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Statistical analysis for windowed Fourier ridge algorithm in fringe pattern analysis

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Based on the windowed Fourier transform, the windowed Fourier ridges (WFR) algorithm and the windowed Fourier filtering algorithm (WFF) have been developed and proven effective for fringe pattern analysis. The WFR algorithm is able to estimate local frequency and phase by assuming the phase distribution in a local area to be a quadratic polynomial. In this paper, a general and detailed statistical analysis is carried out for the WFR algorithm when an exponential phase field is disturbed by additive white Gaussian noise. Because of the bias introduced by the WFR algorithm for phase estimation, a phase compensation method is proposed for the WFR algorithm followed by statistical analysis. The mean squared errors are derived for both local frequency and phase estimates using a first-order perturbation technique. These mean square errors are compared with Cramer–Rao bounds, which shows that the WFR algorithm with phase compensation is a suboptimal estimator. The above theoretical analysis and comparison are verified by Monte Carlo simulations. Furthermore, the WFR algorithm is shown to be slightly better than the WFF algorithm for quadratic phase. © 2012 Optical Society of America
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1. Introduction

The windowed Fourier transform has been proposed for fringe pattern analysis. Two algorithms, WFR and WFF, were developed [1]. It is useful in many applications, including noise suppression, phase and local frequency retrieval, strain estimation, phase unwrapping, phase-shifter calibration, fault detection, and fringe segmentation [1]. The WFR algorithm is a parameter estimator for phase, local frequency, and amplitude from an exponential phase field (EPF), when the phase distribution in a local area is quadratic and the amplitude is constant [2]. This paper focuses on the statistical analysis for the WFR algorithm on local frequency and phase estimates when an EPF is disturbed by additive white Gaussian noise. There exist other similar algorithms for parameter estimation from an EPF, such as the high-order ambiguity function (HAF) algorithm [3,4,5] and the cubic

phase function (CPF) algorithm [6,7]. They will be briefly compared at the end of the paper.

Cramer–Rao bound (CRB) provides the lower bound for the mean squared error (MSE) of an unbiased deterministic parameter from a distribution [8,9]. For a parameter estimate, the ratio of the CRB to the corresponding variance of any estimator is called the efficiency of that estimator. An estimator that attains the CRB is called an efficient estimator [10]. Naturally, it is of interest to evaluate the efficiency of the WFR algorithm by comparing the variance of the local frequency and phase estimates with the corresponding CRBs.

Some theoretical analysis for the WFR algorithm has already been carried out in our previous paper [2]. This paper aims at a more comprehensive analysis with the following contributions: (i) the analysis of the local frequency estimate in [2] was qualitative and a typical fringe pattern was utilized. In this paper, the statistical properties of the local frequency estimate are analyzed quantitatively using the first-order perturbation [3]; (ii) it was found that the

phase estimate was biased [2] and needed to be compensated. To compensate the phase, the second-order derivative of the phase should be estimated. A method for phase compensation is proposed by using local frequencies at two points to estimate the second-order derivative of the phase. The statistical analysis for the phase compensation is given; (iii) with the above analysis, the efficiency of the WFR is discussed, which is lack in [2]; and (iv) the statistical analysis is verified by Monte Carlo simulations.

Both theoretical analysis and Monte Carlo simulations show that the WFR algorithm with phase compensation is a suboptimal estimator as both MSEs of the local frequency and phase estimates are larger than their corresponding CRBs. However, the phase estimate by the WFR algorithm with phase compensation is very close to the CRB and is worse by at most 4.2 times when $c_2 \leq 0.02$ for a data length of 151. Furthermore, Monte Carlo simulations show that the WFR algorithm with phase compensation is slightly better than the WFF algorithm for phase estimate when the phase is quadratic. For brevity, the 1D WFR algorithm with phase compensation applying to a 1D EPF is analyzed, while its extension to 2D is straightforward.

2. Windowed Fourier Ridges Algorithm

By using techniques such as phase shifting [11], carrier technique with Fourier transform [12], and digital holography [13], fringe pattern(s) can be converted to an EPF $f(x)$ [2], which can be generally represented as follows:

$$f(x) = f_0(x) + n(x), \quad (1)$$

where $f_0(x)$ is an intrinsic signal and $n(x)$ is the noise. The intrinsic signal can be expressed as

$$f_0(x) = b \exp[j\varphi(x)], \quad (2)$$

where $j = \sqrt{-1}$; b is the amplitude and is assumed to be locally constant; $\varphi(x)$ is the phase and is assumed to be locally quadratic as

$$\varphi(x) = c_0 + c_1x + 0.5c_2x^2, \quad (3)$$

with c_i as the i -th derivative of $\varphi(x)$ at $x = 0$, i.e.,

$$c_i = \varphi^{(i)}(0), \quad i = 0, 1, 2. \quad (4)$$

The local frequency at x is defined as

$$\omega(x) = \frac{d\varphi}{dx} = c_1 + c_2x. \quad (5)$$

For a noiseless EPF, it is obvious that

$$f(x) = f_0(x), \quad (6)$$

whose forward windowed Fourier transform is as follows [1,2]:

$$Sf(u; \xi) = Sf_0(u; \xi) = \int_{-\infty}^{\infty} f(x)g(u-x) \exp(-j\xi x) dx, \quad (7)$$

where the window is selected as Gaussian,

$$g(x-u) = \frac{1}{(\pi\sigma^2)^{1/4}} \exp\left[-\frac{(x-u)^2}{2\sigma^2}\right], \quad (8)$$

with σ as its kernel size. Equation (7) indicates that harmonics with linear phase $\exp(j\xi x)$ are used to represent an EPF with a quadratic phase $\exp[j\varphi(x)]$. Through some derivations [2], the following results can be obtained:

$$Sf(u; \xi) = A(u; \xi) \exp[j\Phi(u; \xi)], \quad (9)$$

where

$$A(u; \xi) = b \left(\frac{4\pi\sigma^2}{1 + \sigma^4 c_2^2} \right)^{1/4} \exp \left\{ -\frac{\sigma^2 [\xi - \omega(u)]^2}{2(1 + \sigma^4 c_2^2)} \right\}, \quad (10)$$

$$\begin{aligned} \Phi(u; \xi) = \varphi(u) - \xi u - \frac{\sigma^4 c_2 [\xi - \omega(u)]^2}{2(1 + \sigma^4 c_2^2)} \\ + 0.5 \arctan(\sigma^2 c_2). \end{aligned} \quad (11)$$

It is obvious that $|Sf(u; \xi)|$ is maximized when $\xi = \omega(u)$. Thus, the following WFR algorithm can be established:

$$\begin{aligned} \hat{\omega}(u) &= \arg \max_{\xi} |Sf(u; \xi)| \\ &= \arg \max_{\xi} |Sf(u; \xi) \exp(j\xi u)|, \end{aligned} \quad (12)$$

$$\begin{aligned} \hat{\varphi}(u) &= \text{angle}\{Sf[u; \hat{\omega}(u)] \exp[j\hat{\omega}(u)u]\} \\ &\quad - 0.5 \arctan(\sigma^2 \hat{c}_2), \end{aligned} \quad (13)$$

where $\hat{\omega}(u)$, $\hat{\varphi}(u)$, and $\hat{c}_2$ are estimates of $\omega(u)$, $\varphi(u)$, and c_2 , respectively. In Eqs. (12) and (13), $Sf(u; \xi) \exp(j\xi u)$ is used instead of $Sf(u; \xi)$, as the former is obtained directly in the implementation by using convolution operation [1].

In Eq. (13), the term $0.5 \arctan(\sigma^2 \hat{c}_2)$ needs to be estimated and deducted from $\text{angle}\{Sf[u; \hat{\omega}(u)] \exp[j\hat{\omega}(u)u]\}$, which is called phase compensation throughout this paper. To achieve the phase compensation, $\hat{c}_2$ has to be estimated. According to Eq. (5), the local frequency is linear. If frequencies at two points u_1 and u_2 , i.e., $\hat{\omega}(u_1)$ and $\hat{\omega}(u_2)$, have been estimated, $\hat{c}_2$ can be further estimated as

$$\hat{c}_2 = [\hat{\omega}(u_2) - \hat{\omega}(u_1)] / (u_2 - u_1), \quad (14)$$

For the intrinsic signal, since there is no noise, $\hat{c}_2 = c_2$, $\hat{\omega}(u) = \omega(u)$, and $\hat{\varphi}(u) = \varphi(u)$.

3. Estimation of Local Frequency and Phase from a Noisy Signal

A. Local Frequency Estimation

Consider a noisy EPF in Eq. (1) and assume that the noise is white complex Gaussian as

$$n(x) = n_r(x) + jn_i(x), \quad (15)$$

where both $n_r(x)$ and $n_i(x)$ are assumed to have a mean of zero and a variance of σ_n^2 . The windowed Fourier spectrum of $f(x)$ in Eq. (1) is

$$Sf(u; \xi) = Sf_0(u; \xi) + Sn(u; \xi), \quad (16)$$

where $Sf_0(u; \xi)$ is the same as Eq. (7), while

$$Sn(u; \xi) = \int_{-\infty}^{\infty} n(x)g(x-u)\exp(-j\xi x)dx. \quad (17)$$

When the WFR algorithm of Eqs. (12) and (13) is applied to the noisy signal $f(x)$, the estimated local frequency $\hat{\omega}(u)$ will produce an error $\delta\omega(u)$. As derived in Appendix B (Appendix A serves as a preparation for Appendix B), the mean of $\delta\omega$ is zero and its variance is as follows:

$$E[(\delta\omega)^2] = \text{Var}(\delta\omega) = \frac{(1 + \sigma^4 c_2^2)^{5/2}}{8\sqrt{\pi}\sigma^3 \text{SNR}}, \quad (18)$$

where $\text{SNR} = b^2/2\sigma_n^2$ is the signal-to-noise ratio. The variance of $\hat{\omega}(u)$ increases with c_2 , which is intuitively understandable since the window Fourier basis with linear phase is used to match the signal with quadratic phase. It is also intuitive that the variance of $\hat{\omega}(u)$ decreases with SNR. From Eq. (18), the optimal value of σ that minimizes $E[(\delta\omega)^2]$ varies with c_2 and $\sigma = 10$ is a good choice, which is used throughout this paper.

B. Phase Estimation

Similar to the local frequency estimation, the phase estimation is also affected by noises. According to Appendix C, the phase estimation error produced by the WFR algorithm and phase compensation can be derived as

$$\delta\varphi \approx (\delta\varphi)_1 + (\delta\varphi)_2 + (\delta\varphi)_3, \quad (19)$$

with

$$(\delta\varphi)_1 = \text{Im}\{Sn[u; \omega(u)]\exp[j\omega(u)u]/V_0\}, \quad (20)$$

$$(\delta\varphi)_2 = -\frac{\sigma^4 c_2}{2(1 + \sigma^4 c_2^2)}[\delta\omega(u)]^2, \quad (21)$$

$$(\delta\varphi)_3 = -\frac{\sigma^2 \delta c_2}{2(1 + \sigma^4 c_2^2)}, \quad (22)$$

$$V_0 = b \left(\frac{4\pi\sigma^2}{1 + \sigma^4 c_2^2} \right)^{1/4} \exp\{j[\varphi(u) + 0.5 \arctan(\sigma^2 c_2)]\}, \quad (23)$$

where $\text{Im}(\cdot)$ is the imaginary part. As seen from the above, $(\delta\varphi)_1$ is due to the noise $n(x)$, $(\delta\varphi)_2$ is caused by the local frequency error $\delta\omega(u)$, and $(\delta\varphi)_3$ is introduced by the phase compensation.

Also according to Appendix C, the mean and MSE of phase estimate are

$$E(\delta\varphi) \approx -\frac{\sigma c_2(1 + \sigma^4 c_2^2)^{3/2}}{16\sqrt{\pi}\text{SNR}}, \quad (24)$$

$$E[(\delta\varphi)^2] \approx \frac{(1 + \sigma^4 c_2^2)^{1/2}}{4\sqrt{\pi}\sigma\text{SNR}} [1 + k_1(c_2, \text{SNR}, \sigma) + k_2(c_2, \sigma, s) + k_3(c_2, \sigma, s, l)], \quad (25)$$

where

$$k_1(c_2, \text{SNR}, \sigma) = \frac{3\sigma^3 c_2^2(1 + \sigma^4 c_2^2)^{5/2}}{64\sqrt{\pi}\text{SNR}}, \quad (26)$$

$$k_2(c_2, \sigma, s) = \frac{(1 + \sigma^4 c_2^2)^2}{s^2 \sigma^2} \left\{ 1 - \left[1 - \frac{s^2}{2\sigma^2} (1 + \sigma^4 c_2^2) \right] \times \exp \left[-\frac{s^2(1 + \sigma^4 c_2^2)}{4\sigma^2} \right] \right\}, \quad (27)$$

$$k_3(c_2, \sigma, s, l) = \frac{1}{4s} \left\{ l \exp \left[-\frac{l^2(1 + \sigma^4 c_2^2)}{4\sigma^2} \right] - (l - s) \times \exp \left[-\frac{(l - s)^2(1 + \sigma^4 c_2^2)}{4\sigma^2} \right] \right\}, \quad (28)$$

where $l = u - u_1$ and $s = u_2 - u_1$ for a concerned point u and two points u_1 and u_2 for $\hat{c}_2$ estimation.

For an EPF with a data length of N , $c_2 \leq \pi/N$ is required so that the signal is not aliased. For example, $c_2 \approx 0.02$ for $N = 151$. With this data length, if the signal is not aliased, simulation shows that both $k_1(c_2, \text{SNR}, \sigma)$ and $k_2(c_2, \sigma, s)$ are ignorable when $c_2 \leq 0.02$, $\sigma \leq 10$. As for $k_3(c_2, \sigma, s, l)$, it contributes up to about 0.1 when $s \geq 50$; thus, is also ignorable. Overall, Eq. (25) is approximated as

$$E[(\delta\varphi)^2] \approx \frac{(1 + \sigma^4 c_2^2)^{1/2}}{4\sqrt{\pi}\sigma\text{SNR}}, \quad (29)$$

which depends on parameters of c_2 , σ , and SNR. Similarly to $E[(\delta\omega)^2]$, $E[(\delta\varphi)^2]$ increases with c_2 and decreases with SNR. The bias in Eq. (24) is small comparing with the MSE in Eq. (29). Thus, the MSE and variance of $\delta\varphi$ are approximately the same.

C. Comparison with the Cramer–Rao Bounds

The previous analysis suits a continuous signal but can be seen as an approximation for discrete data. Thus, it is necessary to emphasize the units of various parameters: rad for $\varphi(x)$, rad/pixel for $\omega(x)$, rad/(pixel)² for c_2 , pixel for σ . They are omitted since no confusion will be raised.

Given a window size σ , the Gaussian window is theoretically infinitely long. It is truncated to $6\sigma + 1$ in Ref. [2]. In this paper, it is truncated to $10\sigma + 1$ for better verification between theoretical analysis and Monte Carlo simulations. These two different truncations will be compared in Section 4.2. In the WFR algorithm with phase compensation, sliding window approach is adopted. The data within $[u - 5\sigma, u + 5\sigma]$ are used to estimate the local frequency at u by a WFR. The data length is thus $10\sigma + 1$. In order to compensate the phase error, the data within $[u, u + 10\sigma]$ is used for another WFR to estimate the frequency at $u + 5\sigma$. From two frequency estimates, $\hat{c}_2$ can be computed by Eq. (14) and the phase can be further compensated by Eq. (13). Thus, data at $[u - 5\sigma, u + 10\sigma]$ are actually used for the phase estimation, and the data length is thus $15\sigma + 1$.

The CRBs for local frequency and phase estimation have been derived in Ref. [14]. With the above data lengths, CRBs for local frequency and phase are summarized in Table 1, along with the ratio of the MSEs by the WFR algorithm with phase compensation over the CRBs.

It is seen that the results of the WFR algorithm with phase compensation are worse than the corresponding CRBs and thus they are suboptimal. The ratio MSE/CRB of both local frequency and phase estimates increase with c_2 and σ . Two examples are given in Table 1 with ($c_2 = 0.001$, $\sigma = 10$) and ($c_2 = 0.02$, $\sigma = 10$), respectively. It is interesting to see that, although the MSE of the local frequency estimate is not close to its CRB, the MSE of the phase estimate is. This is because phase error caused by the local frequency estimate error, as given in Eqs. (21) and (22), is not significant.

4. Simulations

In this section, Monte Carlo simulations will be carried out in order to (i) verify the above theoretical analysis, (ii) to numerically compare local frequency and phase estimates with CRBs, (iii) to give an intuitive illustration about the effectiveness of the WFR algorithm with phase compensation, and (iv) to compare the WFR algorithm with phase compensation with its sister algorithm WFF as well as other similar algorithms.

A. Monte Carlo Simulations and Comparison with CRBs

To verify the theoretical MSEs in previous sections, the WFR algorithm with phase compensation is applied to a quadratic phase field contaminated by additive white Gaussian noise. The parameters are chosen as $b = 1$, $c_0 = 1$, $c_1 = 4 \times 10^{-3}$, $c_2 = 1 \times 10^{-3}$, and $N = 151$. Such a phase, without noise, is shown in Fig. 1(a). The SNR of the signal varies from -10 dB to 20 dB with a 1 dB interval. For each SNR value, 500 Monte Carlo simulations are performed. For the parameters of the WFR algorithm, the frequency is searched from -1 to 1 with a step of 0.00001 , and the size of Gaussian window is $\sigma = 10$.

In Fig. 2, the achievable MSEs of the WFR algorithm with phase compensation are compared with the corresponding CRBs. The solid lines indicate the CRBs; the dash-dot lines depict the theoretically predicted MSEs; and the plus sign markers show the measured MSEs. The measured and theoretical MSEs agree very well for both local frequency and phase estimates. In addition, they are all suboptimal compared with their CRBs, which has been revealed by previous theoretical analysis. For the local frequency estimate, the MSE is worse than its CRB by less than 12 times (about 10 dB), while for the phase estimate, the MSE is worse than its CRB by only less than 2 times (about 3 dB). The MSEs in Fig. 2 are consistent with the Table 1. Taking phase as an example, the efficiency, i.e., CRB/MSE, is about 50%, which is not high. However, it can be seen from Fig. 2(b) that the MSE at 0 dB is 0.01 rad^2 , which is satisfactory in many applications. The threshold SNRs are often used to characterize an algorithm. Below the threshold SNR, the MSE dramatically increases. It can be observed from Figs. 2(a) and 2(b) that for the data with $N = 151$, the threshold SNR is about -5 dB for frequency estimate and about -3 dB for phase estimate, respectively.

B. Influences of c_2 and Window Truncation

It has been shown in Section 3 that the MSEs of the frequency and phase estimates by the WFR algorithm with phase compensation are affected by c_2 , which is also verified by the Monte Carlo simulations. In Figs. 3(a) and 3(b), the solid lines are CRBs; the plus sign, asterisk and diamond markers show the measured MSEs when c_2 are 0.001, 0.01, and 0.02, respectively. The simulated wrapped phase with $c_2 = 0.02$ is shown in Fig. 1(b) to illustrate how fast the phase changes. Theoretical MSEs are not shown in Fig. 3 as they agree well with the measured ones. It can be seen from Figs. 3(a) and 3(b) that when c_2 increases from 0.001 to 0.02, the

Table 1. Theoretical MSEs of Local Frequency and Phase Estimates by the WFR Algorithm with Phase Compensation and Their CRBs

	CRB	MSE/CRB	MSE/CRB ($c_2 = 0.001$, $\sigma = 10$)	MSE/CRB ($c_2 = 0.02$, $\sigma = 10$)
Local frequency estimate	$\frac{6}{(10c_1+1)^3 \text{SNR}}$	$11.75(1 + \sigma^4 c_2^2)^{5/2}$	12	656.85
Phase estimate	$\frac{10}{(15\sigma+1) \text{SNR}}$	$1.88(1 + \sigma^4 c_2^2)^{1/2}$	1.89	4.20

Fig. 1. (Color online) Simulated wrapped phase maps. (a) $c_2 = 0.001$, (b) $c_2 = 0.02$.

Fig. 2. (Color online) Theoretical and experimental MSEs by the WFR algorithm with phase compensation. (a) Local frequency estimate, (b) phase estimate.

Fig. 3. (Color online) The influence of c_2 in the WFR algorithm with phase compensation. (a) Local frequency estimate, (b) phase estimate.

MSE of the local frequency estimate increases by 54 times (about 17 dB), but the MSE of the phase estimate increases by only about 2 times (about 3 dB). Thus, the phase estimate is seen to be more robust to the increment of c_2 . Furthermore, the threshold SNR is increased to 0 dB for the local frequency estimate and 3 dB for the phase estimate when the data length is 151. It should be mentioned that with the advancement of detectors, data can be acquired with higher resolution and thus lower c_2 value.

For the WFR algorithm, since the size of the Gaussian window is truncated to $6\sigma + 1$ instead of $10\sigma + 1$ in real practices, it is of interest to see the impact by the truncated window size. The SNR of the signal varies from -10 dB to 100 dB with a 5 dB interval. The Monte Carlo simulation results for the WFR algorithm with phase compensation with window sizes of $6\sigma + 1 = 61$ and $10\sigma + 1 = 101$ are shown in Fig. 4 as plus signs markers and diamond markers, respectively. It can be seen that this truncated size contributes an MSE of 10^{-6} rad² (-60 dB), which is considered not significant in practice.

C. WFR with Phase Compensation Versus WFF

The WFR algorithm with phase compensation is also compared with its sister algorithm WFF by Monte Carlo simulations. The description of the WFF has been documented in detail in [2] and is omitted to make the paper concise. The signal simulated in Sec. 4.1 is used again. For the parameters of the WFF algorithm, the frequency varies from -1 to 1 with a step of 0.00001 , the size of the Gaussian window is $\sigma = 10$, and the threshold is set as 3 . The truncated Gaussian window size for the WFF algorithm is selected as $4\sigma + 1$ since larger window size does not improve the result significantly [2]. Figure 5 shows the phase estimates by the WFR algorithm with phase compensation and WFF algorithm, where solid lines are CRBs; plus signs markers indicate the mea-

Fig. 4. (Color online) The influence of window truncation, $6\sigma + 1 = 61$ versus $10\sigma + 1 = 101$.

Fig. 5. (Color online) Phase estimate by the WFR algorithm with phase compensation and WFF algorithm.

sured MSEs by the WFR algorithm with phase compensation, which is denoted as WFR in the figure; and diamond markers show the measured MSEs by the WFF algorithm. It can be seen that the WFF algorithm for phase estimate is close to the CRB and slightly worse than the WFR algorithm with phase compensation. This is not surprising as the WFR retrieves parameter from the highest spectrum coefficient, i.e., the ridge, while the WFF utilizes the coefficients higher than a threshold but could be lower than the ridge.

D. Discussions

The HAF and the CPF are two typical algorithms for parameter estimation from an EPF. Both of them need a preprocessing procedure to decouple the parameters to simplify the optimization problem.

Fig. 6. (Color online) Comparison of phase estimate by the WFR algorithm with phase compensation, HAF, and CPF algorithms.

Details of these two algorithms can be found in [3,6]. The preprocessing procedure reduces the robustness of those two algorithms and makes the estimation more difficult, especially when the input signal is very noisy. The WFR algorithm with phase compensation processes the original signal directly and is thus simpler. Comparing with the HAF and CPF, the WFR algorithm with phase compensation produces larger MSEs, but it can process more noisy signals successfully. For instance, to process the same signal as Section 4.1, the threshold SNR for both HAF and CPF algorithm is 0 dB while the threshold SNR for the WFR algorithm with phase compensation is -3 dB, which is shown in Fig. 6. Although the threshold SNR for the WFR algorithm with phase compensation is also dependent on c_2 , it can be seen from the experiments that when $c_2 < 0.01$, the threshold SNR for the WFR algorithm with phase compensation is less than both HAF and CPF algorithm. For large c_2 (i.e., $c_2 \geq 0.01$), congruence operation can be used to make the WFR algorithm with phase compensation estimator optimal [15].

5. Conclusion

In this paper, statistical analysis is given for the WFR algorithm with phase compensation when the quadratic exponential phase field is contaminated by additive white Gaussian noise. The Monte Carlo simulations are used to verify the theoretical analysis. The MSEs for local frequency and phase estimates depend on the second-order derivative of the phase, the Gaussian window size, and the SNR. Theoretical analysis and Monte Carlo simulations show that the MSEs for the local frequency and phase estimates are suboptimal comparing with the corresponding CRBs. When c_2 increases, the MSEs and threshold SNRs for both local frequency and phase estimates increase accordingly. However, the MSEs for the phase estimate are close to the CRB. The performance of the WFR algorithm with phase compensation is compared with the WFF algorithm. For quadratic phase signal, the WFR algorithm with phase compensation is slightly better than the WFF algorithm.

Appendix A: Perturbation Analysis Preliminaries

This part follows the perturbation analysis in Ref. [3]. Given a complex valued function $g_N(\Omega)$ that depends on a variable Ω and on the number of data points N , let

$$f_N(\Omega) = |g_N(\Omega)|^2 = g_N(\Omega)g_N^*(\Omega), \quad (\text{A1})$$

where the symbol $*$ denotes the conjugate of a complex number. Assume that $f_N(\Omega)$ has a global maximum at $\Omega = \Omega_0$. If $g_N(\Omega)$ is perturbed by $\delta g_N(\Omega)$, then the global maximum point will shift to $\Omega = \Omega_0 + \delta\Omega$. A first-order approximation for $\delta\Omega$ is as follows:

$$\delta\Omega \approx -\frac{B}{A}, \quad (\text{A2})$$

where

$$A = 2 \operatorname{Re} \left[g_N(\Omega_0) \frac{\partial^2 g_N^*(\Omega_0)}{\partial \Omega^2} + \frac{\partial g_N(\Omega_0)}{\partial \Omega} \frac{\partial g_N^*(\Omega_0)}{\partial \Omega} \right], \quad (\text{A3})$$

$$B = 2 \operatorname{Re} \left[g_N(\Omega_0) \frac{\partial \delta g_N^*(\Omega_0)}{\partial \Omega} + \frac{\partial g_N(\Omega_0)}{\partial \Omega} \delta g_N^*(\Omega_0) \right]. \quad (\text{A4})$$

The MSE of $\delta\Omega$ is

$$E[(\delta\Omega)^2] \approx \frac{E(B^2)}{A^2}. \quad (\text{A5})$$

Appendix B: Mean Squared Error of $\hat{\omega}(U)$

When Eqs. (12) and (16) are used to estimate $\omega(u)$, the additive noises cause a shift $\delta\omega$ that can be estimated according to Eq. (A2) in Appendix A. With $Sf_0(u; \xi)$ corresponding to $g_N(\Omega)$, $Sn(u; \xi)$ corresponding to $\delta g_N(\Omega)$, and $\hat{\omega}(u) = \omega(u) + \delta\omega(u)$ corresponding to $\Omega = \Omega_0 + \delta\Omega$, the shift $\delta\omega(u)$ can be derived from Eq. (A2) as follows:

$$\begin{aligned} \delta\omega(u) &\approx -\frac{B}{A} \\ &= -\frac{(1 + \sigma^4 c_2^2)^{5/4}}{b\sigma^2(4\pi\sigma^2)^{1/4}} \operatorname{Im} \left(\exp\{j[c_0 - 0.5c_2u^2 \right. \\ &\quad \left. + 0.5 \arctan(\sigma^2 c_2)]\} \int_{-\infty}^{\infty} n^*(x)g(x-u) \right. \\ &\quad \left. \times (x-u) \exp(j\omega x) dx \right), \end{aligned} \quad (\text{B1})$$

for which the following intermediate results are needed:

$$\begin{aligned} Sf_0(u; \omega) &= b \left(\frac{4\pi\sigma^2}{1 + \sigma^4 c_2^2} \right)^{1/4} \exp\{j[c_0 - 0.5c_2u^2 \\ &\quad + 0.5 \arctan(\sigma^2 c_2)]\}, \end{aligned} \quad (\text{B2})$$

$$\frac{\partial Sf_0(u; \omega)}{\partial \xi} = -ju Sf_0(u; \omega), \quad (\text{B3})$$

$$\frac{\partial^2 Sf_0^*(u; \omega)}{\partial \xi^2} = -Sf_0^*(u; \omega) \left[u^2 + \frac{\sigma^2}{(1 + j\sigma^2 c_2)} \right], \quad (\text{B4})$$

$$Sn^*(u; \omega) = \int_{-\infty}^{\infty} n^*(x)g(u-x) \exp(j\omega x) dx, \quad (\text{B5})$$

$$\frac{\partial S n^*(u; \omega)}{\partial \xi} = j \int_{-\infty}^{\infty} n^*(x) g(u-x) x \exp(j\omega x) dx. \quad (\text{B6})$$

Since $n_r(x)$ and $n_i(x)$ are independent with zero-mean, it can be seen from Eq. (B1) that

$$E[\delta\omega(u)] = 0. \quad (\text{B7})$$

Given two points u_1 and u_2 , using Eq. (B1) and a long derivation, we have

$$E[\delta\omega(u_1) \cdot \delta\omega(u_2)] \approx \frac{(1 + \sigma^4 c_2^2)^{5/2}}{8\sqrt{\pi}\sigma^3 \text{SNR}} \left[1 - \frac{(1 + \sigma^4 c_2^2)s^2}{2\sigma^2} \right] \times \exp \left[-\frac{(1 + \sigma^4 c_2^2)s^2}{4\sigma^2} \right], \quad (\text{B8})$$

where $\text{SNR} = \frac{b^2}{2\sigma_n^2}$ and $s = u_2 - u_1$. By setting $u_1 = u_2$, Eq. (B8) becomes

$$E\{[\delta\omega(u_1)]^2\} = E\{[\delta\omega(u_2)]^2\} = V_\omega \approx \frac{(1 + \sigma^4 c_2^2)^{5/2}}{8\sqrt{\pi}\sigma^3 \text{SNR}}, \quad (\text{B9})$$

from which the variance of the local frequency is seen independent of the location u .

When c_2 is estimated by Eq. (14), its error is

$$\delta c_2 = [\delta\omega(u_2) - \delta\omega(u_1)]/(u_2 - u_1), \quad (\text{B10})$$

which has a mean of zero. The variance of δc_2 can be derived as

$$E[(\delta c_2)^2] \approx \frac{2}{s^2} V_\omega \left\{ 1 - \left[1 - \frac{s^2}{2\sigma^2} (\sigma^4 c_2^2 + 1) \right] \times \exp \left[-\frac{s^2 (\sigma^4 c_2^2 + 1)}{4\sigma^2} \right] \right\}. \quad (\text{B11})$$

Appendix C: Mean Squared Error of $\hat{\varphi}(\mathbf{U})$

In this Appendix, the MSE of the phase estimate is derived when the intrinsic signal is perturbed by additive white Gaussian noise. As seen from Section 2, the ridge locates at

$$\xi = \hat{\omega}(u) = c_1 + c_2 u + \delta\omega(u). \quad (\text{C1})$$

The windowed Fourier spectrum of Eq. (1) is

$$\begin{aligned} \hat{V} &= S f[u; \hat{\omega}(u)] \exp[j\hat{\omega}(u)u] \\ &= S f_0[u; \hat{\omega}(u)] \exp[j\hat{\omega}(u)u] \\ &\quad + S n[u; \hat{\omega}(u)] \exp[j\hat{\omega}(u)u]. \end{aligned} \quad (\text{C2})$$

Following the derivations in Refs [3,6], the noises are assumed to be small for the first-order perturbation analysis. Thus, for the first term on the right hand of Eq. (C2) regarding the intrinsic signal, the second-

order Taylor expansion of exponential function is used, while for the second term regarding noise, the zero-order Taylor expansion is employed. This approximation leads Eq. (C2) to the following equation:

$$\begin{aligned} \hat{V} &\approx S f_0[u; \omega(u)] \exp[j\omega(u)u] + S n[u; \omega(u)] \exp[j\omega(u)u] \\ &\quad - 0.5 \int_{-\infty}^{\infty} f_0(x) (\delta\omega)^2 (u-x)^2 \\ &\quad \times \exp[j\omega(u-x)] g(u-x) dx, \end{aligned} \quad (\text{C3})$$

which can be further derived as

$$\hat{V} \approx V_0 (1 + \tau - \eta), \quad (\text{C4})$$

where

$$\begin{aligned} V_0 &= S f_0[u; \omega(u)] \exp[j\omega(u)u] \\ &= b \left(\frac{4\pi\sigma^2}{1 + \sigma^4 c_2^2} \right)^{1/4} \exp\{j[\varphi(u) + 0.5 \arctan(\sigma^2 c_2)]\}, \end{aligned} \quad (\text{C5})$$

$$\tau = S n[u; \omega(u)] \exp[j\omega(u)u] / V_0, \quad (\text{C6})$$

$$\eta = \frac{\sigma^2 (1 + j\sigma^2 c_2)}{2(1 + \sigma^4 c_2^2)} [\delta\omega(u)]^2. \quad (\text{C7})$$

The following approximation can be used when D is small:

$$\log[C(1 + D)] \approx \log C + D. \quad (\text{C8})$$

Taking $D = \tau - \eta$ gives

$$\begin{aligned} \log \hat{V} &\approx \log \left[b \left(\frac{4\pi\sigma^2}{1 + \sigma^4 c_2^2} \right)^{1/4} \right] \\ &\quad + j[\varphi(u) + 0.5 \arctan(\sigma^2 c_2)] + \tau - \eta. \end{aligned} \quad (\text{C9})$$

The phase of $\hat{V}$ is

$$\begin{aligned} \hat{\Phi} &= \text{angle}(\hat{V}) = \text{Im}[\log(\hat{V})] \\ &\approx \varphi(u) + \text{Im}(\tau) - \text{Im}(\eta) + 0.5 \arctan(\sigma^2 c_2). \end{aligned} \quad (\text{C10})$$

According to Eq. (13), $0.5 \arctan(\sigma^2 \hat{c}_2)$ is deducted from $\text{angle}(\hat{V})$ and thus,

$$\begin{aligned} \delta\varphi &\approx \text{Im}(\tau) - \text{Im}(\eta) + 0.5 \arctan(\sigma^2 c_2) \\ &\quad - 0.5 \arctan(\sigma^2 \hat{c}_2) \\ &= \text{Im}(\tau) - \frac{\sigma^4 c_2}{2(1 + \sigma^4 c_2^2)} [\delta\omega(u)]^2 - \frac{0.5\sigma^2 \delta c_2}{1 + \sigma^4 c_2^2} \\ &= (\delta\varphi)_1 + (\delta\varphi)_2 + (\delta\varphi)_3, \end{aligned} \quad (\text{C11})$$

where $\arctan(a) - \arctan(b) = \arctan(\frac{a-b}{1+ab})$, $\frac{\sigma^2 \delta c_2}{1 + \sigma^4 (c_2 \delta c_2 + c_2^2)} \ll 1$ and $\delta c_2 \ll c_2$ are used. For simplicity, the three terms in the right hand of Eq. (C11) are denoted as $(\delta\varphi)_1$, $(\delta\varphi)_2$, and $(\delta\varphi)_3$, respectively.

It is not difficult to see that $E[(\delta\varphi)_1] = E[(\delta\varphi)_3] = 0$. Thus, according to Eq. (B9),

$$\begin{aligned} E(\delta\varphi) &\approx E[(\delta\varphi)_2] = E\left\{-\frac{\sigma^4 c_2}{2(1 + \sigma^4 c_2^2)} [\delta\omega(u)]^2\right\} \\ &= -\frac{\sigma c_2 (1 + \sigma^4 c_2^2)^{3/2}}{16\sqrt{\pi} \text{SNR}}. \end{aligned} \quad (\text{C12})$$

The MSE of the phase consists of six terms as follows:

$$\begin{aligned} E[(\delta\varphi)^2] &= E[(\delta\varphi)_1^2] + E[(\delta\varphi)_2^2] + E[(\delta\varphi)_3^2] \\ &\quad + 2E[(\delta\varphi)_1(\delta\varphi)_2] + 2E[(\delta\varphi)_2(\delta\varphi)_3] \\ &\quad + 2E[(\delta\varphi)_1(\delta\varphi)_3]. \end{aligned} \quad (\text{C13})$$

For the first term of Eq. (C13), according to Ref. [2],

$$E[(\delta\varphi)_1^2] = \text{Var}[\text{Im}(\tau)] = \frac{(1 + \sigma^4 c_2^2)^{1/2}}{4\sqrt{\pi} \sigma \text{SNR}}. \quad (\text{C14})$$

Assume the probability density function of $\delta\omega$ is Gaussian, the second term can be approximated as follows according to page 27 in Ref. [10]:

$$\begin{aligned} E[(\delta\varphi)_2^2] &= \frac{\sigma^8 c_2^2}{4(1 + \sigma^4 c_2^2)^2} E\{[\delta\omega(u)]^4\} \\ &\approx \frac{\sigma^8 c_2^2}{4(1 + \sigma^4 c_2^2)^2} 3(E\{[\delta\omega(u)]^2\})^2 \\ &= \frac{3\sigma^2 c_2^2 (1 + \sigma^4 c_2^2)^3}{256\pi (\text{SNR})^2}. \end{aligned} \quad (\text{C15})$$

By using Eq. (B11), the third term can be derived as

$$\begin{aligned} E[(\delta\varphi)_3^2] &= \frac{\sigma(1 + \sigma^4 c_2^2)^{1/2}}{16\sqrt{\pi} s^2 \text{SNR}} \left\{1 - \left[1 - \frac{s^2}{2\sigma^2} (\sigma^4 c_2^2 + 1)\right] \right. \\ &\quad \times \exp\left[-\frac{s^2(\sigma^4 c_2^2 + 1)}{4\sigma^2}\right]\}. \end{aligned} \quad (\text{C16})$$

Because of the symmetric distribution of $\delta\omega$, $E[(\delta\varphi)_2(\delta\varphi)_3]$ can be evaluated to be zero [Ref. 10, page 27]. The term $E[(\delta\varphi)_1(\delta\varphi)_2]$ can be evaluated to be zero by using Eqs. (C7) and (B1), of which the long derivation is skipped. For the last term, the covariance between $(\delta\varphi)_1$ and $(\delta\varphi)_3$ can be derived as

$$\begin{aligned} E[(\delta\varphi)_1(\delta\varphi)_3] &= \frac{(1 + \sigma^4 c_2^2)^{1/2}}{16\sqrt{\pi} \sigma s \text{SNR}} \left\{ (u - u_1) \right. \\ &\quad \times \exp\left[-\frac{(u - u_1)^2 (1 + \sigma^4 c_2^2)}{4\sigma^2}\right] - (u - u_2) \\ &\quad \times \exp\left[-\frac{(u - u_2)^2 (1 + \sigma^4 c_2^2)}{4\sigma^2}\right]\}. \end{aligned} \quad (\text{C17})$$

Consequently, the MSE of the phase estimate can be summarized as

$$\begin{aligned} E[(\delta\varphi)^2] &\approx \frac{(1 + \sigma^4 c_2^2)^{1/2}}{4\sqrt{\pi} \sigma \text{SNR}} [1 + k_1(c_2, \text{SNR}, \sigma) \\ &\quad + k_2(c_2, \sigma, s) + k_3(c_2, \sigma, s, l)], \end{aligned} \quad (\text{C18})$$

with

$$k_1(c_2, \text{SNR}, \sigma) = \frac{3\sigma^3 c_2^2 (1 + \sigma^4 c_2^2)^{5/2}}{64\sqrt{\pi} \text{SNR}}, \quad (\text{C19})$$

$$\begin{aligned} k_2(c_2, \sigma, s) &= \frac{(1 + \sigma^4 c_2^2)^2}{s^2 \sigma^2} \left\{1 - \left[1 - \frac{s^2}{2\sigma^2} (1 + \sigma^4 c_2^2)\right] \right. \\ &\quad \times \exp\left[-\frac{s^2(1 + \sigma^4 c_2^2)}{4\sigma^2}\right]\}, \end{aligned} \quad (\text{C20})$$

$$\begin{aligned} k_3(c_2, \sigma, s, l) &= \frac{1}{4s} \left\{ l \exp\left[-\frac{l^2(1 + \sigma^4 c_2^2)}{4\sigma^2}\right] - (l - s) \right. \\ &\quad \times \exp\left[-\frac{(l - s)^2(1 + \sigma^4 c_2^2)}{4\sigma^2}\right]\}, \end{aligned} \quad (\text{C21})$$

where $l = u - u_1$ and $s = u_2 - u_1$.

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References

1. Q. Kemao, "Two-dimensional windowed Fourier transform for fringe pattern analysis: Principles, application and implementation," *Opt. Laser Eng.* **45**, 304–317 (2007).
2. Q. Kemao, H. Wang, and W. Gao, "Windowed Fourier transform for fringe pattern analysis: theoretical analyses," *Appl. Opt.* **47**, 5408–5419 (2008).
3. S. Peleg and B. Porat, "Linear FM signal parameter estimation from discrete-time observations," *IEEE Trans. Aerosp. Electron. Syst.* **27**, 607–616 (1991).
4. B. Friedlander and J. M. Francos, "Model based phase unwrapping of 2-D signals," *IEEE Trans. Signal Process.* **44**, 2999–3007 (1996).
5. S. S. Gorthi and P. Rastogi, "Improved high-order ambiguity-function method for the estimation of phase from interferometric fringes," *Opt. Lett.* **34**, 2575–2577 (2009).
6. P. O'Shea, "A fast algorithm for estimating the parameters of a quadratic FM signal," *IEEE Trans. Signal Process.* **52**, 385–393 (2004).
7. S. S. Gorthi, G. Rajshekhar, and P. Rastogi, "Strain estimation in digital holographic interferometry using piecewise polynomial phase approximation based method," *Opt. Express* **18**, 560–565 (2010).
8. H. Cramer, *Mathematical Methods of Statistics* (Princeton University, 1999).
9. C. R. Rao, "Information and the accuracy attainable in the estimation of statistical parameters," *Bull. Calcutta Math. Soc.* **37**, 81–91 (1945).
10. A. C. Tamhane and D. D. Dunlop, *Statistics and Data Analysis: From Elementary to Intermediate* (Prentice-Hall, 2000).
11. K. Creath, "Phase-measurement interferometry techniques," in *Progress in Optics* (Elsevier, 1988), Vol. XXVI, pp. 349–393.

12. M. Takeda, H. Ina, and S. Kobayashi, "Fourier transform methods of fringe-pattern analysis for computer-based topography and interferometry," *J. Opt. Soc. Am.* **72**, 156–160 (1982).
13. U. Schnars and W. Jueptner, *Digital Holography: Digital Hologram Recording, Numerical Reconstruction, and Related Techniques* (Springer, 2005).
14. S. Peleg and B. Porat, "The achievable accuracy in estimating the instantaneous phase and frequency of a constant amplitude signal," *IEEE Trans. Signal Process.* **41**, 2216–2224 (1993).
15. Q. Kemao, W. Gao, and H. Wang, "Windowed Fourier filtered and quality guided phase unwrapping algorithm: On locally high-order polynomial phase," *Appl. Opt.* **49**, 1075–1079 (2010).