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Performance of the MAP/G/1 Queue Under the Dyadic Control of Workload and Server Idleness

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Abstract This pap
queue under the dyadic control of the D-policy and multiple server vacations. We derive the probability generating function of the queue length and the mean queue length. We then present computational experiences and compare the MAP queue with the Poisson queue.

Keywords MAP/G/1 queue · Dyadic control · Workload and server idleness

1 Introduction

The control of queueing systems has been the subject of vast amount of research papers for the last thirty years. Readers are referenced to Crabill et al. (1977) and Tadj and Choudhury (2005) for the discussions and references.

Among the control schemes, the N-policy of Yadin and Naor (1963), The D-policy of Balachandran (1973) and the T-policy of Heyman (1977) (and vacation system of Levy and Yechiali (1975) as a generalization) have received most attention. In contrast with the well-known N-policy where the server begins to serve the customers only when there are N customers accumulated in the system, the D-policy controls the queueing system by the workload of the waiting customers. The T-policy was developed based on the idea of utilizing the idleness of the server. These control policies can be employed in actual manufacturing settings to control the number of start-ups per unit time and thereby reduce the overall long-run average operating cost per unit time.

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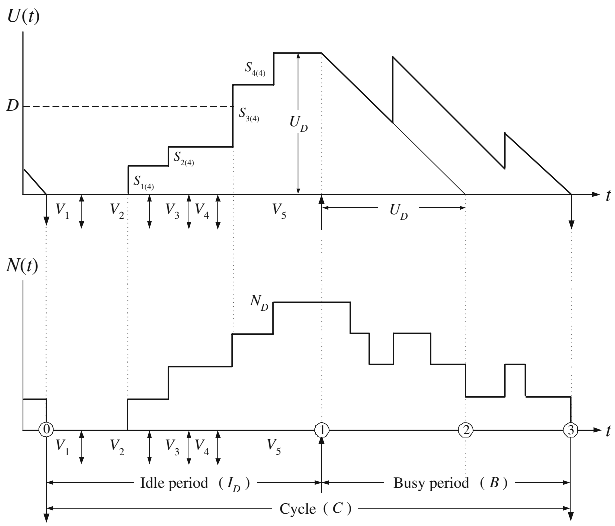
This paper studies the queue length of the MAP/G/1 queue under the mixed control of the D-policy and multiple server vacations (i.e., generalized T-policy). In our system, customers arrive according to the MAP (Markovian arrival process). It is known that the MAP can represent a variety of processes which include, as special cases, the Poisson process, the phase-type renewal processes, the MMPP (Markov modulated Poisson process) and superpositions of these. Readers are advised to see Lucantoni et al. (1990), Lucantoni (1991, 1993), Neuts (1981, 1989) and Ramaswami (1980) for the formal definition of the MAP and early analyses of MAP-related queues.

In our system under study, the server leaves for repeated vacations as soon as the system becomes empty. It resumes its service only when the cumulative workload is found to be greater than the predetermined threshold D at the end of a vacation. Figure 1 shows both the workload process and the queue length process on the synchronized time scale.

The behavioral complexity of our system is in the relationship between the service times of the customers who arrive during the idle period (i.e., vacation period). For example, suppose that the threshold D is crossed by the third customer and there are four customers at the start of the busy period (see the upper part of Fig. 1). Obviously the service times $S_{1(4)}$, $S_{2(4)}$ and $S_{3(4)}$ of the first three customers are not independent. Moreover, they are stochastically different from the ordinary service time random variable because $S_{1(4)}$ and $S_{2(4)}$ are smaller than D. The server must spend $U_D = S_{1(4)} + S_{2(4)} + S_{3(4)} + S_{4(4)}$ amount of time on serving the four 'special' customers during each of which it is necessary to keep track of the arrivals of the 'ordinary' customers who have iid ordinary service times.

Due to the dependencies of the service times, the well-known decomposition property of Fuhrmann and Cooper (1985), Shanthikumar (1988) and Doshi (1990) for the M/G/1 queues with generalized vacations can not be applicable to the D-policy queueing systems.

Fig. 1 The queue length and workload process



Studies on the D-policy queueing systems were pioneered by Balachandran (1973), Balachandran and Tijms (1975), Boxma (1976), and Tijms (1976) for the M/G/1 queue. Their primary concern was in the optimal control of D under a linear cost structure. While these authors used the mean workload, Chae and Park (1999) used the mean queue length to determine the optimal value of D.

Boxma (1976) showed that the optimal D-policy is superior over the optimal N-policy for all service time distributions if the cost function consists of the startup cost and linear workload holding cost. But, Artalejo (2002) showed that the same is no longer true if the mean queue length is used in the cost function.

Gakis et al. (1995) derived the distributions of the idle and busy periods under simple and dyadic policies. Sivazlian (1979) provided an approximate formula for optimal D in terms of the first three moments of the service time. His formula was exact under exponential service times. Rhee (1997) developed a new methodology to find the expected busy periods for controllable M/G/1 queueing models.

Li and Niu (1992) considered the GI/G/1/D-policy queue and derived the waiting time distributions in transform-free style. Lillo and Martin (2000) claimed the superiority of the D-policy over the N-policy if the mean queue length is used, but their argument was based on the erroneously derived mean queue length. Feinberg and Kella (2002) considered switching costs, running costs and holding costs per unit time and proved the optimality of the D-policies. Readers are referenced to Tijms (1976, 1986) for approximate numerical results and related analysis.

Due to the behavioral and analytical complexities inherent in the D-policy queueing systems, the study on the queue length could not be found until Rubin and Zhang (1988) studied the switch-on policies for communication systems.

Dshalalow (1998) carried out an extensive study on the queue length process of the batch-arrival modified-D-policy queues with vacations. Chae and Park (2001) derived the probability generating function of the queue length of the M/G/1 queue under the D-policy. Artalejo (2001) derived the complete queue length distribution.

Lee et al. (2006) developed a methodology that can be applied to obtain the queue length and waiting time distributions under a unified framework.

Lee and Song (2004) and Lee et al. (2004) studied the queue length and the waiting time of the MAP/G/1/D-policy queue.

2 The system, objective and notation

In this paper, we consider the queueing system with the following specifications:

(1) Customers arrive according to the MAP (Markovian Arrival process) with parameter matrices ($\boldsymbol{C}$, $\boldsymbol{D}$). At their arrivals, they take a random sample from the service time distribution function $S(x)$.
(2) The idle server leaves for repeated vacations as soon as the system becomes empty. It resumes its service only when the cumulative workload is found to be greater than the predetermined threshold D at the end of a vacation.
(3) The service times and vacation times are identically and independently distributed (iid) and are independent of the arrival process. Without loss of generality we assume that the service times and vacation times are absolutely continuous.

The objective of this paper is to derive the distribution of the queue length and the mean value, and see the effects of the MAP arrivals.

Throughout the paper, we will use the notation as follows.

S: service time random variable,
$S(x)$: distribution function (DF) of S,
$S^{(n)}(x) = Pr(S_1 + S_2 + \cdots + S_n \leq x)$, $(S^{(0)}(x) = 1)$: DF of the n-fold convolution of S with itself,
$s^{(n)}(x)$: probability density function (pdf) of $S^{(n)}(x)$,
$E(S)$: expected value of S,
V: vacation time random variable,
$V(x)$: DF of V,
N_D: queue length (i.e., number of customers) at the start of a busy period (point ① in Fig. 1),
U_D: workload at point ① $(U_D > D)$,
$(\boldsymbol{E})_{ij}$: (i, j)-element of matrix $\boldsymbol{E}$,
$(\boldsymbol{C}, \boldsymbol{D})$: parameter matrices of the underlying Markov chain (UMC) of the MAP arrival process,
m: dimension of the phase of the UMC,
$J(t)$: phase of the UMC at time t,
$\pi_i = \lim_{t\to\infty} Pr[J(t) = i]$, $(i = 1, 2, \ldots, m)$,
$\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$,
$\mathbf{e}$: $(m \times 1)$ vector of 1's,
$\lambda = \boldsymbol{\pi}\boldsymbol{D}\mathbf{e}$: customer arrival rate,
$\rho = \lambda E(S)$: traffic intensity.

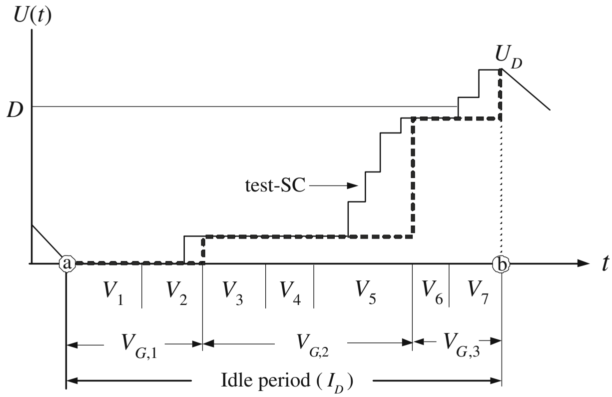
3 Analysis of the idle period

For analytical simplicity, let us call the customers who arrive during the idle period the Special Customers (SC) and the customers who arrive during the busy period the Ordinary Customers (OC). The reason for this categorization is that, under the condition that there are $N_D = n$ customers at the start of the busy period, the service times of these SCs are neither identical nor independent. Moreover, their distributions are different from the ordinary service time S. Thus, the number of OCs who arrive during the service time of a SC is different from the number of OCs who arrive during an ordinary service time S. This fact is crucial to obtaining the queue length distribution.

Deriving the queue length distribution of the MAP-related queueing systems starts with the derivation of the queue length distribution at an arbitrary departure. Consider an arbitrary SC (we will call this customer the test-SC) who departs the system. The customers left behind by this test-SC are the SCs who arrive during the remaining idle period since its arrival and the OCs who arrive until its service is finished. The number of these customers again depend on the amount of work at the arrival instance of the test-SC during the idle period. Thus, the analysis of the idle period is a necessity to the analysis of the queue length process of the whole system.

With the presence of multiple vacations, we first need to find the workload at the start of the vacation during which the test-SC arrives. For this purpose, we use the

Fig. 2 The workload grand vacation process (WGVP)



workload grand vacation process (WGVP) which is depicted in thick dotted lines in Fig. 2. This WGVP was first used in Lee et al. (2006).

Let us take a look at Fig. 2. The server leaves for a vacation as soon as the workload becomes zero (at ⓐ) and this is the starting point of the first grand vacation (GV) $V_{G,1}$. $V_{G,1}$ lasts until the change in the work level is observed at the end of a vacation (Thus, a GV consists of one or more vacations. In the figure, $V_{G,1}$ consists of two vacations). At the end of $V_{G,1}$, if the workload is still less than D, the second GV $V_{G,2}$ begins and lasts until the change in the work level is observed again after a vacation (In the figure, $V_{G,2}$ consists of three vacations). This process continues until the workload at the end of a GV becomes greater than D. Then the idle period ends and the busy period begins (at ⓑ).

We note that during a grand vacation, the level of the grand vacation process do not change. We also note that a grand vacation is stochastically equivalent to the idle period in the simple MAP/G/1 queue with multiple vacations and $D = 0$.

Let us define the following joint probabilities.

$(\tilde{V}_n)_{ij}$: the joint probability that n customers arrive during a vacation and the UMC phase at the end of the vacation is j under the condition that the UMC phase at the start of the vacation is i.

$(G_n^{GV})_{ij}$: the joint probability that n customers arrive during a GV and the UMC phase at the end of the GV is j under the condition that the UMC phase at the start of the GV is i.

Let $\tilde{\boldsymbol{V}}_n$ and $\boldsymbol{G}_n^{GV}$ be the matrices of $(\tilde{V}_n)_{ij}$ and $(G_n^{GV})_{ij}$. Denoting $\tilde{\boldsymbol{V}}(z)$ as the matrix generating function (GF) of $\{\tilde{\boldsymbol{V}}_n\}$, we have (Lucantoni et al. 1990)

$$\tilde{\boldsymbol{V}}(z) = \sum_{n=0}^{\infty} \tilde{\boldsymbol{V}}_n z^n = \int_0^{\infty} e^{(\boldsymbol{C}+\boldsymbol{D}z)x} dV(x), \tag{3.1}$$

$$\boldsymbol{G}_n^{GV} = \sum_{k=0}^{\infty} (\tilde{\boldsymbol{V}}_0)^k \tilde{\boldsymbol{V}}_n = (\boldsymbol{I} - \tilde{\boldsymbol{V}}_0)^{-1} \tilde{\boldsymbol{V}}_n. \tag{3.2}$$

Let (n, x) be the state of the grand vacation process (GVP) with queue length n and the workload $x(< D)$. Let us define $[R_n(x)]_{ij}$ as the probability that the GVP ever visits (n, x) and the UMC phase is j at the start of the GV under the condition that at the end of the busy period (i.e., at the start of the first GV) the UMC phase is i. Then $(R_n)_{ij} = \int_0^D (R_n(x))_{ij}\, dx$ is the probability that the GVP ever visits the queue length n. For the matrices $\boldsymbol{R}_n(x)$ and $\boldsymbol{R}_n$ of $[R_n(x)]_{ij}$ and $(R_n)_{ij}$ and the matrix GF $\boldsymbol{R}(z) = \sum_{n=0}^{\infty} \boldsymbol{R}_n z^n$, we have the following theorem.

Theorem 3.1 *We have,* $(\boldsymbol{R}_0 = \boldsymbol{I})$,

$$\boldsymbol{R}_n = \sum_{k=1}^{n} \boldsymbol{R}_{n-k} \left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_k, \ (n \geq 1), \tag{3.3}$$

$$\boldsymbol{R}_n(x) = \boldsymbol{R}_n \cdot s^{(n)}(x), \ (n \geq 0, x < D), \tag{3.4}$$

$$\boldsymbol{R}(z) = \left[\boldsymbol{I} - \left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \left(\tilde{\boldsymbol{V}}(z) - \tilde{\boldsymbol{V}}_0\right)\right]^{-1}. \tag{3.5}$$

Proof Equation 3.3 can be obtained by conditioning on the state of the previous GV. The GVP visits level (n, x) if and only if the sum of the service times of the n customers is x which accounts for Eq. 3.4. Equation 3.5 can be obtained in a straightforward manner from Eq. 3.3. □

Now, the joint distribution of the queue length N_D and the workload U_D at the start of the busy period are in order. They are necessary to find the distributions of the queue length and the workload that pile up during the remaining idle period after the arrival of the test-SC. Let J_0 be the UMC phase at the end of the previous busy period and J_B be the UMC phase at the start of the current busy period. Let us define $\Omega_{ij}(n, x)$ as follows.

$$\Omega_{ij}(n, x)dx = Pr(N_D = n, \ x < U_D \leq x + dx, J_B = j\,|J_0 = i), \ (n \geq 1, x > D). \tag{3.6}$$

Denoting $\boldsymbol{\Omega}(n, x)$ as the matrix of $\Omega_{ij}(n, x)$, we have the following theorem.

Theorem 3.2 *We have,*

$$\boldsymbol{\Omega}(n, x) = \begin{cases} \left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_1 \cdot s(x), & (n = 1, \ x > D) \\ \left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_n \cdot s^{(n)}(x) + \int_{w=0}^{D} \sum_{k=1}^{n-1} \boldsymbol{R}_k \left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_{n-k} & \\ \qquad \times s^{(k)}(w) s^{(n-k)}(x - w) dw. & (n \geq 2, \ x > D). \end{cases} \tag{3.7}$$

Proof The first equation is for the case in which the threshold is crossed by the first customer during the first grand vacation. The first term in the second equation is for the case in which the threshold is crossed by the nth customer who arrives during the first grand vacation. The second term is for the case in which there are two or more grand vacations and is obtained by conditioning on the state of the previous grand vacation. □

Now, we have the mean length of the idle period, the busy period and the cycle in the following theorem.

Theorem 3.3 *Let $\boldsymbol{\kappa}$ be the phase probability vector at the end of a busy period. Let I_D, B and C be the lengths of the idle period, the busy period and the cycle. Then, we have*

$$E(I_D) = \left[\sum_{n=0}^{\infty} \boldsymbol{\kappa} \boldsymbol{R}_n \left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0 \right)^{-1} \mathbf{e} \right] \cdot E(V), \tag{3.8a}$$

$$E(B) = \frac{\rho E(I_D)}{1-\rho}, \quad E(C) = \frac{E(I_D)}{1-\rho}. \tag{3.8b}$$

Proof Equation 3.8a comes from the fact that $\left[\left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0 \right)^{-1} \right]_{ij}$ is the mean number of vacations contained in a grand vacation which ends with phase j under the condition that the grand vacation starts with phase i. Equation 3.8b comes from the fact that our system is work conserving and the server is idle with probability $(1-\rho)$. □

$\boldsymbol{\kappa}$ in Eq. 3.8a will be obtained in Eq. 4.41.

4 Queue length at an arbitrary departure

In this section, we derive the queue length distribution at an arbitrary departure point. Let us define the joint probabilities as follows:

$x_{k,i} = Pr$ (at an arbitrary departure, the queue length (i.e., the total number of customers regardless of their types) is k and UMC phase is i),
$x^{sc}_{k,i} = Pr$ (at an arbitrary departure, the queue length is k, UMC phase is i and the departing customer is a SC),
$x^{oc}_{k,i} = Pr$ (at an arbitrary departure, the queue length is k, UMC phase is i and the departing customer is an OC).

Let us define the vectors:

$$\boldsymbol{x}_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,m}), \ \boldsymbol{x}^{sc}_k = (x^{sc}_{k,1}, x^{sc}_{k,2}, \ldots, x^{sc}_{k,m}), \text{ and } \boldsymbol{x}^{oc}_k = (x^{oc}_{k,1}, x^{oc}_{k,2}, \ldots, x^{oc}_{k,m}).$$

Then, we have

$$\boldsymbol{x}_k = \boldsymbol{x}^{sc}_k + \boldsymbol{x}^{oc}_k, \ (k \geq 0). \tag{4.1a}$$

Defining the vector GFs as follows,

$$\boldsymbol{X}(z) = \sum_{k=0}^{\infty} z^k \boldsymbol{x}_k, \quad \boldsymbol{X}_{sc}(z) = \sum_{k=0}^{\infty} z^k \boldsymbol{x}^{sc}_k, \quad \boldsymbol{X}_{oc}(z) = \sum_{k=0}^{\infty} z^k \boldsymbol{x}^{oc}_k,$$

we have

$$\boldsymbol{X}(z) = \boldsymbol{X}_{sc}(z) + \boldsymbol{X}_{oc}(z). \tag{4.1b}$$

4.1 Queue length at an arbitrary SC departure

In this section we derive the vector GF $\boldsymbol{X}_{sc}(z)$ of the queue length at an arbitrary SC departure. The number of customers left behind by the departing test-SC is the sum of the following two (see Fig. 3):

(1) the number N_{RI} of SCs who arrive since the arrival of the test-SC, and
(2) the number of OCs who arrive during the sum U_{TC} of the service times of the SCs before and including the test-SC.

Obviously N_{RI} and U_{TC} are not independent. We first note that U_{TC} depends on the workload U_0 at the start of the vacation V_* during which the test-SC arrives. We also note that U_0 is equal to the workload U^*_{GV} at the start of the grand vacation V^*_G to which V_* belongs.

Let us define the joint probability as follows,

$$\alpha(w, n, j)dw = Pr(w < U_{TC} \leq w + dw, N_{RI} = n, J_B = j). \tag{4.2}$$

$\alpha(w, n, j)$ depends on different situations at the arrival instance of the test-SC as follows:

Case-(1): Test-SC arrives during the first GV.
Case-(2): Test-SC arrives during the second GV or later.

For each case, we have three disjoint situations:

(a) Workload at the end of the GV does not exceed D.
(b) Workload just after the arrival of the test-SC (this includes the service time of the test-SC itself) does not exceed D, but the workload at the end of the GV exceeds D.
(c) The threshold D is crossed before the arrival of the test-SC or by the test-SC itself.

Let $U_{G,i}(x)$ be the probability that the workload at the start of an arbitrary GV is less than or equal to $x(< D)$ and the UMC phase is i. Let $\boldsymbol{U}_G(x) = \left(U_{G,1}(x), \ldots, U_{G,m}(x)\right)$ be the vector. We note that $U_{G,i}(0)$ is the probability that an arbitrary GV is the first GV and the UMC phase is i at the start of the GV. We have the following theorem.

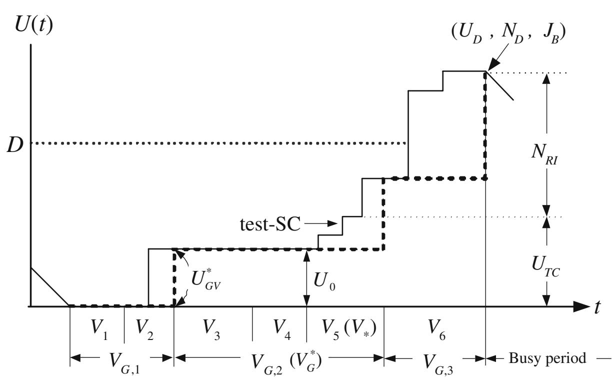


Fig. 3 Situation at the arrival instance of the test-SC

Theorem 4.1 *We have,*

$$\boldsymbol{U}_G(0) = \frac{\boldsymbol{\kappa}}{\sum_{n=0}^{\infty} \boldsymbol{\kappa} \boldsymbol{R}_n \mathbf{e}}, \tag{4.3}$$

$$\boldsymbol{u}_G(x) = \frac{d}{dx}\boldsymbol{U}_G(x) = \frac{\sum_{n=1}^{\infty} \boldsymbol{\kappa} \boldsymbol{R}_n(x)}{\sum_{n=0}^{\infty} \boldsymbol{\kappa} \boldsymbol{R}_n \mathbf{e}}, \ (0 < x \le D). \tag{4.4}$$

Proof We recall that $[\boldsymbol{\kappa} \boldsymbol{R}_n(x)]_j$ is the probability of visiting a GV which has n customers and the workload of x that starts with UMC phase j. Thus, the denominator $\sum_{n=0}^{\infty} \boldsymbol{\kappa} \boldsymbol{R}_n \mathbf{e}$ is the mean number of GVs during an idle period. Equation 4.3 comes from the fact that zero workload exists only during the first GV. Equation 4.4 comes from the definition of $\boldsymbol{R}_n(x)$. □

We note that $\boldsymbol{u}_G(x)dx$ is the vector probability that the workload is at the start of an arbitrary GV belongs to $[x, x+dx)$, $(x > 0)$. Thus,

$$\boldsymbol{u}_G = \int_0^D d\boldsymbol{U}_G(x) = \frac{\sum_{n=0}^{\infty} \boldsymbol{\kappa} \boldsymbol{R}_n}{\sum_{n=0}^{\infty} \boldsymbol{\kappa} \boldsymbol{R}_n \mathbf{e}} \tag{4.5}$$

becomes the UMC phase probability at the start of an arbitrary GV.

If we define A_G as the number of customers that arrive during an arbitrary GV, we have

$$h_n = Pr(A_G = n) = \boldsymbol{u}_G \left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_n \mathbf{e}, \tag{4.6}$$

$$A_G(z) = \sum_{n=1}^{\infty} z^n Pr(A_G = n) = \boldsymbol{u}_G \left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \left[\tilde{\boldsymbol{V}}(z) - \tilde{\boldsymbol{V}}_0\right] \mathbf{e}, \tag{4.7a}$$

$$\begin{aligned} h = E(A_G) &= \frac{d}{dz} A_G(z)\Big|_{z=1} \\ &= \boldsymbol{u}_G \left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \left[\lambda E(V)\boldsymbol{I} + (\mathbf{e}\boldsymbol{\pi} + \boldsymbol{C} + \boldsymbol{D})^{-1} (\tilde{\boldsymbol{V}} - \boldsymbol{I})\boldsymbol{D}\right] \mathbf{e}. \end{aligned} \tag{4.7b}$$

Now, let us define $G_V(x, k, l, j)$ as the probability that at the start of V_G^*, the workload is less than or equal to $x(< D)$, the test-SC is the kth customer among those arriving during V_G^*, l customer arrives during the remaining time of V_G^* and the UMC phase at the end of V_G^* is j. Then, we have the following theorem.

Theorem 4.2 *We have,*

$$G_V(x, k, l, j) = \left[\boldsymbol{U}_G(x) \cdot \frac{\left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_{k+l}}{h}\right]_j. \tag{4.8a}$$

Proof Consider a discrete-time renewal process that is generated by iid random variable A_G. Let a_{k+l} be the probability that the test-SC belongs to the group of size $(k+l)$. Then, we have $a_{k+l} = \frac{(k+l)h_{k+l}}{h}$. The test-SC is the kth customer with probability $\frac{1}{k+l}$. Thus, within an arbitrary grand vacation, the test-SC is the kth customer and there are l additional customers behind it with probability $a_{k+l} \cdot \frac{1}{k+l} = \frac{h_{k+l}}{h} = \frac{\boldsymbol{u}_G\left(\boldsymbol{I}-\tilde{\boldsymbol{V}}_0\right)^{-1}\tilde{\boldsymbol{V}}_{k+l}\mathbf{e}}{h}$. Then, $G_V(x, k, l, j)$ becomes

$$G_V(x, k, l, j) = \boldsymbol{U}_G(x) \cdot \frac{\boldsymbol{u}_G\left(\boldsymbol{I}-\tilde{\boldsymbol{V}}_0\right)^{-1}\tilde{\boldsymbol{V}}_{k+l}\mathbf{e}}{h} \cdot \frac{\left[\left(\boldsymbol{I}-\tilde{\boldsymbol{V}}_0\right)^{-1}\tilde{\boldsymbol{V}}_{k+l}\right]_j}{\boldsymbol{u}_G\left(\boldsymbol{I}-\tilde{\boldsymbol{V}}_0\right)^{-1}\tilde{\boldsymbol{V}}_{k+l}\mathbf{e}}$$

which reduces to Eq. 4.8a. □

Let us define the vectors $\boldsymbol{G}_V(0, k, l)$ and $\boldsymbol{g}_V(x, k, l)$, $(x > 0)$ as follows:

$$\boldsymbol{G}_V(0, k, l) = \left(G_V(0, k, l, 1), \cdots, G_V(0, k, l, m)\right), \tag{4.8b}$$

$$\boldsymbol{g}_V(x, k, l) = \left(\frac{d}{dx}G_V(x, k, l, 1), \cdots, \frac{d}{dx}G_V(x, k, l, m)\right). \tag{4.8c}$$

We also define $\boldsymbol{\Omega}_D^{(n)}$ as the matrix probability that there are n customers at the start of the busy period given the UMC phase at the end of the previous busy period. Then, from Eq. 3.7, we get $\boldsymbol{\Omega}_D^{(n)} = \int_D^\infty \boldsymbol{\Omega}(n, x)dx$.

Now we are ready to express $\alpha(w, n, j)$'s defined in Eq. 4.2 for those different cases.

Case-(1-a) (See Fig. 4) The test-SC is the kth customer and l more customers arrive during the first GV with probability $\boldsymbol{G}_V(0, k, l)$. Then, the sum of the service times of the k customers is w and the sum of service times of the l customers is r with respective probabilities $s^{(k)}(w)$ and $s^{(l)}(r)$, and at the end of the GV, the idle period starts all over again with threshold $D - (W + r)$. Thus, we have

$$\alpha(w, n, j) = \left[\int_{r=0}^{D-w}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\boldsymbol{G}_V(0, k, l)s^{(k)}(w)s^{(l)}(r)\boldsymbol{\Omega}_{D-(w+r)}^{(n-l)}dr\right]_j, (n \geq 1, w \leq D), \tag{4.9}$$

Fig. 4 Case-(1-a)

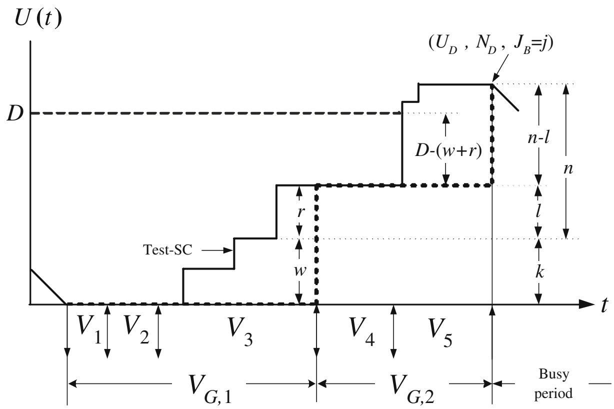
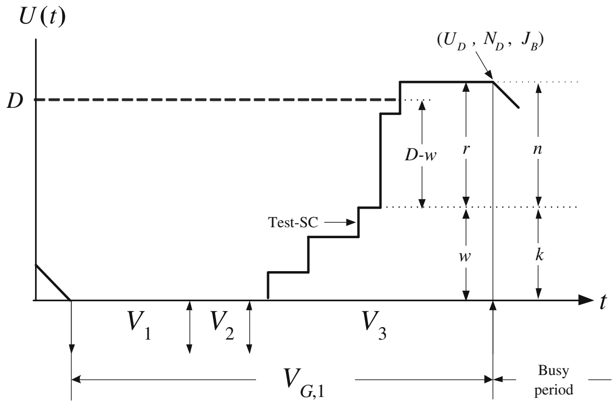


Fig. 5 Case-(1-b)



where $\boldsymbol{\Omega}_x^{(n)}$ is the matrix probability of the queue length n at the start of the busy period under the x-policy and multiple vacations.

In the analogous way, for the remaining cases, we get the followings:

Case-(1-b) (See Fig. 5)

$$\alpha(w, n, j) = \left[\int_{r=D-w}^{\infty} \sum_{k=1}^{\infty} \boldsymbol{G}_V(0, k, n) s^{(k)}(w) s^{(n)}(r) dr \right]_j, \ (n \geq 1, w \leq D), \tag{4.10}$$

Case-(1-c) (See Fig. 6)

$$\alpha(w, n, j) = \left[\sum_{k=1}^{\infty} \boldsymbol{G}_V(0, k, n) s^{(k)}(w) \right]_j, \ (n \geq 0, w > D), \tag{4.11}$$

Fig. 6 Case-(1-c)

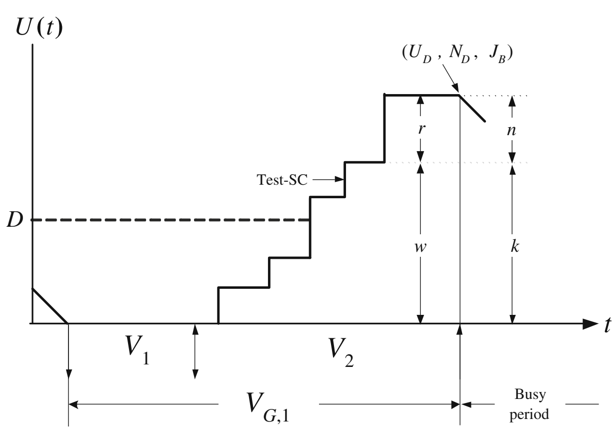
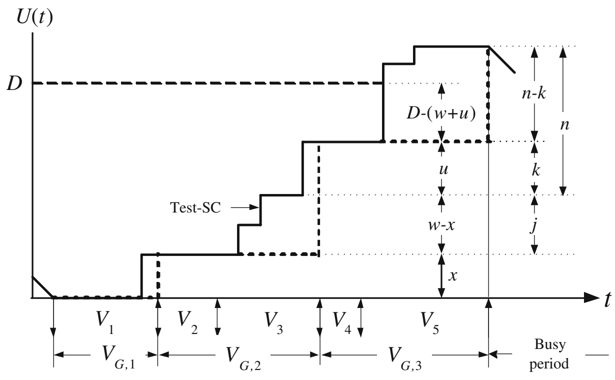


Fig. 7 Case-(2-a)



Case-(2-a) (See Fig. 7)

$$\alpha(w,n,j) = \left[\int_{r=0}^{D-w} \sum_{k=1}^{\infty} \sum_{l=0}^{n-1} \left(\int_{x=0}^{w} \boldsymbol{g}_V(x,k,l) s^{(k)}(w-x)dx \right) s^{(l)}(r) \boldsymbol{\Omega}_{D-(w+r)}^{(n-l)} dr \right]_j, \quad (n \geq 1, w \leq D), \tag{4.12}$$

Case-(2-b) (See Fig. 8)

$$\alpha(w,n,j) = \left[\int_{r=D-w}^{\infty} \sum_{k=1}^{\infty} \left(\int_{x=0}^{w} \boldsymbol{g}_V(x,k,n) s^{(k)}(w-x)dx \right) s^{(n)}(r)dr \right]_j, \ (n \geq 1, w \leq D), \tag{4.13}$$

Case-(2-c) (See Fig. 9)

$$\alpha(w,n,j) = \left[\sum_{k=1}^{\infty} \left(\int_{x=0}^{D} \boldsymbol{g}_V(x,k,n) s^{(k)}(w-x)dx \right) \right]_j, \ (n \geq 0, w > D). \tag{4.14}$$

Now, we are ready to derive the GF of the above $\alpha(w,n,j)$'s. Recall that w in $\alpha(w,n,j)$ denotes the total amount of work just after the arrival of the test-SC

Fig. 8 Case-(2-b)

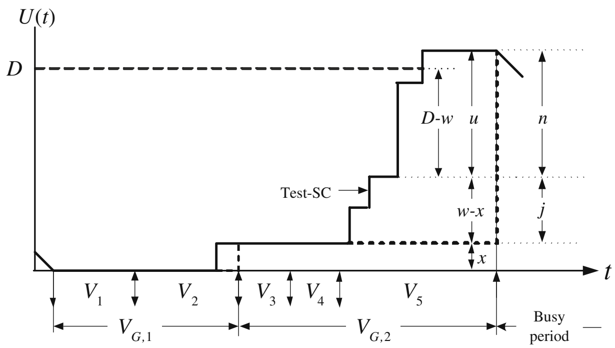
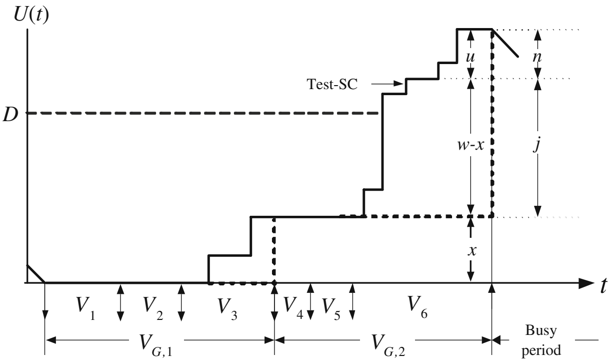


Fig. 9 Case-(2-c)



(including the service time of the test-SC itself). We note that this is equal to the time length from the start of the busy period until the end of the service of the test-SC. Thus the matrix GF of the number of OCs that arrive until the end of its service becomes (Lucantoni 1991)

$$e^{(\boldsymbol{C}+\boldsymbol{D}z)w}. \tag{4.15}$$

Let us define $\boldsymbol{\Omega}_D(z) = \sum_{n=1}^{\infty} \boldsymbol{\Omega}_D^{(n)} z^n$ as the matrix GF of the queue length at the start of the busy period. We use Eqs. 4.9–4.14 together with Eqs. 4.8a, 4.8b, 4.8c and 4.15 to obtain the vector GFs of the queue length just after the departure of the test-SC for each of the above cases as follows.

Case-(1-a)

$$\boldsymbol{\alpha}_{1,a}(z) = \int_{w=0}^{D} \int_{r=0}^{D-w} \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} z^l \boldsymbol{U}_G(0) \frac{\left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_{k+l}}{h} \times s^{(k)}(w) s^{(l)}(r) \boldsymbol{\Omega}_{D-(w+r)}(z) e^{(\boldsymbol{C}+\boldsymbol{D}z)w} dr dw, \tag{4.16}$$

Case-(1-b)

$$\boldsymbol{\alpha}_{1,b}(z) = \int_{w=0}^{D} \int_{r=D-w}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} z^n \boldsymbol{U}_G(0) \frac{\left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_{k+n}}{h} s^{(k)}(w) s^{(n)}(r) e^{(\boldsymbol{C}+\boldsymbol{D}z)w} dr dw, \tag{4.17}$$

Case-(1-c)

$$\boldsymbol{\alpha}_{1,c}(z) = \int_{w=D}^{\infty} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} z^n \boldsymbol{U}_G(0) \frac{\left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_{k+n}}{h} s^{(k)}(w) e^{(\boldsymbol{C}+\boldsymbol{D}z)w} dw, \tag{4.18}$$

Case-(2-a)

$$\boldsymbol{\alpha}_{2,a}(z) = \int_{w=0}^{D} \int_{r=0}^{D-w} \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} z^l \left[\int_{x=0}^{w} \boldsymbol{u}_G(x) \frac{\left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_{k+l}}{h} s^{(k)}(w-x) dx \right] \times s^{(l)}(r) \boldsymbol{\Omega}_{D-(w+r)}(z) e^{(\boldsymbol{C}+\boldsymbol{D}z)w} dr dw, \tag{4.19}$$

Case-(2-b)

$$\boldsymbol{\alpha}_{2,b}(z) = \int_{w=0}^{D}\int_{r=D-w}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} z^n \left[\int_{x=0}^{w} \boldsymbol{u}_G(x)\frac{\left(\boldsymbol{I}-\tilde{\boldsymbol{V}}_0\right)^{-1}\tilde{\boldsymbol{V}}_{k+n}}{h}s^{(k)}(w-x)dx\right]$$

$$\times s^{(n)}(r)e^{(\boldsymbol{C}+\boldsymbol{D}z)w}drdw, \tag{4.20}$$

Case-(2-c)

$$\boldsymbol{\alpha}_{2,c}(z) = \int_{w=D}^{\infty}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} z^n \left[\int_{x=0}^{D} \boldsymbol{u}_G(x)\frac{\left(\boldsymbol{I}-\tilde{\boldsymbol{V}}_0\right)^{-1}\tilde{\boldsymbol{V}}_{k+n}}{h}s^{(k)}(w-x)dx\right]e^{(\boldsymbol{C}+\boldsymbol{D}z)w}dw. \tag{4.21}$$

An arbitrary customer is a SC with probability

$$\phi_{sc} = \frac{E(N_D)}{E(N_{\text{cycle}})}, \tag{4.22}$$

where $E(N_D)$ is the mean number of SCs served during a cycle and $E(N_{\text{cycle}})$ is the mean number of total customers who are served during a cycle. We note that $E(N_D)$ and $E(N_{\text{cycle}})$ can be obtained from

$$E(N_D) = \frac{d}{dz}\left[\boldsymbol{\kappa}\boldsymbol{\Omega}_D(z)\right]\Big|_{z=1}\mathbf{e}, \tag{4.23}$$

and Lucantoni (1991),

$$E(N_{\text{cycle}}) = \boldsymbol{\kappa}\boldsymbol{\kappa}^*, \tag{4.24}$$

where $\boldsymbol{\kappa}^*$ is the mean number of customers that are served during a cycle given the phase at the busy period ending point. $\boldsymbol{\kappa}^*$ will be derived later in Eq. 4.35.

Now, finally the vector GF $\boldsymbol{X}_{sc}(z)$ of the queue length at the end of the departure of an arbitrary SC (including the probability that the departure point is that of an SC) can be obtained by summing Eq. 4.16–4.21

$$\boldsymbol{X}_{sc}(z) = \phi_{sc}\cdot\{\boldsymbol{\alpha}_{1,a}(z)+\boldsymbol{\alpha}_{1,b}(z)+\boldsymbol{\alpha}_{1,c}(z)+\boldsymbol{\alpha}_{2,a}(z)+\boldsymbol{\alpha}_{2,b}(z)+\boldsymbol{\alpha}_{2,c}(z)\}. \tag{4.25}$$

4.2 Queue length at an arbitrary OC departure

In this section, we derive the vector queue length GF $\boldsymbol{X}_{oc}(z)$ at an arbitrary OC-departure contained in Eq. 4.1b. Note that the services of OCs begin only after the services of all SCs are finished.

To obtain the probability vector $\boldsymbol{x}_k^{oc} = (x_{k,1}^{oc},\ldots,x_{k,m}^{oc})$ of the queue length just after the 'current' OC-departure, we consider the queue length at the 'previous' departure. We have two possible cases.

(Case 1) (The previous departure is an OC-departure).
In this case, if the queue length is j at the previous departure, the queue length at the current OC-departure becomes k if and only if $(k-j+1)$

customers arrive during the current service time. Thus, we have k customers with probability

$$\boldsymbol{x}_k^{oc} = \sum_{j=1}^{k+1} \boldsymbol{x}_j^{oc} \int_0^\infty \boldsymbol{P}(k-j+1, x)dS(x), \tag{4.26}$$

where $\boldsymbol{P}(n, x)$ is the matrix probability that n customers arrive during x.

(Case 2) (The previous departure is an SC-departure).
Let $\boldsymbol{x}_{j,\text{last}}^{sc}$ be the vector probability that the previous departure is a SC-departure (this is the last departure among the N_D SC departures) and the queue length is j. We have k customers at the current OC-departure with probability

$$\boldsymbol{x}_k^{oc} = \sum_{j=1}^{k+1} \boldsymbol{x}_{j,\text{last}}^{sc} \int_0^\infty \boldsymbol{P}(k-j+1, x)dS(x). \tag{4.27}$$

Combining Eq. 4.26 and 4.27, and defining the vector GF $\boldsymbol{X}_{sc}^{\text{last}}(z) = \sum_{k=0}^{\infty} z^k \boldsymbol{x}_{k,\text{last}}^{sc}$, we get

$$\begin{aligned}
\boldsymbol{X}_{oc}(z) &= \sum_{k=0}^{\infty} z^k \sum_{j=1}^{k+1} \boldsymbol{x}_j^{oc} \int_{x=0}^\infty \boldsymbol{P}(k-j+1, x)dS(x) \\
&\quad + \sum_{k=0}^{\infty} z^k \sum_{j=1}^{k+1} \boldsymbol{x}_{j,\text{last}}^{sc} \int_{x=0}^\infty \boldsymbol{P}(k-j+1, x)dS(x) \\
&= z^{-1} \cdot \left[\boldsymbol{X}_{oc}(z) - \boldsymbol{x}_0^{oc}\right] \int_{x=0}^\infty e^{(\boldsymbol{C}+\boldsymbol{D}z)x} dS(x) \\
&\quad + z^{-1} \cdot \left[\boldsymbol{X}_{sc}^{\text{last}}(z) - \boldsymbol{x}_{0,\text{last}}^{sc}\right] \int_{x=0}^\infty e^{(\boldsymbol{C}+\boldsymbol{D}z)x} dS(x) \\
&= \left[\boldsymbol{X}_{sc}^{\text{last}}(z) - \boldsymbol{x}_0\right] \boldsymbol{A}(z) \left[z\boldsymbol{I} - \boldsymbol{A}(z)\right]^{-1},
\end{aligned} \tag{4.28}$$

where

$$\boldsymbol{A}(z) = \int_0^\infty e^{(\boldsymbol{C}+\boldsymbol{D}z)x} dS(x) \tag{4.29}$$

is the matrix GF of the number of customers that arrive during a service time.

Now, we need to obtain the vector generating function $\boldsymbol{X}_{sc}^{\text{last}}(z)$ and the vector $\boldsymbol{x}_0$ contained in Eq. 4.28. To derive $\boldsymbol{X}_{sc}^{\text{last}}(z)$, we note that the last special customer is the one who arrive during the last grand vacation and the customers left by this SC are the OCs who arrive during the total sum of the service times of the SCs. Thus, using Eqs. 4.11 and 4.14, we get

$$\begin{aligned}
\boldsymbol{X}_{sc}^{\text{last}}(z) = \phi_{sc} \cdot \int_{w=D}^\infty \sum_{k=1}^{\infty} \Bigg\{ &\boldsymbol{G}_V(0, k, 0)s^{(k)}(w) \\
&+ \left[\int_{x=0}^{D} \boldsymbol{g}_V(x, k, 0)s^{(k)}(w-x)dx\right] \Bigg\} e^{(\boldsymbol{C}+\boldsymbol{D}z)w} dw.
\end{aligned} \tag{4.30}$$

$\boldsymbol{G}_V(0, k, 0)$ and $\boldsymbol{g}_V(x, k, 0)$ can be obtained from Eqs. 4.8b and 4.8c.

To obtain $\boldsymbol{x}_0$, we need to obtain $\boldsymbol{K}(z)$ which is the matrix GF of the number of customers that are served during a cycle. We have the following theorem.

Theorem 4.3 *We have,*

$$\boldsymbol{K}(z) = \int_{x=D}^{\infty} \sum_{k=1}^{\infty} z^k \left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_k \cdot e^{[\boldsymbol{C}+\boldsymbol{D}\boldsymbol{G}(z)]x} dS^{(k)}(x)$$
$$+ \int_{x=D}^{\infty} \sum_{l=1}^{\infty} \sum_{k=l+1}^{\infty} z^k \boldsymbol{R}_l \left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_{k-l} \cdot e^{[\boldsymbol{C}+\boldsymbol{D}\boldsymbol{G}(z)]x} \int_{w=0}^{D} s^{(l)}(w) s^{(k-l)}(x-w) dw dx, \tag{4.31}$$

where $\boldsymbol{G}(z)$ is the matrix GF that represents the number of customers that are served during a fundamental period (Neuts 1981, 1989).

Proof Let us define $Q_{i,j}(k_1, k_2)$ as the probability that there are k_1 SCs at the busy period starting point and there are k_2 OCs at the end of the service of the last SC with UMC phase j under the condition that the UMC phase is i at the start of the idle period. Let $\boldsymbol{Q}_{(k_1,k_2)}$ be the matrix of $Q_{i,j}(k_1, k_2)$ with respect to i and j. Then, using Eq. 3.7, we get

$$\boldsymbol{Q}_{(k_1,k_2)} = \int_D^{\infty} \boldsymbol{\Omega}(k_1, x) \cdot \boldsymbol{P}(k_2, x) dx. \tag{4.32}$$

Defining the matrix GF $\boldsymbol{Q}(z_1, z_2)$ of $\boldsymbol{Q}_{(k_1,k_2)}$, we get

$$\begin{aligned}
\boldsymbol{Q}(z_1, z_2) &= \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \boldsymbol{Q}_{(k_1,k_2)} z_1^{k_1} z_2^{k_2} \\
&= \int_D^{\infty} \sum_{k_1=1}^{\infty} z_1^{k_1} (\boldsymbol{I} - \tilde{\boldsymbol{V}}_0)^{-1} \tilde{\boldsymbol{V}}_{k_1} \cdot e^{(\boldsymbol{C}+\boldsymbol{D}z_2)x} dS^{(k_1)}(x) \\
&\quad + \int_{x=D}^{\infty} \sum_{l=1}^{\infty} \sum_{k_1=l+1}^{\infty} z_1^{k_1} \boldsymbol{R}_l (\boldsymbol{I} - \tilde{\boldsymbol{V}}_0)^{-1} \tilde{\boldsymbol{V}}_{k_1-l} \cdot e^{(\boldsymbol{C}+\boldsymbol{D}z_2)x} \\
&\quad \times \int_{w=0}^{D} s^{(l)}(w) s^{(k_1-l)}(x-w) dw dx,
\end{aligned} \tag{4.33}$$

where we used $\sum_{k_2=0}^{\infty} z_2^{k_2} \cdot \boldsymbol{P}(k_2, x) = e^{(\boldsymbol{C}+\boldsymbol{D}z_2)x}$. Now using z in place of z_1 and $\boldsymbol{G}(z)$ in place of z_2 yields Eq. 4.31. □

The matrix $\boldsymbol{K}$ that denotes the phase shift probability during the cycle can be obtained as

$$\begin{aligned}\boldsymbol{K} = \boldsymbol{K}(z)|_{z=1} = & \int_{D}^{\infty} \sum_{k=1}^{\infty} \left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_k \cdot e^{(\boldsymbol{C}+\boldsymbol{DG})x} dS^{(k)}(x) \\ & + \int_{x=D}^{\infty} \sum_{l=1}^{\infty} \sum_{k=l+1}^{\infty} \boldsymbol{R}_l \left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_{k-l} \cdot e^{(\boldsymbol{C}+\boldsymbol{DG})x} \\ & \times \int_{w=0}^{D} s^{(l)}(w) s^{(k-l)}(x-w) dw dx. \end{aligned} \tag{4.34}$$

The vector $\boldsymbol{\kappa}^*$ that denotes the mean number of customers that are served during a cycle becomes

$$\begin{aligned}\boldsymbol{\kappa}^* = & \frac{d}{dz}\boldsymbol{K}(z)\Big|_{z=1} \mathbf{e} = \int_{x=D}^{\infty} \sum_{n=1}^{\infty} n(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0)^{-1} \tilde{\boldsymbol{V}}_n e^{(\boldsymbol{C}+\boldsymbol{DG})x} \mathbf{e} dS^{(n)}(x) \\ & + \int_{x=D}^{\infty} \sum_{n=1}^{\infty} (\boldsymbol{I} - \tilde{\boldsymbol{V}}_0)^{-1} \tilde{\boldsymbol{V}}_n \left[\sum_{k=1}^{\infty} \frac{x^k}{k!} (\boldsymbol{C} + \boldsymbol{DG})^{k-1} \boldsymbol{D\mu}\right] dS^{(n)}(x) \\ & + \int_{x=D}^{\infty} \sum_{l=1}^{\infty} \sum_{n=l+1}^{\infty} \boldsymbol{R}_l (\boldsymbol{I} - \tilde{\boldsymbol{V}}_0)^{-1} \tilde{\boldsymbol{V}}_{n-l} e^{(\boldsymbol{C}+\boldsymbol{DG})x} \mathbf{e} \int_{w=0}^{D} s^{(l)}(w) s^{(n-l)}(x-w) dw dx \\ & + \int_{x=D}^{\infty} \sum_{l=1}^{\infty} \sum_{n=l+1}^{\infty} \boldsymbol{R}_l (\boldsymbol{I} - \tilde{\boldsymbol{V}}_0)^{-1} \tilde{\boldsymbol{V}}_{n-l} \sum_{k=1}^{\infty} \frac{x^k}{k!} (\boldsymbol{C} + \boldsymbol{DG})^{k-1} \boldsymbol{D\mu} \\ & \times \int_{w=0}^{D} s^{(l)}(w) s^{(n-l)}(x-w) dw dx \end{aligned} \tag{4.35}$$

where we used

$$\frac{d}{dz} e^{[\boldsymbol{C}+\boldsymbol{DG}(z)]x}\Big|_{z=1} \mathbf{e} = \sum_{k=1}^{\infty} \frac{x^k}{k!} (\boldsymbol{C} + \boldsymbol{DG})^{k-1} \boldsymbol{D\mu} \tag{4.36}$$

in which $\boldsymbol{\mu}$ is the mean number of customers that are served during a fundamental period which is given by (Lucantoni 1991)

$$\boldsymbol{\mu} = \frac{d}{dz}\boldsymbol{G}(z)\Big|_{z=1} \mathbf{e} = (\boldsymbol{I} - \boldsymbol{G} + \mathbf{e}\boldsymbol{g})[\boldsymbol{I} - \boldsymbol{A} + (\mathbf{e} - \boldsymbol{\beta})\boldsymbol{g}]^{-1}\mathbf{e}. \tag{4.37}$$

In Eq. 4.37, $\boldsymbol{A} = \int_0^{\infty} e^{(\boldsymbol{C}+\boldsymbol{D})x} dS(x)$ is the phase change probability during a service time, $\boldsymbol{G} = \boldsymbol{G}(z)|_{z=1}$, and $\boldsymbol{g} = (g_1, g_2, \ldots, g_m)$ is the stationary vector of $\boldsymbol{G}$ which satisfies

$$\boldsymbol{g} = \boldsymbol{g}\boldsymbol{G}, \;\; \boldsymbol{g}\mathbf{e} = 1. \tag{4.38}$$

$\boldsymbol{\beta}$ in Eq. 4.37 is the mean number of customers that arrive during a service time and is given by (Lucantoni 1991)

$$\boldsymbol{\beta} = \frac{d}{dz}\boldsymbol{A}(z)\Big|_{z=1} \mathbf{e} = \rho\mathbf{e} + (\mathbf{e}\boldsymbol{\pi} + \boldsymbol{C} + \boldsymbol{D})^{-1}(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{D}\mathbf{e}. \tag{4.39}$$

Now, finally $\boldsymbol{x}_0$ can be obtained from (Lucantoni 1991)

$$\boldsymbol{x}_0 = \frac{\boldsymbol{\kappa}}{\boldsymbol{\kappa}\boldsymbol{\kappa}^*}, \tag{4.40}$$

where $\boldsymbol{\kappa}$ is the UMC phase probability at the end of an arbitrary busy period which satisfies

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}\boldsymbol{K}, \ \ \boldsymbol{\kappa}\mathbf{e} = 1. \tag{4.41}$$

Using Eqs. 4.30 and 4.40 in Eq. 4.28 allows us to obtain the complete $\boldsymbol{X}_{oc}(z)$.

Now, at last, the vector GF $\boldsymbol{X}(z)$ of the queue length at an arbitrary departure can be obtained by using Eqs. 4.25 and 4.28 in Eq. 4.1b.

5 The queue length at an arbitrary time

Let $y_{k,i}$ be the probability that the queue length is k at an arbitrary time in steady-state and the UMC phase is i. Let $\boldsymbol{y}_k$ be the vector

$$\boldsymbol{y}_k = (y_{k,1}, \ldots, y_{k,m}). \tag{5.1}$$

The vector GF $\boldsymbol{Y}(z) = \sum_{k=0}^{\infty} \boldsymbol{y}_k z^k$ can be obtained from Takine and Takahashi (1998).

$$\boldsymbol{Y}(z)\,(\boldsymbol{C} + \boldsymbol{D}z) = \lambda(z-1)\boldsymbol{X}(z). \tag{5.2}$$

6 The mean queue lengths

To derive the mean queue length, let us denote $\boldsymbol{M}$ and $\boldsymbol{M}^{(n)}$ for the matrix GF $\boldsymbol{M}(z)$ as

$$\boldsymbol{M} = \boldsymbol{M}(z)|_{z=1}, \quad \boldsymbol{M}^{(n)} = \frac{d^n}{dz^n}\boldsymbol{M}(z)\bigg|_{z=1}.$$

The mean queue length L_d at an arbitrary departure can be obtained from Eq. 4.1b as follows,

$$L_d = \boldsymbol{X}^{(1)}\mathbf{e} = \left(\boldsymbol{X}_{sc}^{(1)} + \boldsymbol{X}_{oc}^{(1)}\right)\mathbf{e}. \tag{6.1}$$

From Eq. 4.25, we get, after a tedious and laborious manipulation,

$$\boldsymbol{X}_{sc}^{(1)}\mathbf{e} = \phi_{sc}\{A + B + C\}, \tag{6.2}$$

where

$$A = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} n \cdot \left[\boldsymbol{U}_G(0) + \int_{x=0}^{D} \boldsymbol{u}_G(x)dx\right] \frac{\left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_{k+n}}{h} \mathbf{e}$$

$$+ \int_{w=0}^{D}\int_{r=0}^{D-w} \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \left[\boldsymbol{U}_G(0)s^{(k)}(w) + \int_{x=0}^{w} \boldsymbol{u}_G(x)s^{(k)}(w-x)dx\right]$$

$$\times \frac{\left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_{k+n}}{h} \boldsymbol{\Omega}_{D-(w+r)}^{(1)} \mathbf{e} \cdot s^{(n)}(r)drdw,$$

$$B = \int_{w=0}^{D}\int_{r=0}^{D-w} \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \left[\boldsymbol{U}_G(0)s^{(k)}(w) + \int_{x=0}^{w} \boldsymbol{u}_G(x)s^{(k)}(w-x)dx\right]$$

$$\times \frac{\left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_{k+n}}{h} \boldsymbol{\Omega}_{D-(w+r)} \left[\frac{d}{dz} e^{(\boldsymbol{C}+\boldsymbol{D}z)w}\Big|_{z=1}\right] \mathbf{e}s^{(n)}(r)drdw$$

$$+ \int_{w=0}^{D}\int_{r=D-w}^{\infty} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left[\boldsymbol{U}_G(0)s^{(k)}(w) + \int_{x=0}^{w} \boldsymbol{u}_G(x)s^{(k)}(w-x)dx\right]$$

$$\times \frac{\left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_{k+n}}{h} \left[\frac{d}{dz} e^{(\boldsymbol{C}+\boldsymbol{D}z)w}\Big|_{z=1}\right] \mathbf{e}s^{(n)}(r)drdw,$$

$$C = \int_{w=D}^{\infty} \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \left[\boldsymbol{U}_G(0)s^{(k)}(w) + \int_{x=0}^{w} \boldsymbol{u}_G(x)s^{(k)}(w-x)dx\right]$$

$$\times \frac{\left(\boldsymbol{I} - \tilde{\boldsymbol{V}}_0\right)^{-1} \tilde{\boldsymbol{V}}_{k+n}}{h} \left[\frac{d}{dz} e^{(\boldsymbol{C}+\boldsymbol{D}z)w}\Big|_{z=1}\right] \mathbf{e}dw.$$

To derive $\boldsymbol{X}_{oc}^{(1)}\mathbf{e}$, we rewrite Eq. 4.28 as follows

$$\boldsymbol{X}_{oc}(z)\left[z\boldsymbol{I} - \boldsymbol{A}(z)\right] = \boldsymbol{B}(z), \tag{6.3}$$

where

$$\boldsymbol{B}(z) = \left[\boldsymbol{X}_{sc}^{\text{last}}(z) - \boldsymbol{x}_0\right] \boldsymbol{A}(z). \tag{6.4}$$

Taking a derivative of Eq. 6.3 and using $z = 1$, we get

$$\boldsymbol{X}_{oc}^{(1)}(\boldsymbol{I} - \boldsymbol{A}) + \boldsymbol{X}_{oc}\left(\boldsymbol{I} - \boldsymbol{A}^{(1)}\right) = \boldsymbol{B}^{(1)}. \tag{6.5}$$

Adding $\boldsymbol{X}_{oc}^{(1)}\mathbf{e}\boldsymbol{\pi}$ to both sides of Eq. 6.5 and using $\boldsymbol{\pi}(\boldsymbol{I} - \boldsymbol{A} + \mathbf{e}\boldsymbol{\pi})^{-1} = \boldsymbol{\pi}$, we get

$$\boldsymbol{X}_{oc}^{(1)} = \boldsymbol{X}_{oc}^{(1)}\mathbf{e}\boldsymbol{\pi} + \left[\boldsymbol{B}^{(1)} - \boldsymbol{X}_{oc}(\boldsymbol{I} - \boldsymbol{A}^{(1)})\right](\boldsymbol{I} - \boldsymbol{A} + \mathbf{e}\boldsymbol{\pi})^{-1}. \tag{6.6}$$

Taking the second derivative of Eq. 6.3 and using $z = 1$, we get

$$\boldsymbol{X}_{oc}^{(1)}\boldsymbol{\beta} = \boldsymbol{X}_{oc}^{(1)}\mathbf{e} - \frac{1}{2}\left[\boldsymbol{X}_{oc}\boldsymbol{A}^{(2)}\mathbf{e} + \boldsymbol{B}^{(2)}\mathbf{e}\right]. \tag{6.7}$$

Postmultiplying Eq. 6.6 by $\boldsymbol{\beta}$ and using $\boldsymbol{\pi\beta} = \rho$ (Neuts 1989), we get

$$\boldsymbol{X}_{oc}^{(1)}\boldsymbol{\beta} = \rho \boldsymbol{X}_{oc}^{(1)}\mathbf{e} + \left[\boldsymbol{B}^{(1)} - \boldsymbol{X}_{oc}(\boldsymbol{I} - \boldsymbol{A}^{(1)})\right](\boldsymbol{I} - \boldsymbol{A} + \mathbf{e}\boldsymbol{\pi})^{-1}\boldsymbol{\beta}. \tag{6.8}$$

From Eqs. 6.7 and 6.8, we get

$$\boldsymbol{X}_{oc}^{(1)}\mathbf{e} = \frac{1}{2(1-\rho)}\Big\{\boldsymbol{B}^{(2)}\mathbf{e} + \boldsymbol{X}_{oc}\boldsymbol{A}^{(2)}\mathbf{e} + 2\left[\boldsymbol{B}^{(1)} - \boldsymbol{X}_{oc}(\boldsymbol{I} - \boldsymbol{A}^{(1)})\right](\boldsymbol{I} - \boldsymbol{A} + \mathbf{e}\boldsymbol{\pi})^{-1}\boldsymbol{\beta}\Big\}. \tag{6.9}$$

Using Eqs. 6.2 and 6.9 in Eq. 6.1 yields the mean queue length L_d.

The mean queue length at an arbitrary time can be obtained by using Eq. 5.2. From Eq. 5.2, we get

$$\boldsymbol{Y}^{(1)}(\boldsymbol{C} + \boldsymbol{D}) = \lambda \boldsymbol{X} - \boldsymbol{\pi}\boldsymbol{D}. \tag{6.10}$$

Adding $\boldsymbol{Y}^{(1)}\mathbf{e}\boldsymbol{\pi}$ to Eq. 6.10 and using $\boldsymbol{\pi}(\mathbf{e}\boldsymbol{\pi} + \boldsymbol{C} + \boldsymbol{D})^{-1} = \boldsymbol{\pi}$, we get

$$\boldsymbol{Y}^{(1)} = \boldsymbol{Y}^{(1)}\mathbf{e}\boldsymbol{\pi} + [\lambda\boldsymbol{X} - \boldsymbol{\pi}\boldsymbol{D}](\mathbf{e}\boldsymbol{\pi} + \boldsymbol{C} + \boldsymbol{D})^{-1}. \tag{6.11}$$

Taking the second derivative of Eq. 5.2 and using $z = 1$ yields

$$\boldsymbol{Y}^{(2)}(\boldsymbol{C} + \boldsymbol{D}) = 2\left[\lambda\boldsymbol{X}^{(1)} - \boldsymbol{Y}^{(1)}\boldsymbol{D}\right]. \tag{6.12}$$

Multiplying Eq. 6.12 by $\mathbf{e}$ and using $(\boldsymbol{C} + \boldsymbol{D})\mathbf{e} = \mathbf{0}$, we get

$$\boldsymbol{Y}^{(1)}\boldsymbol{D}\mathbf{e} = \lambda\boldsymbol{X}^{(1)}\mathbf{e}. \tag{6.13}$$

Multiplying Eq. 6.11 by $\boldsymbol{D}\mathbf{e}$, we get

$$\boldsymbol{Y}^{(1)}\boldsymbol{D}\mathbf{e} = \lambda\boldsymbol{Y}^{(1)}\mathbf{e} + [\lambda\boldsymbol{X} - \boldsymbol{\pi}\boldsymbol{D}](\mathbf{e}\boldsymbol{\pi} + \boldsymbol{C} + \boldsymbol{D})^{-1}\boldsymbol{D}\mathbf{e}. \tag{6.14}$$

Using Eqs. 6.13 and 6.14, we have the mean queue length

$$L = \boldsymbol{Y}^{(1)}\mathbf{e} = \boldsymbol{X}^{(1)}\mathbf{e} - \left[\boldsymbol{X} - \frac{\boldsymbol{\pi}\boldsymbol{D}}{\lambda}\right](\mathbf{e}\boldsymbol{\pi} + \boldsymbol{C} + \boldsymbol{D})^{-1}\boldsymbol{D}\mathbf{e}, \tag{6.15}$$

where $\boldsymbol{X}$ can be obtained from

$$\boldsymbol{X} = \boldsymbol{X}_{sc} + \boldsymbol{X}_{oc}. \tag{6.16}$$

$\boldsymbol{X}_{sc}$ in Eq. 6.16 can be obtained from Eq. 4.25. To derive $\boldsymbol{X}_{oc}$, we add $\boldsymbol{X}_{oc}\mathbf{e}\boldsymbol{\pi}$ to Eq. 6.3 and use 6.4 to get

$$\boldsymbol{X}_{oc} - \boldsymbol{X}_{oc}\mathbf{e}\boldsymbol{\pi} = \left(\boldsymbol{X}_{oc}^{\text{last}} - \boldsymbol{x}_0\right)\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{A} + \mathbf{e}\boldsymbol{\pi})^{-1}. \tag{6.17}$$

Using Eq. 6.16 and adding $\boldsymbol{X}\mathbf{e} = 1$ to Eq. 6.17 yields

$$\boldsymbol{X}_{oc} = \boldsymbol{\pi} - \boldsymbol{X}_{sc}\mathbf{e}\boldsymbol{\pi} + \left(\boldsymbol{X}_{oc}^{\text{last}} - \boldsymbol{x}_0\right)\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{A} + \mathbf{e}\boldsymbol{\pi})^{-1}. \tag{6.18}$$

7 Computational experience

In this section, as a computational experience, we compare our system with the M/G/1 queue under the same threshold values, mean arrival rates and mean service

Table 1 Comparison with the Poisson queue ($D = 0.5$)

ρ	L(MAP)	L(Poisson)	$\frac{L(\text{MAP})}{L(\text{Poisson})}$
0.1	8.5001	9.3012	0.9139
0.3	5.3712	4.9879	1.0771
0.5	7.1998	4.6778	1.5391
0.7	13.1761	5.6474	2.3331
0.9	44.7232	12.1177	3.6907
0.95	92.4706	22.0821	4.1876

times. We will assume the following parameter matrices $\boldsymbol{C} = \begin{pmatrix} -10.0 & 1.0 \\ 0.4 & -0.8 \end{pmatrix}$, $\boldsymbol{D} = \begin{pmatrix} 9.0 & 0 \\ 0 & 0.4 \end{pmatrix}$. From $\boldsymbol{\pi}(\boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{0}$ and $\boldsymbol{\pi}\mathbf{e} = 1$, the stationary vector probability of the UMC becomes $\boldsymbol{\pi} = (\pi_1, \pi_2) = (2/7, 5/7)$ and the arrival rate becomes $\lambda = \boldsymbol{\pi} \boldsymbol{D}\mathbf{e} = 20/7$.

For simplicity, we assume that the vacation time follows the exponential distribution with mean $E(V) = 1.0$. We also assume that the service times follows the exponential distribution. We will change the mean service time $E(S)$ so that we have different values of traffic intensity $\rho = \lambda E(S)$.

Tables 1 and 2 show the comparison of the mean queue lengths of the two different systems when $D = 0.5$ and $D = 2.0$. The last columns show the ratio of the two mean queue lengths.

Tables 1 and 2 are sketched in Figs. 10 and 11. We see in the figures that when the traffic intensity ρ is very low, increased traffic intensity does not necessarily mean increased mean queue length. This peculiar phenomenon can not be seen in usual queueing systems. This occurs when the mean service time is very small compared to the threshold value D. To be more specific, when ρ is very low, it takes many arrivals to surpass the threshold and we have larger mean queue lengths during the idle period, which, in turn, increases the total mean queue length.

It is seen in the figures that when the traffic is heavy, a naive Poisson assumption results in a severe underestimation of the mean queue length.

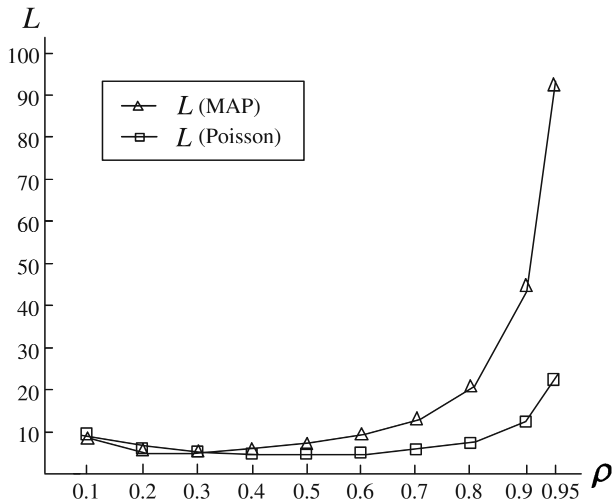
8 Summary, discussion and further research

In this paper, we analyzed the MAP/G/1 queue under the D-policy and multiple vacations. We first derived the queue length distribution at an arbitrary departure

Table 2 Comparison with the Poisson queue ($D = 2.0$)

ρ	L(MAP)	L(Poisson)	$\frac{L(\text{MAP})}{L(\text{Poisson})}$
0.1	29.1618	30.5761	0.9537
0.3	11.6143	11.7878	0.9853
0.5	10.6584	8.5033	1.2534
0.7	15.5118	8.1661	1.8995
0.9	46.4949	13.8942	3.3464
0.95	94.3903	23.7201	3.9793

Fig. 10 MAP vs Poisson ($D = 0.5$)

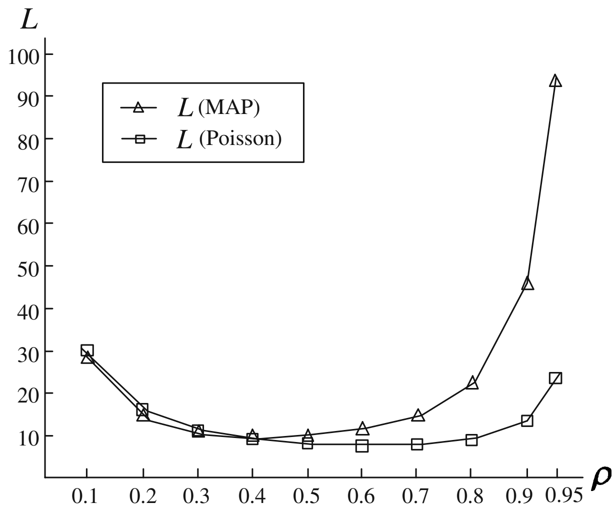


and then obtained the queue length distribution and the mean queue length at an arbitrary time.

We lastly presented computational experiences and compared the MAP queue with the Poisson queue. Our computation shows that naive Poisson assumptions may lead to a severe underestimation of the mean queue length.

The major computational obstacles of the results presented in this paper seems to reside in the multiple integrations and summations that are encountered in the computational process of the mean queue length. Our experience shows that if the service times follow the exponential or the Erlang distribution of low order, they did not pose serious problems. But if the service time distribution is in a complex

Fig. 11 MAP vs Poisson ($D = 2.0$)



functional form, we may need some numerical integrations which may be a time-consuming job.

A future extension of this work would be to discuss the approximate numerical analysis of queueing models operating under the *D*-policy when the arrival flow is modelled as a MAP.

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