1}}$ denote all the divisors of M' such that $1 = d_1 < d_2 < \cdots < d_{2^{t-1}} = M'$. Let $\phi': J_0(L)^{2^{t-1}} \to J_0(LM')$ be the map $v_{d_1}^* \times \cdots \times v_{d_{2^{t-1}}}^*$, and $\phi: J_0(L)^{2^t} \to J_0(LM)$ may be taken to be the map $v_{d_1}^* \times \cdots \times v_{d_{2^{t-1}}}^* \times v_{d_1 q_t}^* \times \cdots \times v_{d_{2^{t-1}} q_t}^*$. Then the diagram

$$
\begin{array}{ccc}
J_0(L)^{2^{t-1}} \times J_0(L)^{2^{t-1}} & \searrow^{\phi} & \\
{\scriptstyle \phi' \times \phi'} \downarrow & & \\
J_0(LM') \times J_0(LM') & \xrightarrow{v_1^* \times v_{q_t}^*} & J_0(LM)
\end{array}
$$

is clearly commutative.

When L is odd, Theorem 3 shows that $(v_1^* \times v_{q_t}^*)^{-1}(\Sigma(LM)) = \Sigma(LM')^2$. By the inductive hypothesis, $(\phi')^{-1}(\Sigma(LM')) = \Sigma(L)^{2^{t-1}}$. Therefore, $\phi^{-1}(\Sigma(LM)) = \Sigma(L)^{2^t}$.

If L is even, then Theorem 3 shows that the index of $\Sigma(LM')^2$ in $(v_1^* \times v_{q_t}^*)^{-1}(\Sigma(LM))$ is at most two.

By the inductive hypothesis, $(\phi')^{-1}(\Sigma(LM'))$ and $\Sigma(L)^{2^{t-1}}$ are equal up to a 2-group. Therefore, $\phi^{-1}(\Sigma(LM))$ and $\Sigma(L)^{2^t}$ are equal up to a 2-group. ∎

THEOREM 7. *Let N be a positive integer and let $p \geqslant 5$ be a prime not dividing N. Let $M = q_1 \cdots q_t$ (q_i distinct primes) be relatively prime to Np. Let ϕ be the map*

$$\phi: J_0(Np)^{r \cdot 2^t} \to J_0(Np^r M).$$

(i) *If $N \in \{1, 3, 5, 9\}$ or if $N \in \{2, 4, 6, 8\}$ and $M = 1$, then $\phi^{-1}(\Sigma(Np^r M)) = \Sigma(Np)^{r \cdot 2^t}$.*

(ii) *If $N \in \{2, 4, 6, 8\}$ and $M \neq 1$, then $\phi^{-1}(\Sigma(Np^r M))$ and $\Sigma(Np)^{r \cdot 2^t}$ are equal up to a 2-group.*

Proof. We prove this theorem in two cases. We also assume throughout this proof that $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$.

(I) When $M = 1$ ($t = 0$). In this case, ϕ becomes the map $\gamma: J_0(Np)^r \to J_0(Np^r)$, the kernel of which is given in Theorem 2.

Consider the composition of maps

$$J_0(Np)^r \times J_0(Np)^r \xrightarrow{\gamma \times \gamma} J_0(Np^r) \times J_0(Np^r) \xrightarrow{v_1^* \times v_{p^r}^*} J_0(Np^{2r}).$$

It is clear that this composition is the map $\gamma': J_0(Np)^{2r} \to J_0(Np^{2r})$ defined analogously to γ.

Now suppose $y \in \Sigma(Np^r)$ is $\gamma(x)$ for some $x \in J_0(Np)^r$. The element $(x, -x) \in J_0(Np)^r \times J_0(Np)^r$ yields $(y, -y) = (\gamma \times \gamma)(x, -x)$ and, since the degeneracy maps coincide on the Shimura subgroup ([9], Theorem 4),
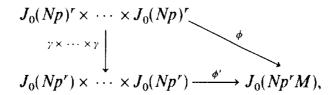
$\gamma'(x, -x) = (v_1^* \times v_{p'}^*)(y, -y) = 0$. Then by Theorem 2, we deduce that $x \in \Sigma(Np)^r$, i.e., $\gamma^{-1}(\Sigma(Np^r)) = \Sigma(Np)^r$.

(II) When $M = q_1 \cdots q_t$ ($t \geqslant 1$). Let $1 = d_1 < d_2 < \cdots < d_{2^t} = M$ be all the positive divisors of M, and let ϕ': $J_0(Np^r)^{2^t} \to J_0(Np^r M)$ be the map $v_{d_1}^* \times \cdots v_{d_{2^t}}^*$.

As above, let γ be the map $v_1^* \times \cdots \times v_{p^r-1}^*$: $J_0(Np)^r \to J_0(Np^r)$. Let ϕ: $J_0(Np)^{r \cdot 2^t} \to J_0(Np^r M)$ be the map

$$(v_{d_1}^* \times \cdots \times v_{d_1 p^r-1}^*) \times (v_{d_2}^* \times \cdots \times v_{d_2 p^r-1}^*) \times \cdots \times (v_{d_{2^t}}^* \times \cdots \times v_{d_{2^t} p^r-1}^*).$$

Then the diagram

$$
\begin{array}{ccc}
J_0(Np)^r \times \cdots \times J_0(Np)^r & & \\
\gamma \times \cdots \times \gamma \downarrow & \searrow^{\phi} & \\
J_0(Np^r) \times \cdots \times J_0(Np^r) & \xrightarrow{\phi'} & J_0(Np^r M),
\end{array}
$$

is clearly commutative.

If N is odd, then $(\phi')^{-1}(\Sigma(Np^r M)) = \Sigma(Np^r)^{2^t}$ by Theorem 6. Since $\gamma^{-1}(\Sigma(Np^r)) = \Sigma(Np)^r$, it follows that $\phi^{-1}(\Sigma(Np^r M)) = \Sigma(Np)^{r \cdot 2^t}$.

If N is even, then $(\phi')^{-1}(\Sigma(Np^r M))$ and $\Sigma(Np^r)^{2^t}$ are equal up to a 2-group, so $\phi^{-1}(\Sigma(Np^r M))$ and $\Sigma(Np)^{r \cdot 2^t}$ are also equal up to a 2-group. ∎

As a corollary to Theorems 6 and 7, we can compute the kernel of ϕ.

Let K_ϕ be the kernel of ϕ: $J_0(L)^{2^t} \to J_0(LM)$, where $M = q_1 \cdots q_t$ (q_i distinct primes) is relatively prime to L. We shall prove that K_ϕ is as described in Theorem 4.

The case where $M = q_1$ is a prime is precisely Theorem 4.3 of [12].

Since the degeneracy maps coincide on $\Sigma(L)$ ([9], Theorem 4), we clearly have $\Sigma(L)_0^{2^t} \subseteq K_\phi$.

If L is odd, then $K_\phi \subseteq \Sigma(L)^{2^t}$ by Theorem 6. Let $x = (x_1, \ldots, x_{2^t}) \in K_\phi$. Then $0 = \phi(x) = v_1^*(x_1 + \cdots + x_{2^t})$ by [9], Theorem 4. Therefore, $x_1 + \cdots + x_{2^t} = 0$, i.e., $x \in \Sigma(L)_0^{2^t}$. Hence, $K_\phi = \Sigma(L)_0^{2^t}$.

If L is even, then up to a 2-group we have $K_\phi \subseteq \Sigma(L)^{2^t}$. The same argument then shows that K_ϕ and $\Sigma(L)_0^{2^t}$ are equal up to a 2-group.

This proves Theorem 4.

Similarly we can prove

THEOREM 8. *Let N be a positive integer and let $p \geqslant 5$ be a prime not dividing N. Let $M = q_1 \cdots q_t$ (q_i distinct primes) be such that $(Np, M) = 1$. Let K_ϕ be the kernel of the map ϕ: $J_0(Np)^{r \cdot 2^t} \to J_0(Np^r M)$.*

(i)   *If* $N \in \{1, 3, 5, 9\}$ *or if* $N \in \{2, 4, 6, 8\}$ *and* $M = 1$, *then* $K_\phi = \Sigma(Np)_0^{r \cdot 2^t}$.

(ii)   *If* $N \in \{2, 4, 6, 8\}$ *and* $M \neq 1$, *then* K_ϕ *and* $\Sigma(Np)_0^{r \cdot 2^t}$ *are equal up to a 2-group.*

6. Behaviour of Shimura Subgroups under Degeneracy Maps

In this section we study the restriction of degeneracy maps to Shimura subgroups and the interaction between different Shimura subgroups under such maps.

THEOREM 9.   *Let* $M_1, ..., M_r$ *be pairwise relatively prime positive integers, and let* N *be any positive integer. Then the map*

$$\alpha: \Sigma(M_1) \times \cdots \times \Sigma(M_r) \to \Sigma(M_1 \cdots M_r N),$$

defined by the formula $\alpha(x_1, ..., x_r) = v_1^* x_1 + \cdots + v_1^* x_r$, *is injective.*

Remarks.   1.   Since the degeneracy maps coincide on the Shimura subgroups [9, Theorem 4]), α is actually independent of the choice of the degeneracy maps used in its definition.

2.   Theorem 9 may be viewed as a generalisation of [8, Section 1].

Proof.   The obvious projection map

$$(\mathbf{Z}/M_1 \cdots M_r N\mathbf{Z})^\times \to (\mathbf{Z}/M_1\mathbf{Z})^\times \times \cdots \times (\mathbf{Z}/M_r\mathbf{Z})^\times \qquad (12)$$

is surjective. Applying $\mathrm{Hom}(\ , \mathbf{U})$ to (12), we may obtain the following diagram (which is easily checked to be commutative)

$$
\begin{array}{ccc}
\Sigma(M_1) \times \cdots \times \Sigma(M_r) & \xrightarrow{\;\;\alpha\;\;} & \Sigma(M_1 \cdots M_r N) \\
\downarrow & & \downarrow \\
0 \longrightarrow \mathrm{Hom}((\mathbf{Z}/M_1\mathbf{Z})^\times, \mathbf{U}) \times \cdots \times \mathrm{Hom}((\mathbf{Z}/M_r\mathbf{Z})^\times, \mathbf{U}) & \longrightarrow & \mathrm{Hom}((\mathbf{Z}/M_1 \cdots M_r N\mathbf{Z})^\times, \mathbf{U}),
\end{array}
$$

$$(13)$$

where the vertical maps are the canonical injection

$$\Sigma(L) \hookrightarrow \mathrm{Hom}((\mathbf{Z}/L\mathbf{Z})^\times, \mathbf{U}) \qquad (14)$$

in [9], (2).

Injectivity of α follows immediately from (13).   ∎

We derive immediately two corollaries.

COROLLARY 1. *Let $M_1, ..., M_r$ and N be as in Theorem 9. Let t_i be the number of divisors of $(M_1 \cdots M_r N)/M_i$. Then the map*

$$\beta: \Sigma(M_1)^{t_1} \times \cdots \times \Sigma(M_r)^{t_r} \to \Sigma(M_1 \cdots M_r N),$$

defined via all the possible degeneracy maps, has as its kernel $\Sigma(M_1)_0^{t_1} \times \cdots \times \Sigma(M_r)_0^{t_r}$.

Proof. This follows immediately from Theorem 9 and the fact that the different degeneracy maps, acting on the Shimura subgroup, have the same action. ∎

Remark. Corollary 1 is a generalisation of [8, Theorem 1].

COROLLARY 2. *Let L, M be relatively prime positive integers, and let N be any positive integer. Then the map*

$$\alpha: \Sigma(L) \times \Sigma(M) \to \Sigma(LMN)$$

is injective.

Corollary 2, which is obtained from Theorem 9 by setting $r = 2$, serves as a motivation for the following theorem.

THEOREM 10. *Given positive integers L, M and N. Let $P = \text{lcm}(L, M)$ and $G = \text{gcd}(L, M)$. Then the kernel of*

$$\alpha: \Sigma(L) \times \Sigma(M) \to \Sigma(PN)$$

contains $\Sigma(G)_0^2$, where $\Sigma(G)$ is regarded as a subgroup of $\Sigma(L)$ (resp. $\Sigma(M)$) through the degeneracy maps. Moreover, the prime-to-6 part of $\ker \alpha$ is equal to the prime-to-6 part of $\Sigma(G)_0^2$.

Proof. Clearly $\Sigma(G)_0^2$ is contained in $\ker \alpha$.

For the second statement, let $p_1, ..., p_t$ denote all the common prime divisors of L and M. Then we write $L = (\prod p_i^{a_i}) L'$, $M = (\prod p_i^{b_i}) M'$, where $a_i, b_i > 0$ and $(L', p_i) = 1 = (M', p_i)$ for all i. Moreover, $(L', M') = 1$.

For each $1 \leqslant i \leqslant t$, let $c_i = \max(a_i, b_i)$ and let $d_i = \min(a_i, b_i)$. Then

$$P = \left(\prod p_i^{c_i}\right) L'M' \quad \text{and} \quad G = \prod p_i^{d_i}.$$

Since the map $\alpha: \Sigma(L) \times \Sigma(M) \to \Sigma(PN)$ factors through $\Sigma(P)$, we may assume $N = 1$. For each positive integer Z, let $m(Z)$ be as in subsection 2.1. Let $\tilde{Z} = Z/m(Z)$. Let

$$\theta_1: (\mathbf{Z}/\tilde{L}\mathbf{Z})^\times \times (\mathbf{Z}/\tilde{M}\mathbf{Z})^\times \to ((\mathbf{Z}/\tilde{G}\mathbf{Z})^\times)_0^2$$

be defined by $\theta_1(x, y) = (xy^{-1} \bmod \tilde{G}, (xy^{-1})^{-1} \bmod \tilde{G})$. It is clear that θ_1 is surjective.

Let

$$\theta_2 \colon (\mathbf{Z}/\tilde{P}\mathbf{Z})^\times \to (\mathbf{Z}/\tilde{L}\mathbf{Z})^\times \times (\mathbf{Z}/\tilde{M}\mathbf{Z})^\times$$

be the obvious projection map $\theta_2(x) = (x \bmod \tilde{L}, x \bmod \tilde{M})$. We claim that the sequence

$$(\mathbf{Z}/\tilde{P}\mathbf{Z})^\times \xrightarrow{\ \theta_2\ } (\mathbf{Z}/\tilde{L}\mathbf{Z})^\times \times (\mathbf{Z}/\tilde{M}\mathbf{Z})^\times \xrightarrow{\ \theta_1\ } ((\mathbf{Z}/\tilde{G}\mathbf{Z})^\times)_0^2 \longrightarrow 0 \qquad (15)$$

is exact. To verify this, we only need to check that (15) is exact at $(\mathbf{Z}/\tilde{L}\mathbf{Z})^\times \times (\mathbf{Z}/\tilde{M}\mathbf{Z})^\times$.

It follows immediately from the definition of θ_1 and θ_2 that $\operatorname{Im} \theta_1 \subseteq \ker \theta_2$. Conversely, suppose that $(x, y) \in (\mathbf{Z}/\tilde{L}\mathbf{Z})^\times \times (\mathbf{Z}/\tilde{M}\mathbf{Z})^\times$ is such that $x \equiv y \bmod \tilde{G}$. We want to find $z \in (\mathbf{Z}/\tilde{P}\mathbf{Z})^\times$ such that $z \equiv x \bmod \tilde{L}$ and $z \equiv y \bmod \tilde{M}$.

In other words, we want $z \in (\mathbf{Z}/\tilde{P}\mathbf{Z})^\times$ such that

$$z \equiv x \bmod \tilde{L}'$$

$$z \equiv x \bmod p_i^{a_i - [a_i/2]} \qquad 1 \leqslant i \leqslant t$$

$$z \equiv y \bmod \tilde{M}'$$

$$z \equiv y \bmod p_i^{b_i - [b_i/2]} \qquad 1 \leqslant i \leqslant t.$$

The existence of z is guaranteed by the facts that $x \equiv y \bmod p_i^{d_i - [d_i/2]}$ and

$$\tilde{P} = \tilde{L}'\tilde{M}' \left(\prod_i p_i^{c_i - [c_i/2]} \right),$$

and applying the Chinese Remainder Theorem to $\tilde{L}'$, $\tilde{M}'$ and $p_i^{c_i - [c_i/2]}$ for $1 \leqslant i \leqslant t$. Therefore we have shown that (15) is exact.

Applying $\operatorname{Hom}(\ , \mathbf{U})$ to (15), we obtain an exact sequence

$$0 \to [\operatorname{Hom}((\mathbf{Z}/\tilde{G}\mathbf{Z})^\times, \mathbf{U})]_0^2 \to \operatorname{Hom}((\mathbf{Z}/\tilde{L}\mathbf{Z})^\times, \mathbf{U}) \times \operatorname{Hom}((\mathbf{Z}/\tilde{M}\mathbf{Z})^\times, \mathbf{U})$$

$$\to \operatorname{Hom}((\mathbf{Z}/\tilde{P}\mathbf{Z})^\times, \mathbf{U}). \qquad (16)$$

From [9, Section 2.3], we see that for every positive integer N, there exists a subgroup $J'(N)$ of $(\mathbf{Z}/\tilde{N}\mathbf{Z})^\times$ of exponent dividing six such that

$$\Sigma(N) \simeq \operatorname{Hom}((\mathbf{Z}/\tilde{N}\mathbf{Z})^\times / J'(N), \mathbf{U}). \qquad (17)$$

Observing that the prime-to-6 parts of $\mathrm{Hom}((\mathbf{Z}/\tilde{N}\mathbf{Z})^{\times}, \mathrm{U})$ and $\mathrm{Hom}((\mathbf{Z}/\tilde{N}\mathbf{Z})^{\times}/J'(N), \mathrm{U})$ are isomorphic, we obtain from (16) and (17) the exact sequence

$$0 \longrightarrow (\Sigma(G)_0^2)_{(6)} \longrightarrow (\Sigma(L) \times \Sigma(M))_{(6)} \xrightarrow{\;\alpha\;} \Sigma(P)_{(6)},$$

where the subscript $_{(6)}$ denotes the prime-to-6 part. This completes the proof of Theorem 10. ∎

Remark. Actually, the quotient

$$\mathrm{Hom}((\mathbf{Z}/\tilde{N}\mathbf{Z})^{\times}, \mathrm{U})/\mathrm{Hom}((\mathbf{Z}/\tilde{N}\mathbf{Z})^{\times}/J'(N), \mathrm{U})$$

has exponent dividing six. The argument above then shows that $\ker \alpha$ is equal to $\Sigma(G)_0^2$ up to a group of exponent dividing six.

7. Some Consequences

In this final section, we discuss some applications of the above results to the study of the action of degeneracy maps on Jacobians of modular curves.

PROPOSITION 4. *Let L be a positive integer. Let $N > 1$ be a square-free integer relatively prime to L. Let M be any divisor of LN. Let t be the number of prime divisors of N, and let s be the number of divisors of LN/M. Let δ be the map*

$$\delta: J_0(L)^{2^t} \times J_0(M)^s \to J_0(LN),$$

obtained from all the possible degeneracy maps. Let

$$H = \left\{ (x_1, \ldots, x_{2^t}, y_1, \ldots, y_s) \in \Sigma(L)^{2^t} \times \Sigma(M)^s \,\middle|\, \sum x_i = -\sum y_j \in \Sigma(G) \right\},$$

where $G = \gcd(L, M)$.

Then H is contained in $(\ker \delta) \cap (J_0(L)^{2^t} \times \Sigma(M)^s)$, and the prime-to-6 parts of these two groups are equal.

Proof. It is clear that H is contained in $(\ker \delta) \cap (J_0(L)^{2^t} \times \Sigma(M)^s)$.

Let $\delta_1: J_0(L)^{2^t} \to J_0(LN)$ be the restriction of δ to $J_0(L)^{2^t} \times 0$, and let $\delta_2: J_0(M)^s \to J_0(LN)$ be the restriction of δ to $0 \times J_0(M)^s$. Therefore, $\delta = \delta_1 \times \delta_2$.

Suppose $(x, y) \in J_0(L)^{2^t} \times \Sigma(M)^s$ satisfies $\delta(x, y) = 0$. Since $y \in \Sigma(M)^s$, we have $\delta_2(y) \in \Sigma(LN)$. Hence $\delta_1(x) \in \Sigma(LN)$. By Theorem 6, $\delta_1^{-1}(\Sigma(LN)) = \Sigma(L)^{2^t}$ if L is odd, and this equality is true up to a 2-group if L is even. Therefore, the prime-to-2 part of $(\ker \delta) \cap (J_0(L)^{2^t} \times \Sigma(M)^s)$ is contained in $\Sigma(L)^{2^t} \times \Sigma(M)^s$. Proposition 4 then follows readily from Theorem 10. ∎

By taking $M = N$, we obtain immediately

COROLLARY 1. *Let* L *be a positive integer and let* $M = q_1 \cdots q_t$ *(q_i distinct primes) be an integer such that* $(L, M) = 1$. *Let* s *be the number of divisors of* L. *Then for*

$$\delta \colon J_0(L)^{2^t} \times J_0(M)^s \to J_0(LM),$$

we have $(\ker \delta) \cap (J_0(L)^{2^t} \times \Sigma(M)^s) = \Sigma(L)_0^{2^t} \times \Sigma(M)_0^s$ *if* L *is odd, and the same is true up to a 2-group if* L *is even.*

Proof. Proposition 4 shows that $\Sigma(L)_0^{2^t} \times \Sigma(M)_0^s$ is contained in $(\ker \delta) \cap (J_0(L)^{2^t} \times \Sigma(M)^s)$.

From the proof of Proposition 4, we see that $(\ker \delta) \cap (J_0(L)^{2^t} \times \Sigma(M)^s)$ is contained in $\Sigma(L)^{2^t} \times \Sigma(M)^s$ if L is odd, and the same is true up to a 2-group if L is even. This corollary then follows from Corollary 2 of Theorem 9. ∎

PROPOSITION 5. *Let* $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$ *and let* $p \geq 5$ *be a prime not dividing* N. *Let* $M = q_1 \cdots q_t$ *(q_i distinct primes) be relatively prime to* Np. *Let* L *be a divisor of* N. *For an integer* $r \geq 1$, *let* $0 \leq k < r$ *be an integer. Let*

$$\delta \colon J_0(Np)^{r \cdot 2^t} \times J_0(Lp^k M)^s \to J_0(Np^r M)$$

be the map defined via all the degeneracy maps. Let

$$H = \left\{ (x_1, \ldots, x_{r \cdot 2^t}, y_1, \ldots, y_s) \in \Sigma(Np)^{r \cdot 2^t} \times \Sigma(Lp^k M)^s \; \middle| \right.$$

$$\left. \sum x_i = -\sum y_j \in \Sigma(Lp) \right\}.$$

Then H *is contained in* $(\ker \delta) \cap (J_0(Np)^{r \cdot 2^t} \times \Sigma(Lp^k M)^s)$, *and the prime-to-6 parts of the two groups are equal.*

The proof is similar to that of Proposition 4, with Theorem 7 replacing Theorem 6.

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