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Ling, San; Solé, Patrick

2003

Ling, S., & Solé, P. (2003). Good self-dual quasi-cyclic codes exist. *IEEE Transactions on Information Theory*, 49(4), 1052-1053.

<https://hdl.handle.net/10356/96309>

<https://doi.org/10.1109/TIT.2003.809501>

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Good Self-Dual Quasi-Cyclic Codes Exist

San Ling and Patrick Solé, *Member, IEEE*

Abstract—We show that there are long binary quasi-cyclic self-dual (either Type I or Type II) codes satisfying the Gilbert–Varshamov bound.

Index Terms—Cubing construction, Gilbert–Varshamov bound, quasi-cyclic codes, self-dual codes.

I. INTRODUCTION

It has been known for 30 years that good long self-dual codes exist [6], and for more than a quarter century [1] that there are good long quasi-cyclic codes of rate $1/2$.

In this correspondence, we show that good long self-dual quasi-cyclic codes exist. Building on well-known mass formulas for self-dual binary and quaternary codes, we derive a Gilbert–Varshamov bound for long binary self-dual quasi-cyclic codes.

The proof uses the cubing construction of [5], [3] and the proof technique of [6].

As suggested by one referee, it might have been possible to build on [1] to derive this asymptotic result. However, [1] uses quasi-cyclic codes of index 2 while we use quasi-cyclic codes of index $n/3$, where n denotes the length. In some sense, we provide information on a different asymptotic ensemble of codes than [1].

II. KNOWN FACTS AND NOTATIONS

A code is said to be **quasi-cyclic of index ℓ** or ℓ -quasi-cyclic if and only if it is invariant under T^ℓ , where T denotes the cyclic shift. If $\ell = 1$, such a code is just a cyclic code. We assume that all binary codes are equipped with the Euclidean inner product and all the $\mathbf{F}_4$ -codes are equipped with the Hermitian inner product. The latter condition is necessary, when using the cubing construction, to ensure that the resulting binary code is Euclidean self-dual. Self-duality in the following discussion is with respect to these respective inner products. A binary self-dual code is said to be of Type II if and only if all its weights are multiples of 4 and of Type I otherwise. We first recall some background material on mass formulas for self-dual binary and quaternary codes.

Proposition 2.1: Let ℓ be an even positive integer.

- i) The number of self-dual binary codes of length ℓ is given by

$$N(2, \ell) = \prod_{i=1}^{\frac{\ell}{2}-1} (2^i + 1).$$

- ii) Let $\mathbf{v}$ be a codeword of length ℓ and even Hamming weight, other than $\mathbf{0}$ and $\mathbf{1}$. The number of self-dual binary codes of length ℓ containing $\mathbf{v}$ is given by

$$M(2, \ell) = \prod_{i=1}^{\frac{\ell}{2}-2} (2^i + 1).$$

- iii) The number of self-dual $\mathbf{F}_4$ -codes of length ℓ is given by

$$N(4, \ell) = \prod_{i=0}^{\frac{\ell}{2}-1} (2^{2i+1} + 1).$$

- iv) The number of self-dual $\mathbf{F}_4$ -codes of length ℓ containing a given nonzero codeword of length ℓ and even Hamming weight is given by

$$M(4, \ell) = \prod_{i=0}^{\frac{\ell}{2}-2} (2^{2i+1} + 1).$$

Proof: i) and iii) are well-known facts, cf. [7]. ii) is an immediate consequence of [6, Theorem 2.1] with $s = 2$. (Note that every self-dual binary code must contain the all-one vector $\mathbf{1}$.) iv) follows from [2, Theorem 1] with $n_1 = \ell$ and $k_1 = 1$. $\square$

Proposition 2.2: Let ℓ be a positive integer divisible by 8.

- i) The number of Type II binary codes of length ℓ is given by

$$T(2, \ell) = 2 \prod_{i=1}^{\frac{\ell}{2}-2} (2^i + 1).$$

- ii) Let $\mathbf{v}$ be a codeword of length ℓ and Hamming weight divisible by 4, other than $\mathbf{0}$ and $\mathbf{1}$. The number of Type II binary codes of length ℓ containing $\mathbf{v}$ is given by

$$S(2, \ell) = 2 \prod_{i=1}^{\frac{\ell}{2}-3} (2^i + 1).$$

Proof: i) is found in [7] and ii) is exactly [6, Corollary 2.4]. $\square$

III. MAIN RESULT

Let C_1 denote a binary code of length ℓ and C_2 a quaternary code of length ℓ . We construct a binary code C of length 3ℓ by the cubing construction [3]. Define a map

$$\Phi: C_1 \times C_2 \longrightarrow \mathbf{F}_2^{3\ell}$$

by the rule

$$\Phi(\mathbf{x}, \mathbf{a} + \mathbf{b}\omega) := (\mathbf{x} + \mathbf{a}, \mathbf{x} + \mathbf{b}, \mathbf{x} + \mathbf{a} + \mathbf{b})$$

where $\mathbf{a}, \mathbf{b}$ are binary vectors of length ℓ , and we write $\mathbf{F}_4 = \{0, 1, \omega, \omega^2\}$. Then we can define the code C as $\text{Im}(\Phi)$

$$C := \{\Phi(\mathbf{x}, \mathbf{a} + \mathbf{b}\omega) \mid \mathbf{x} \in C_1, \mathbf{a} + \mathbf{b}\omega \in C_2\}.$$

Now a direct calculation shows that

$$\Phi(\mathbf{x}, \omega^2(\mathbf{a} + \mathbf{b}\omega)) = (\mathbf{x} + \mathbf{a} + \mathbf{b}, \mathbf{x} + \mathbf{a}, \mathbf{x} + \mathbf{b})$$

is a shift of $\Phi(\mathbf{x}, \mathbf{a} + \mathbf{b}\omega)$ by ℓ places. Therefore, C is ℓ -quasi-cyclic. Furthermore, it is easy to check that C is self-dual if and only if both C_1 and C_2 are, and C is of Type II if and only if C_1 is of Type II and C_2 is self-dual.

We assume henceforth that C is a self-dual code constructed in the above way. Any codeword $\mathbf{c}$ in C must necessarily have even Hamming weight. Suppose that $\mathbf{c}$ corresponds to the pair $(\mathbf{c}_1, \mathbf{c}_2)$, where $\mathbf{c}_1 \in C_1$ and $\mathbf{c}_2 \in C_2$. Since C_1 and C_2 are self-dual, it follows that

S. Ling is with the Department of Mathematics, National University of Singapore, Singapore 117543, Republic of Singapore (e-mail: matlings@nus.edu.sg).

P. Solé is with CNRS, I3S, ESSI, 06903 Sophia Antipolis, France (e-mail: ps@essi.fr).

Communicated by S. Litsyn, Associate Editor for Coding Theory.

Digital Object Identifier 10.1109/TIT.2003.809501

c_1 and c_2 must both have even Hamming weights. When $c \neq 0$, there are three possibilities for the pair (c_1, c_2) :

- 1) $c_1 \neq 0, c_2 \neq 0$;
- 2) $c_1 = 0, c_2 \neq 0$;
- 3) $c_1 \neq 0, c_2 = 0$.

We try to enumerate the number of words c in each of these categories for a given weight d (d even).

For type 2, if the Hamming weight of c is d , then c_2 has Hamming weight $d/2$. Since c_2 has even Hamming weight, it follows that d is divisible by 4 in order for this case to occur. It is easy to see that the number $A_2(\ell, d)$ of such words c is given by $\binom{\ell}{d/2} 3^{d/2} (4|d)$. For d not divisible by 4, set $A_2(\ell, d) = 0$.

The argument to obtain the number of words of type 3 is similar. It is easy to show that the number $A_3(\ell, d)$ of such words is given by $\binom{\ell}{d/3} (6|d)$. When d is not divisible by 6, $A_3(\ell, d) = 0$.

For $A_1(\ell, d)$, the number of words of type 1, we simply give an upper bound. The total number of words in $F_2^{3\ell}$ of weight d is $\binom{3\ell}{d}$, so

$$A_1(\ell, d) \leq \binom{3\ell}{d} - A_2(\ell, d) - A_3(\ell, d).$$

Combining the preceding observations and Proposition 2.1, the number of self-dual binary ℓ -quasi-cyclic codes of length 3ℓ whose minimum weight is $< d$ is bounded above by

$$\sum_{e < d, e \text{ even}} (A_1(\ell, e)M(2, \ell)M(4, \ell) + A_2(\ell, e)N(2, \ell)M(4, \ell) + A_3(\ell, e)M(2, \ell)N(4, \ell)).$$

Theorem 3.1: Let ℓ be an even integer and let d be the largest even integer such that

$$\sum_{\substack{e < d \\ e \equiv 0 \pmod{2}}} \binom{3\ell}{e} + \left(\sum_{\substack{e < d \\ e \equiv 0 \pmod{4}}} \binom{\ell}{e/2} 3^{e/2} \right) 2^{\frac{\ell}{2}-1} + \left(\sum_{\substack{e < d \\ e \equiv 0 \pmod{6}}} \binom{\ell}{e/3} \right) 2^{\ell-1} \leq (2^{\frac{\ell}{2}-1} + 1)(2^{\ell-1} + 1).$$

Then there exists a self-dual binary ℓ -quasi-cyclic code of length 3ℓ with minimum weight of at least d .

If we are interested only in Type II ℓ -quasi-cyclic codes, using Proposition 2.2, we see easily that the number of Type II binary ℓ -quasi-cyclic codes of length 3ℓ whose minimum weight is $< d$ is bounded above by

$$\sum_{e < d, e \equiv 0 \pmod{4}} (A_1(\ell, e)S(2, \ell)M(4, \ell) + A_2(\ell, e)T(2, \ell)M(4, \ell) + A_3(\ell, e)S(2, \ell)N(4, \ell)).$$

Theorem 3.2: Let ℓ be divisible by 8 and let d be the largest multiple of 4 such that

$$\sum_{\substack{e < d \\ e \equiv 0 \pmod{4}}} \binom{3\ell}{e} + \left(\sum_{\substack{e < d \\ e \equiv 0 \pmod{4}}} \binom{\ell}{e/2} 3^{e/2} \right) 2^{\frac{\ell}{2}-2} + \left(\sum_{\substack{e < d \\ e \equiv 0 \pmod{12}}} \binom{\ell}{e/3} \right) 2^{\ell-1} \leq (2^{\frac{\ell}{2}-2} + 1)(2^{\ell-1} + 1).$$

Then there exists a Type II binary ℓ -quasi-cyclic code of length 3ℓ with minimum weight of at least d .

IV. ASYMPTOTIC ANALYSIS

We will require the celebrated entropy function

$$H(x) := -x \log_2(x) - (1-x) \log_2(1-x)$$

defined for $x \in (0, 1)$ and of constant use in estimating binomial coefficients of large arguments [5, pp. 309–310].

We are now in a position to state and prove the asymptotic versions of Theorems 3.1 and 3.2.

Theorem 4.1: There exists an infinite family of self-dual quasi-cyclic binary codes C_i of length $3\ell_i$ and of distance d_i such that the limit δ of $d_i/3\ell_i$ for large i exists and is bounded below as

$$\delta \geq H^{-1}(1/2) = 0.110 \dots$$

Proof: The right-hand side (RHS) of the inequality of Theorem 3.1 is plainly of the order of $2^{3\ell/2}$ for large ℓ . We compare this in turn to each of the three summands in the left-hand side (at the price of a more stringent inequality, congruence conditions on the summation range are neglected). By [5, Ch. 10, Corollary 9], for large ℓ (with $\mu = \delta$ and $n = \ell$), the first and third summands are of order $2^{3\ell H(\delta)}$ and $2^{\ell + \ell H(\delta)}$, respectively. They both are of the order of the RHS for $H(\delta) = 1/2$. By [5, Ch. 10, Lemma 7], for large ℓ (with $\lambda = \delta$ and $n = \ell$), the second summand is of order $2^{\ell f(3\delta/2)}$ for $f(t) := 0.5 + t \log_2(3) + H(t)$, which is of the order of the RHS for

$$\delta = 0.1762 \dots$$

a value $> H^{-1}(1/2)$. $\square$

Similarly, for doubly even codes, we have the following.

Theorem 4.2: There exists an infinite family of Type II quasi-cyclic binary codes C_i of length $3\ell_i$ and of distance d_i such that the limit δ of $d_i/3\ell_i$ for large i exists and is bounded below as

$$\delta \geq H^{-1}(1/2) = 0.110 \dots$$

Proof: Since we neglected the congruence conditions in the preceding analysis, the calculations are exactly the same. $\square$

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