BMAP/G/1 QUEUE UNDER *D*-POLICY: QUEUE LENGTH ANALYSIS

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□ *We study the queue length distribution of a queueing system with BMAP arrivals under D-policy. The idle server begins to serve the customers only when the sum of the service times of all waiting customers exceeds some fixed threshold D. We derive the vector generating functions of the queue lengths both at a departure and at an arbitrary point of time. Mean queue lengths are derived and a numerical example is presented.*

Keywords BMAP/G/1; *D*-policy; Queue length.

Mathematics Subject Classification Primary 60K25, 90B22, 68M20.

1. INTRODUCTION

In this paper, we study the queue length process of the BMAP/G/1/FCFS queue under *D*-policy. Customers arrive according to the batch Markovian arrival process (BMAP) with parameter matrices $\{\boldsymbol{D}_n, (n \geq 0)\}$. The service times are independently and identically distributed (iid). The server becomes idle as soon as the system becomes empty and is reactivated only when the cumulative workload exceeds D. Figure 1 shows the queue length process and the workload process on the same time scale.

The queue length process of the MAP/G/1/FCFS queue under *D*-policy was studied by Lee and Song[13]. They first found the joint distribution of the workload and the queue length at the busy period starting point, and based on the information, derived the queue length generating functions (GF) at a departure and at an arbitrary time. But their approach can no longer be applied to the queue length analysis of the BMAP/G/1 queue under *D*-policy because the batch size depends on the phase of the underlying Markov chain (UMC).

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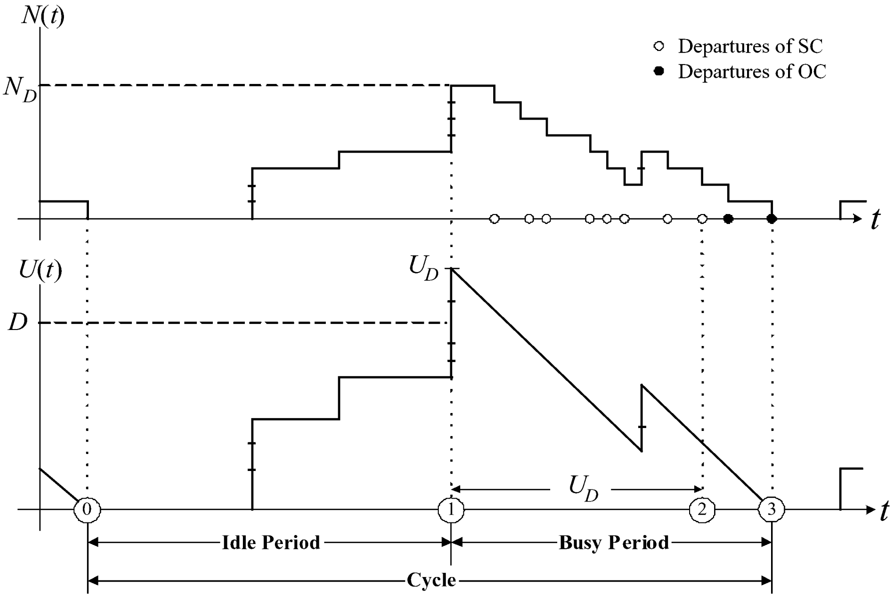

FIGURE 1 The queue length and workload process under *D*-policy.

The *D*-policy was first studied by Balachandran[3], Balachandran and Tijms[4], and Boxma[5] for M/G/1 queues. Their primary concern was in the optimal control of *D*.

Queue length analysis of *D*-policy queue is quite new. It was Dshalalow[10] who carried out the first extensive study on the queue length process of the batch arrival M/G/1 queue (see also Dshalalow[9]). But Dshalalow's *D*-policy was a modified one in the sense that he assumed that the customers who arrived during the idle period were assigned a totally new iid service time when the busy period start. Even though his *D*-policy does not agree with the classical one, his assumption can be considered as a practical modification that can significantly reduce the complexity of the classical *D*-policy.

Artalejo[1,2] derived the complete queue length distribution of M/G/1 under *D*-policy, where readers can also find an excellent literature review. Chae and Park[6,7] derived the PGF of the queue length.

The MAP (Markovian Arrival Process) was first named by Lucantoni et al.[15]. But, the MAP had already been studied under different names. The versatile Markovian point process of Neuts[18] and the N-process of Ramaswami[21] were found to be equivalent to the BMAP. The MAP can represent a variety of processes that include, as special cases, the Poisson process, the phase-type renewal processes, the MMPP (Markov modulated Poisson process) and superpositions of these.

For a nice tutorial and computational algorithms of BMAP/G/1 queues, readers are advised to see Lucantoni[16,17]. Early analyses of MAP/G/1 queues were based on the Matrix Analytic Method (MAM) pioneered by Neuts[19,20]. Lee et al.[11] used the supplementary variable technique combined with eigenvalues and eigenvectors and observed that a factorization property exists for BMAP/G/1 queues with generalized vacations. The factorization property was formally proved by Chang et al.[8] Lee et al.[12] applied the factorization property to the analysis of the production system with BMAP arrival, bilevel threshold control and maintenance. The waiting time of MAP/G/1/FCFS under D-policy was studied by Lee et al.[14].

2. NOTATION

Throughout the paper, we will use the following notation:

S: Service time random variable

$s(x), S(x)$: pdf and distribution function (DF) of S

$S^*(\theta)$: Laplace-Stieltjes transform (LST) of S; $S^*(\theta) = \int_0^\infty e^{-\theta x} dS(x)$

$S^{(n)}(x) = Pr(S_1 + S_2 + \cdots + S_n \le x)$, $(S^{(0)}(0) = 1)$ DF of the n-fold convolution of S

$s^{(n)}(x)$: probability density function (pdf) of $S^{(n)}(x)$

$E(S)$: expected value of S

N_D: queue length (number of customers including the one in service) at the start of a busy period (point ① in Figure 1)

U_D: workload at point ① in Figure 1 ($U_D > D$)

$\{\boldsymbol{D}_n, (n \ge 0)\}$: $(m \times m)$ parameter matrices of the underlying Markov chain (UMC)

$(\boldsymbol{D}_n)_{i,j}$: (i, j)-element of $\boldsymbol{D}_n$

$\boldsymbol{D}(z) = \sum_{n=0}^\infty z^n \boldsymbol{D}_n$: matrix generating function of $\{\boldsymbol{D}_n, (n \ge 0)\}$

$\boldsymbol{D} = \boldsymbol{D}(1) = \sum_{n=0}^\infty \boldsymbol{D}_n$: infinitesimal generator of the UMC

$J(t)$: phase of the UMC at time t

$\pi_i = \lim_{t\to\infty} Pr[J(t) = i], (1 \le i \le m)$

$\boldsymbol{\pi} = \{\pi_1, \pi_2, \ldots, \pi_m\}$: steady-state phase probability vector of UMC

$N_A(t)$: number of customers that arrive during $(0, t]$

$P_{i,j}(k, t) = Pr[N_A(t) = k, J(t) = j \,|\, J(0) = i]$, $(1 \le i \le m, 1 \le j \le m)$

$\boldsymbol{P}(k, t)$: matrix of $P_{i,j}(k, t)$

$\mathbf{e}$: $(m \times 1)$ vector of 1's

$\lambda_g = \boldsymbol{\pi} \sum_{n=1}^{\infty} \boldsymbol{D}_n \mathbf{e}$: arrival rate of groups
$\lambda = \boldsymbol{\pi} \sum_{n=1}^{\infty} n\boldsymbol{D}_n \mathbf{e}$: arrival rate of customers
$\rho = \lambda E(S)$: traffic intensity.

3. QUEUE LENGTH AT AN ARBITRARY DEPARTURE

We first derive the vector GF $\boldsymbol{X}(z)$ of the queue length just after an arbitrary departure (readers are advised to see Figure 1 to understand the departure process). For this purpose, we categorize the customers into two types: the special customer (SC) and the ordinary customer (OC)

SC: the customer who arrives during the idle period (In Figure 1, there are eight SCs).
OC: the customer who arrives when the server is busy (In Figure 1, there are two OCs).

The reason for this categorization is that, under the condition of $N_D = n$ at the start of the busy period, the service times of these SCs are neither identical nor independent of each other. Moreover, their distributions are different from the ordinary service time S. Thus, the number of OCs who arrive during the service times of these SCs is different from the number of OCs who arrive during an ordinary service time S.

Let us define the joint probability of the queue length and the UMC phase at an arbitrary departure as follows:

$$x_{k,i} = Pr(\text{queue length is } k \text{ and UMC phase is } i \text{ just after an arbitrary departure})$$

$$x_{k,i}^{SC} = Pr(\text{an arbitrary departing customer is a SC, he leaves } k \text{ customers behind and the UMC phase is } i \text{ just after his departure})$$

$$x_{k,i}^{OC} = Pr(\text{an arbitrary departing customer is an OC, he leaves } k \text{ customers behind and the UMC phase is } i \text{ just after his departure}).$$

Let us define the vectors,

$$\boldsymbol{x}_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,m}), \quad \boldsymbol{x}_k^{SC} = (x_{k,1}^{SC}, x_{k,2}^{SC}, \ldots, x_{k,m}^{SC}), \quad \boldsymbol{x}_k^{OC} = (x_{k,1}^{OC}, x_{k,2}^{OC}, \ldots, x_{k,m}^{OC}).$$

Then, we have

$$\boldsymbol{x}_k = \boldsymbol{x}_k^{SC} + \boldsymbol{x}_k^{OC} \quad (k \geq 0). \tag{3.1}$$

3.1. Analysis of the Idle Period

To derive the vector GF $\boldsymbol{X}(z) = \sum_{k=0}^{\infty} \boldsymbol{x}_k z^k$ and especially the queue length probability $\boldsymbol{x}_k^{SC}$ at a departure point of an arbitrary SC (this SC will be called the test-SC), it is necessary to analyze the idle period. This is because the number of customers left behind by the test-SC includes the SCs who arrive after him during the idle period and the OCs who arrive during the busy period until the completion of his service. These quantities, then, depend on the workload and the UMC phase at the arrival instance of the test-SC.

To derive the joint distribution of the workload and the UMC phase at the arrival instance of the test-SC, let us consider the BMAP arrival process during the idle period. Let $I_{i,j}(k)$ be the indicator random variable that takes 1 or 0 depending on whether the arrival process visits state (queue length) k and the UMC phase just after the visit is j under the condition that the UMC phase is i at the start of the idle period (i.e., point ⓪ in Figure 1). Conditioning on the size of the first arrival group, we have,

$$Pr[I_{i,j}(k) = 1] = \sum_{l=1}^{k} \sum_{n=1}^{m} [(-\boldsymbol{D}_0)^{-1} \boldsymbol{D}_l]_{i,n} Pr[I_{n,j}(k-l) = 1], \tag{3.2a}$$

where $I_{n,j}(0) = \begin{cases} 0, & (n \neq j) \\ 1, & (n = j) \end{cases}$. If we define the matrix $\boldsymbol{I}_k^* = (Pr[I_{i,j}(k) = 1])$, (3.2a) becomes

$$\boldsymbol{I}_k^* = \sum_{n=1}^{k} (-\boldsymbol{D}_0)^{-1} \boldsymbol{D}_n \boldsymbol{I}_{k-n}^*, \ (\boldsymbol{I}_0^* = \boldsymbol{I}). \tag{3.2b}$$

$\boldsymbol{I}_k^*$ is the matrix probability that the idle period queue length process visits state k.

Remark. $\boldsymbol{I}_k^*$ is the sum of the probabilities of all possibilities that lead to the visit of state k. For example,

$$\boldsymbol{I}_3^* = [(-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_1]^3 + (-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_1(-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_2 + (-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_2(-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_1 + (-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_3$$

Let $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_m)$ be the vector in which κ_i is the probability that the UMC phase is i at the start of the idle period. The mean length of the idle period is given in the following theorem.

Theorem 3.1. *Let I_D be the length of an arbitrary idle period under D-policy. Then, we have*

$$E(I_D) = \boldsymbol{\kappa} \int_0^D \sum_{k=0}^{\infty} \boldsymbol{I}_k^*(x)(-\boldsymbol{D}_0)^{-1}\mathbf{e}\, dx, \quad \text{where } \boldsymbol{I}_k^*(x) = \boldsymbol{I}_k^* \cdot s^{(k)}(x). \tag{3.3}$$

Proof. The probability that the idle period queue length process visits state k and the work is x at the visit is given by $\boldsymbol{I}_k^* s^{(k)}(x)$. It, then, stays there for $(-\boldsymbol{D}_0)^{-1}$ on the average, which proves the theorem.

Let us first define the following probability:

$u_j(x)$: the time-average probability that at an arbitrary point of time during the idle period the workload is x and the UMC phase is in j, $(x \geq 0)$

Let us define the vector,

$$\boldsymbol{u}(x) = (u_1(x), u_2(x), \ldots, u_m(x)). \tag{3.4}$$

Then, we have the following theorem

Theorem 3.2. *We have*

$$\boldsymbol{u}(0) = \frac{\boldsymbol{\kappa}(-\boldsymbol{D}_0)^{-1}}{E(I_D)}, \quad \boldsymbol{u}(x) = \frac{\boldsymbol{\kappa}\sum_{k=1}^{\infty}\boldsymbol{I}_k^*(x)(-\boldsymbol{D}_0)^{-1}}{E(I_D)}, \quad (0 < x \leq D) \tag{3.5}$$

Proof. It is known that the (i,j)-element $[(-\boldsymbol{D}_0)^{-1}]_{i,j}$ of $(-\boldsymbol{D}_0)^{-1}$ is the mean time the UMC stays in phase j until the next arrival under the condition that the UMC phase is in i at the start of the idle period. Thus, we have $u_j(0) = \frac{\sum_{i=1}^{m}\kappa_i[(-\boldsymbol{D}_0)^{-1}]_{i,j}}{E(I_D)}$, which proves the first equation. For the second equation, we note that $u_j(x)$ can be obtained as the time fraction $u_j(x) = \frac{E(T_{x,j})}{E(I_D)}$ where $T_{x,j}$ is the total length of time the workload is x and the UMC phase is j during the idle period. If the arrival process visits k and the workload is x at the visit (with probability $\boldsymbol{I}_k^*(x)$), it stays there for $(-\boldsymbol{D}_0)^{-1}$, and thus, we have $u_j(x) = \frac{(\boldsymbol{\kappa}\sum_{k=1}^{\infty}I_k^*(x)(-\boldsymbol{D}_0)^{-1})_j}{E(I_D)}$, which proves the theorem.

Let us define $\psi_{i,j}(n,x)dx = Pr(N_D = n, x \leq U_D \leq x + dx, J_{busy} = j | J_{idle} = i)$ where N_D, U_D and J_{busy} are the queue length, workload and the phase of the UMC at the busy period starting point, and J_{idle} is the UMC phase at the start of the previous idle period $(x > D, n \geq 1, 1 \leq i,j \leq m)$. Let $\boldsymbol{\Psi}(n,x)$ be the matrix of $\psi_{i,j}(n,x)$. Then, we have the following theorem.

Theorem 3.3. *We have*

$$\boldsymbol{\Psi}(n,x) = (-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_n s^{(n)}(x) + \sum_{k=1}^{n-1}\boldsymbol{I}_k^* \cdot (-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_{n-k}\int_{y=0}^{D} s^{(k)}(y)s^{(n-k)}(x-y)\,dy,$$

$$(n \geq 1, x > D). \tag{3.6}$$

Proof. $(-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_n$ represents the UMC phase probability at the first arrival instance with the group of size n. The first term is for the arrival with one group of size n and the total service time $x(>D)$. To prove the second term, assume that the last state visited before the workload exceeds D is k. This occurs with probability $\boldsymbol{I}_k^*$ and the workload at the visit is y with probability $s^{(k)}(y)$, $y(\leq D)$. Under this condition, the queue length at the busy period starting point becomes n with workload x if and only if the size of the last arrival group is $(n-k)$ and the total work is $(x-y)$ with probability $s^{(n-k)}(x-y)$.

We note that when $n = 1$, the above equation becomes $\boldsymbol{\Psi}(1, x) = (-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_1 s(x)$ under the usual notational convention $\sum_{k=1}^{0}(\ ) = 0$.

3.2. Queue Length at an Arbitrary SC Departure

The number of customers left behind by an arbitrary SC (test-SC) is the sum of the following two (see Figure 2):

(1) the number of OCs who arrive during the sum S_T of the service times of the SCs before and including the test-SC, and
(2) the number N_R of SCs who arrive after the test-SC. (N_R includes the customers who are behind the test-SC within his group. We will call this group the test group.

Note that S_T and N_R are not independent. Let us define the joint probability

$$\alpha(x, n, j) = Pr(x < S_T \leq x + dx, N_R = n, J_{busy} = j). \tag{3.7}$$

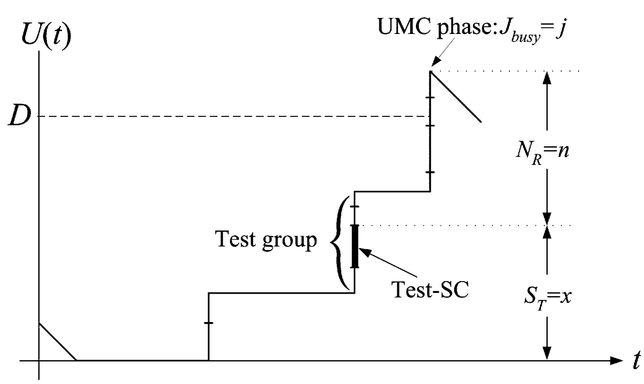

FIGURE 2 Number of customers left behind by test-SC.

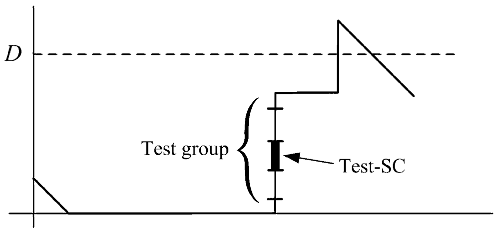

FIGURE 3 Case (1)-(a).

$\alpha(x, n, j)$ depends on different situations at the arrival instance of the test group as follows:

(1) The workload is 0 at the arrival instance of the test group:

   (a) D is not exceeded by the test group (Figure 3).
   (b) D is exceeded by the customers up to the test-SC (Figure 4).
   (c) D is not exceeded by the customers up to the test-SC, but is exceeded by the remaining customers in the test group (Figure 5).

(2) The workload is $y(>0)$ at the arrival instance of the test group:

   (a) D is not exceeded by the test group (Figure 6).
   (b) D is exceeded by the customers up to the test-SC (Figure 7).
   (c) D is not exceeded by the customers up to the test-SC, but is exceeded by the remaining customers in the test group (Figure 8).

To find the workload at the arrival instance of the test group, we use $\boldsymbol{u}(x)$ obtained in (3.5). As this quantity represents the workload and the UMC phase at an arbitrary time during the idle period (i.e., at a "virtual" time point), we need to convert this virtual quantity to the "actual" quantity as in the following theorem.

Theorem 3.4. *Let $\beta(x, i, k, j)$ be the joint probability that an arbitrary actual test group during the idle period sees the workload of x. The test-SC is* ith *within*

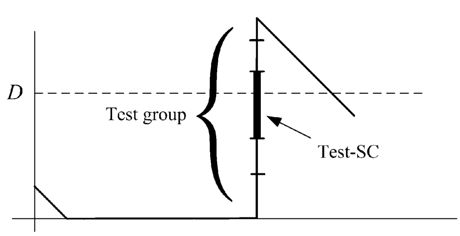

FIGURE 4 Case (1)-(b).

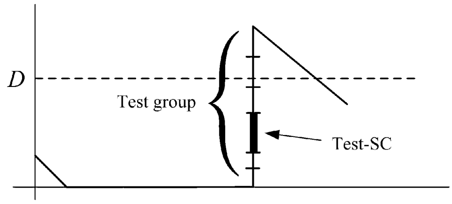

FIGURE 5 Case (1)-(c).

his group, there are k additional customers behind the test-SC in his group and the UMC phase is j just after the arrival. Then, we have

$$\beta(x, i, k, j) = \left(\boldsymbol{u}(x) \frac{\boldsymbol{D}_{i+k}}{\lambda} \right)_j . \tag{3.8}$$

Proof. Let G be the size of an arriving group. Then we have $Pr(G = i + k) = \frac{\boldsymbol{\pi D}_{i+k}\mathbf{e}}{\sum_{n=1}^{\infty} \boldsymbol{\pi D}_n \mathbf{e}} = \frac{\boldsymbol{\pi D}_{i+k}\mathbf{e}}{\lambda_g}$. An arbitrary test customer belongs to the group of size $(i + k)$ with probability

$$a_{i+k} = \frac{(i+k)Pr(G = i+k)}{E(G)} = \frac{(i+k)Pr(G = i+k)}{\lambda/\lambda_g} = \frac{(i+k)\boldsymbol{\pi D}_{i+k}\mathbf{e}}{\lambda_g}.$$

Because the test customer is ith in his group with probability $\frac{1}{i+k}$, the probability that the virtual group sees the workload x, the test customer belongs to the group of size $(i + k)$ and ith within his group is given by

$$\boldsymbol{u}(x) \cdot a_{i+k} \cdot \frac{1}{i+k} = \boldsymbol{u}(x) \frac{\boldsymbol{\pi D}_{i+k}\mathbf{e}}{\lambda}.$$

To convert the virtual probability to the actual probability, we postmultiply by $\frac{\boldsymbol{D}_{i+k}}{\boldsymbol{\pi D}_{i+k}\mathbf{e}}$ and complete the proof.

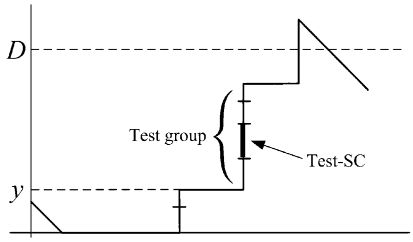

FIGURE 6 Case (2)-(a).

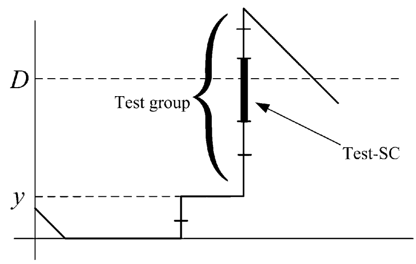

FIGURE 7 Case (2)-(b).

Now, we are ready to find the probability $\alpha(x, n, j)$ defined in (3.7) and depicted in Figure 2 for each of the above six cases.

Case (1)-(a). The workload is 0 at the arrival instance of the test group and D is not exceeded by the test group (Figure 3).

In this case, if we denote the (i, j) element of the matrix $\boldsymbol{\Psi}_D^{(n)} = \int_D^\infty \boldsymbol{\Psi}(n, x)\,dx$ as the probability that under the D-policy, the queue length is n and the UMC phase is j at the busy period starting point under the condition that the UMC phase is i at the start of the previous idle period, we have

$$\alpha(x, n, j) = \left[\int_{w=0}^{D-x} \sum_{i=1}^{\infty} \sum_{k=0}^{n-1} \boldsymbol{u}(0)\frac{\boldsymbol{D}_{i+k}}{\lambda} s^{(i)}(x) s^{(k)}(w) \boldsymbol{\Psi}_{D-x-w}^{(n-k)}\,dw\right]_j,$$
$$(n \geq 1, x \leq D). \tag{3.9}$$

In the above equation, $\boldsymbol{u}(0)\boldsymbol{D}_{i+k}/\lambda$ is the probability that the workload that is seen by the test SC during the idle period is zero and the SC is the ith with k customers behind him in his group. At this point, the workload is $x + w$ with probability $s^{(i)}(x)s^{(k)}(w)$ by these customers and under this condition, the remaining idle period is the same as the $(D - x - w)$-policy queue with $(n - k)$ more customers needed to exceed the threshold.

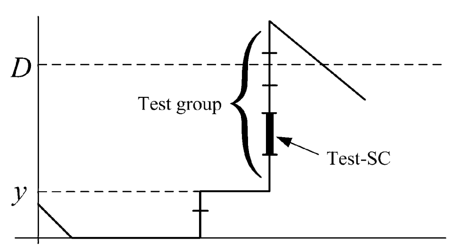

FIGURE 8 Case (2)-(c).

We can apply the same probabilistic arguments as we did in case (1)-(a) to the remaining five cases.

Case (1)-(b). The workload is 0 at the arrival instance of the test group, and D is exceeded by the customers up to the test-SC (Figure 4).

$$\alpha(x,n,j) = \left(\sum_{i=1}^{\infty} \boldsymbol{u}(0) \frac{\boldsymbol{D}_{i+n}}{\lambda} s^{(i)}(x) \right)_j, \quad (n \geq 0, x > D). \tag{3.10}$$

Case (1)-(c). The workload is 0 at the arrival instance of the test group, and $D$is not exceeded by the customers up to the test-SC, but is exceeded by the remaining customers in the test group (Figure 5).

$$\alpha(x,n,j) = \left(\int_{w=D-x}^{\infty} \sum_{i=1}^{\infty} \boldsymbol{u}(0) \frac{\boldsymbol{D}_{i+n}}{\lambda} s^{(i)}(x) s^{(n)}(w)\, dw \right)_j, \quad (n \geq 1, x \leq D). \tag{3.11}$$

Case (2)-(a). The workload is y at the arrival instance of the test group, and D is not exceeded by the test group (Figure 6).

$$\alpha(x,n,j) = \left\{ \int_{w=0}^{D-x} \sum_{i=1}^{\infty} \sum_{k=0}^{n-1} \left[\int_{y=0+}^{x} \boldsymbol{u}(y) \frac{\boldsymbol{D}_{i+k}}{\lambda} s^{(i)}(x-y)\, dy \right] s^{(k)}(w) \boldsymbol{\Psi}_{D-x-w}^{(n-k)}\, dw \right\}_j, \\ (n \geq 1, x \leq D). \tag{3.12}$$

Case (2)-(b). The workload is y at the arrival instance of the test group, and D is exceeded by the customers up to the test-SC.

$$\alpha(x,n,j) = \left(\sum_{i=1}^{\infty} \left[\int_{y=0+}^{D} \boldsymbol{u}(y) \frac{\boldsymbol{D}_{i+n}}{\lambda} s^{(i)}(x-y)\, dy \right] \right)_j, \quad (n \geq 0, x > D) \tag{3.13}$$

Case (2)-(c). The workload is y at the arrival instance of the test group, and D is not exceeded by the customers up to the test-SC, but is exceeded by the remaining customers in the test group.

$$\alpha(x,n,j) = \left(\int_{w=D-x}^{\infty} \sum_{i=1}^{\infty} \left[\int_{y=0+}^{x} \boldsymbol{u}(y) \frac{\boldsymbol{D}_{i+n}}{\lambda} s^{(i)}(x-y)\, dy \right] s^{(n)}(w)\, dw \right)_j, \\ (n \geq 1, x \leq D) \tag{3.14}$$

Now, the vector GF $\boldsymbol{X}_{SC}(z) = \sum_{k=0}^{\infty} z^k \boldsymbol{x}_k^{SC}$ of the queue length at the departure of the test-SC can be obtained by utilizing each $\alpha(x,n,j)$ of the six different cases. For example, if we define $\boldsymbol{\alpha}(x,n)$ as the vector of

$\alpha(x, n, j)$, the vector GF of the queue length at the departure of the test-SC for the case (1)-(a) becomes

$$(1-\rho)\int_{x=0}^{D}\sum_{n=1}^{\infty} z^n \boldsymbol{\alpha}(x,n) e^{\boldsymbol{D}(z)x}\,dx$$
$$=(1-\rho)\int_{x=0}^{D}\sum_{n=1}^{\infty} z^n \left[\int_{w=0}^{D-x}\sum_{i=1}^{\infty}\sum_{k=0}^{n-1} \boldsymbol{u}(0)\frac{\boldsymbol{D}_{i+k}}{\lambda} s^{(i)}(x) s^{(k)}(w) \boldsymbol{\Psi}_{D-x-w}^{(n-k)}\,dw\right] e^{\boldsymbol{D}(z)x}\,dx,$$

where $(1-\rho)$ is multiplied because the system is observed at an arbitrary time during the idle period (note that our system is work-conserving and the system is idle with probability $(1-\rho)$) and $e^{\boldsymbol{D}(z)x}$ at the end represents the matrix GF of the number of customers that arrive during x, which is given by $e^{\boldsymbol{D}(z)x} = \sum_{n=0}^{\infty} z^n \boldsymbol{P}(n, x)$ (Lucantoni[16,17]).

We can do the same thing for the remaining five cases. If we define the matrix GF $\boldsymbol{\Psi}_D(z) = \sum_{n=1}^{\infty} z^n \boldsymbol{\Psi}_D^{(n)}$ for notational simplicity, the vector GF $\boldsymbol{X}_{SC}(z)$ becomes

$$\begin{aligned}
&\boldsymbol{X}_{SC}(z)\\
&=(1-\rho)\Bigg\{\int_{x=0}^{D}\int_{w=0}^{D-x}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty} z^k \boldsymbol{u}(0)\frac{\boldsymbol{D}_{i+k}}{\lambda} s^{(i)}(x) s^{(k)}(w)\boldsymbol{\Psi}_{D-x-w}(z) e^{\boldsymbol{D}(z)x}\,dw\,dx\\
&\quad+\int_{x=D}^{\infty}\sum_{n=0}^{\infty} z^n \sum_{i=1}^{\infty}\boldsymbol{u}(0)\frac{\boldsymbol{D}_{i+n}}{\lambda} s^{(i)}(x) e^{\boldsymbol{D}(z)x}\,dx\\
&\quad+\int_{x=0}^{D}\int_{w=D-x}^{\infty}\sum_{n=1}^{\infty} z^n \sum_{i=1}^{\infty}\boldsymbol{u}(0)\frac{\boldsymbol{D}_{i+n}}{\lambda} s^{(i)}(x) s^{(n)}(w) e^{\boldsymbol{D}(z)x}\,dw\,dx\\
&\quad+\int_{x=0}^{D}\int_{w=0}^{D-x}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty} z^k \left[\int_{y=0+}^{x}\boldsymbol{u}(y)\frac{\boldsymbol{D}_{i+k}}{\lambda} s^{(i)}(x-y)\,dy\right] s^{(k)}(w)\boldsymbol{\Psi}_{D-x-w}(z)\\
&\quad\times e^{\boldsymbol{D}(z)x}\,dw\,dx\\
&\quad+\int_{x=D}^{\infty}\sum_{n=0}^{\infty} z^n \sum_{i=1}^{\infty}\left[\int_{y=0+}^{D}\boldsymbol{u}(y)\frac{\boldsymbol{D}_{i+n}}{\lambda} s^{(i)}(x-y)\,dy\right] e^{\boldsymbol{D}(z)x}\,dx\\
&\quad+\int_{x=0}^{D}\int_{w=D-x}^{\infty}\sum_{n=1}^{\infty} z^n \sum_{i=1}^{\infty}\left[\int_{y=0+}^{x}\boldsymbol{u}(y)\frac{\boldsymbol{D}_{i+n}}{\lambda} s^{(i)}(x-y)\,dy\right] s^{(n)}(w) e^{\boldsymbol{D}(z)x}\,dw\,dx\Bigg\}
\end{aligned}\tag{3.15}$$

$\boldsymbol{u}(0)$ and $\boldsymbol{u}(y)$ were obtained in (3.5).

3.3. Queue Length at an Arbitrary OC Departure

To obtain the probability vector $\boldsymbol{x}_k^{OC}$ of the queue length just after the "current" OC-departure, we consider the queue length at the "previous" departure. We have two possible cases.

Case 1. The previous departure is an OC-departure. In this case, if the queue length is j at the previous departure, the queue length at the current OC-departure becomes k if and only if $(k-j+1)$ customers arrive during the service time. Thus, we have k customers with probability $\sum_{j=1}^{k+1} \boldsymbol{x}_j^{OC} \int_0^\infty \boldsymbol{P}(k-j+1, x)\, dS(x)$.

Case 2. The previous departure is as SC-departure. Let $\boldsymbol{x}_{j,last}^{SC}$ be the vector probability that the previous departure is an SC-departure (this is the last departure among the SC departures) and the queue length is j. Then, we have k customers at the current OC-departure with probability $\sum_{j=1}^{k+1} \boldsymbol{x}_{j,last}^{SC} \int_0^\infty \boldsymbol{P}(k-j+1, x)\, dS(x)$.

Combining the probabilities of the two cases, we get

$$\boldsymbol{x}_k^{OC} = \sum_{j=1}^{k+1} \left(\boldsymbol{x}_j^{OC} + \boldsymbol{x}_{j,last}^{SC}\right) \int_0^\infty \boldsymbol{P}(k-j+1, x)\, dS(x). \tag{3.16}$$

Let us define the vector GFs as follows;

$$\boldsymbol{X}_{OC}(z) = \sum_{k=0}^{\infty} z^k \boldsymbol{x}_k^{OC}, \quad \boldsymbol{X}_{SC}^{last}(z) = \sum_{k=0}^{\infty} z^k \boldsymbol{x}_{k,last}^{SC}.$$

Multiplying (3.16) by z^k and summing over $k = 0, 1, 2, \ldots$, we get

$$\boldsymbol{X}_{OC}(z) = z^{-1}[\boldsymbol{X}_{OC}(z) - \boldsymbol{x}_0^{OC} + \boldsymbol{X}_{SC}^{last}(z) - \boldsymbol{x}_{0,last}^{SC}]\boldsymbol{A}(z), \tag{3.17}$$

where

$$\boldsymbol{A}(z) = \int_{x=0}^{\infty} e^{\boldsymbol{D}(z)x}\, dS(x) \tag{3.18}$$

is the GF of the number of customers that arrive during a service time.

Using $\boldsymbol{x}_0^{OC} + \boldsymbol{x}_{0,last}^{SC} = \boldsymbol{x}_0$ in (3.17), we get

$$\boldsymbol{X}_{OC}(z) = [\boldsymbol{X}_{SC}^{last}(z) - \boldsymbol{x}_0]\boldsymbol{A}(z)[z\boldsymbol{I} - \boldsymbol{A}(z)]^{-1}. \tag{3.19}$$

An arbitrary departure is the departure of the last SC if and only if he belongs to the last group during the idle period and he is the last customer in his group. Thus, using (3.10) and (3.13), $\boldsymbol{X}_{SC}^{last}(z)$ in (3.19) becomes

$$\boldsymbol{X}_{SC}^{last}(z) = (1-\rho)\int_{D}^{\infty}\left\{\sum_{i=1}^{\infty}\boldsymbol{u}(0)\frac{\boldsymbol{D}_i}{\lambda}s^{(i)}(x) + \sum_{i=1}^{\infty}\left[\int_{y=0+}^{D}\boldsymbol{u}(y)\frac{\boldsymbol{D}_i}{\lambda}s^{(i)}(x-y)\right]\right\} \times e^{\boldsymbol{D}(z)x}\,dx. \tag{3.20}$$

3.4. Queue Length at an Arbitrary Departure

Finally, the queue length at an arbitrary departure can be obtained from

$$\boldsymbol{X}(z) = \sum_{k=0}^{\infty} z^k \boldsymbol{x}_k = \boldsymbol{X}_{SC}(z) + \boldsymbol{X}_{OC}(z), \tag{3.21}$$

where $\boldsymbol{X}_{SC}(z)$ and $\boldsymbol{X}_{OC}(z)$ were obtained in (3.15) and (3.19).

3.5. Determining $\boldsymbol{x}_0$

It is necessary to determine $\boldsymbol{x}_0$ in (3.19). We can use the standard method developed by Neuts[20] and employed by many authors (see, for example, Lucantoni et al.[15] and Lucantoni[16,17]). Let $\boldsymbol{K}(z)$ be the matrix GF of the number of customers that are served during a cycle (i.e., during the interval from ⓪ to ③ in Figure 1). Then, we have the following theorem.

Theorem 3.5. *We have*

$$\boldsymbol{K}(z) = \int_{x=D}^{\infty}\sum_{n=1}^{\infty}(-\boldsymbol{D}_0)^{-1}z^n\boldsymbol{D}_n e^{\boldsymbol{D}[\boldsymbol{G}(z)]x}s^{(n)}(x)\,dx + \int_{x=D}^{\infty}\sum_{j=1}^{\infty}z^j\boldsymbol{\Gamma}_j^*\sum_{n=j+1}^{\infty}z^{n-j}(-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_{n-j}e^{\boldsymbol{D}[\boldsymbol{G}(z)]x}\int_{y=0}^{D}s^{(n-j)}(x-y)s^{(j)}(y)\,dy\,dx \tag{3.22}$$

where $\boldsymbol{G}(z)$ is the matrix GF of the number of customers that are served during a fundamental period (Neuts[20]).

Proof. Let us define the (i,j) element of the matrix $\boldsymbol{Q}_{k_1,k_2}$ as the joint probability that the queue length at ① is k_1 (these are SCs), the queue length at ② is k_2 (these are OCs existing at the end of the last service of SC), and the UMC phase at ② is j under the condition that the UMC phase is i at ⊚. Then, we get, after using the definition of $\boldsymbol{\Psi}(k_1, x)$ used in

Theorem 3.3,

$$
\begin{aligned}
\boldsymbol{Q}_{k_1,k_2} &= \int_{x=D}^{\infty} \boldsymbol{\Psi}(k_1,x)\boldsymbol{P}(k_2,x)\,dx \\
&= \begin{cases}
\displaystyle\int_{x=D}^{\infty} (-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_{k_1}\cdot s^{(k_1)}(x)\boldsymbol{P}(k_2,x)\,dx \quad (k_1=1, k_2\geq 0), \\
\displaystyle\int_{x=D}^{\infty} (-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_{k_1}\cdot s^{(k_1)}(x)\boldsymbol{P}(k_2,x)\,dx \\
\displaystyle\quad + \int_{x=D}^{\infty}\sum_{j=1}^{k_1-1}\Big[\boldsymbol{I}_j^{*}\cdot(-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_{k_1-j}\int_{y=0}^{D} s^{(j)}(y)s^{(k_1-j)}(x-y)\,dy\Big]\boldsymbol{P}(k_2,x), \\
\qquad\qquad (k_1\geq 2, k_2\geq 0)
\end{cases}
\end{aligned}
\tag{3.23}
$$

If we define $\boldsymbol{Q}(z_1,z_2) = \sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty} z_1^{k_1}\boldsymbol{Q}_{k_1,k_2}z_2^{k_2}$, we get

$$
\begin{aligned}
\boldsymbol{Q}(z_1,z_2) &= \sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty} z_1^{k_1}\boldsymbol{Q}_{k_1,k_2}z_2^{k_2} = \int_{x=D}^{\infty}\sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty} z_1^{k_1}(-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_{k_1}s^{(k_1)}(x)\boldsymbol{P}(k_2,x)z_2^{k_2}\,dx \\
&\quad + \int_{x=D}^{\infty}\sum_{k_1=2}^{\infty}\sum_{k_2=0}^{\infty} z_1^{k_1}\sum_{j=1}^{k_1-1}\boldsymbol{I}_j^{*}(-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_{k_1-j}\boldsymbol{P}(k_2,x)z_2^{k_2} \\
&\quad \times \int_{y=0}^{D} s^{(k_1-j)}(x-y)s^{(j)}(y)\,dy\,dx \\
&= \int_{x=D}^{\infty}\sum_{k_1=1}^{\infty}(-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_{k_1}z_1^{k_1}s^{(k_1)}(x)e^{\boldsymbol{D}(z_2)x}\,dx \\
&\quad + \int_{x=D}^{\infty}\sum_{j=1}^{\infty} z_1^{j}\boldsymbol{I}_j^{*}\sum_{k_1=j+1}^{\infty}(-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_{k_1-j}z_1^{k_1-j}e^{\boldsymbol{D}(z_2)x} \\
&\quad \times \int_{y=0}^{D} s^{(k_1-j)}(x-y)s^{(j)}(y)\,dy\,dx
\end{aligned}
\tag{3.24a}
$$

Then, $\boldsymbol{K}(z) = \boldsymbol{Q}(z_1,z_2)|_{z_1=z, z_2=\boldsymbol{G}(z)}$ proves the theorem.

We note that $\boldsymbol{K}(z)$ can be expressed as

$$
\boldsymbol{K}(z) = \int_{x=D}^{\infty}\sum_{n=1}^{\infty} z^n\boldsymbol{\Psi}(n,x)\sum_{k=0}^{\infty}\boldsymbol{P}(k,x)[\boldsymbol{G}(z)]^k\,dx, \tag{3.24b}
$$

which can be shown to be equal to (3.24a).

Let $\boldsymbol{K}$ be the phase transition probability matrix between ◎ and ③. $\boldsymbol{K}$ can be obtained from $\boldsymbol{K} = \boldsymbol{K}(z)|_{z=1}$ and becomes

$$\boldsymbol{K} = \int_{x=D}^{\infty} \sum_{n=1}^{\infty} (-\boldsymbol{D}_0)^{-1} \boldsymbol{D}_n e^{\boldsymbol{D}(\boldsymbol{G})x} s^{(n)}(x)\, dx + \int_{x=D}^{\infty} \sum_{j=1}^{\infty} \boldsymbol{I}_j^* \sum_{n=j+1}^{\infty} (-\boldsymbol{D}_0)^{-1} \boldsymbol{D}_{n-j} e^{\boldsymbol{D}(\boldsymbol{G})x} \int_{y=0}^{D} s^{(n-j)}(x-y) s^{(j)}(y)\, dy\, dx \quad (3.25)$$

The stationary vector $\boldsymbol{K}$ of the UMC probabilities at ◎ can be obtained from

$$\boldsymbol{\kappa} = \boldsymbol{\kappa K}, \quad \boldsymbol{\kappa}\mathbf{e} = 1. \quad (3.26)$$

Let $\boldsymbol{\kappa}^* = (\kappa_1^*, \kappa_2^*, \ldots, \kappa_m^*)^T$ be $(m \times 1)$ vector that represents the mean number of customers that are served during a cycle. Then, we have

$$\begin{aligned}
\boldsymbol{\kappa}^* = \frac{d}{dz}\boldsymbol{K}(z)\Big|_{z=1}\mathbf{e} &= \sum_{n=1}^{\infty} n(-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_n[1 - S^{(n)}(D)]\mathbf{e} \\
&+ \int_{x=D}^{\infty} \sum_{n=1}^{\infty} (-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_n \left[\frac{d}{dz} e^{\boldsymbol{D}[\boldsymbol{G}(z)]x}\right]_{z=1} s^{(n)}(x)\, dx\mathbf{e} \\
&+ \sum_{j=1}^{\infty} \boldsymbol{I}_j^* \sum_{n=j+1}^{\infty} n(-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_{n-j}\mathbf{e} \int_{x=D}^{\infty} \int_{y=0}^{D} s^{(n-j)}(x-y)s^{(j)}(y)\, dy\, dx \\
&+ \sum_{j=1}^{\infty} \boldsymbol{I}_j^* \sum_{n=j+1}^{\infty} (-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_{n-j} \int_{x=D}^{\infty} \left[\frac{d}{dz} e^{\boldsymbol{D}[\boldsymbol{G}(z)]x}\right]_{z=1} \\
&\times \mathbf{e} \int_{y=0}^{D} s^{(n-j)}(x-y)s^{(j)}(y)\, dy\, dx
\end{aligned} \quad (3.27)$$

In (3.27), we have

$$\left[\frac{d}{dz} e^{\boldsymbol{D}[\boldsymbol{G}(z)]x}\right]_{z=1} \mathbf{e} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} [\boldsymbol{D}(\boldsymbol{G})]^n \sum_{k=1}^{\infty} \boldsymbol{D}_k \sum_{i=0}^{k-1} \boldsymbol{G}^i \boldsymbol{\mu},$$

where $\boldsymbol{\mu}$ is given by (Lucantoni[16])

$$\boldsymbol{\mu} = \frac{d}{dz}\boldsymbol{G}(z)\Big|_{z=1}\mathbf{e} = (\boldsymbol{I} - \boldsymbol{G} + \mathbf{e}\boldsymbol{g})[\boldsymbol{I} - \boldsymbol{A} + (\mathbf{e} - \boldsymbol{\beta})\boldsymbol{g}]^{-1}\mathbf{e}$$

in which $\boldsymbol{\beta}$. is given by

$$\boldsymbol{\beta} = \frac{d}{dz}\boldsymbol{A}(z)\Big|_{z=1}\mathbf{e} = \rho\mathbf{e} + (\mathbf{e}\boldsymbol{\pi} + \boldsymbol{D})^{-1}(\boldsymbol{A} - \boldsymbol{I})\left(\sum_{n=1}^{\infty} n\boldsymbol{D}_n\mathbf{e}\right).$$

In the above equation, the (i, j) element of $\boldsymbol{A}$ is the probability that at the end of a service time the UMC phase is j under the condition that the UMC phase is i at the start of the service.

As $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}^*$ have been obtained, $\boldsymbol{x}_0$ in (3.19) can be computed from the following well-known identity (Neuts[20]; Lucantoni[17]).

$$\boldsymbol{x}_0 = \frac{\boldsymbol{\kappa}}{\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}^*}. \tag{3.28}$$

4. QUEUE LENGTH AT AN ARBITRARY TIME

If we denote the vector $\boldsymbol{y}_k$ as the probability that there are k customers at an arbitrary time and define the vector GF $\boldsymbol{Y}(z) = \sum_{k=0}^{\infty} z^k \boldsymbol{y}_k$, we have the following relationship (Takine and Takahashi[22])

$$\boldsymbol{Y}(z)\boldsymbol{D}(z) = \lambda(z-1)\boldsymbol{X}(z). \tag{4.1}$$

We note that above relationship holds, as stated by Takine and Takahashi[22], for a broad class of stationary queues with BMAP arrivals including our D-policy queues.

5. MEAN QUEUE LENGTHS

In this section, we derive the mean queue length. For a matrix or vector GF $\boldsymbol{M}(z)$, we use $\boldsymbol{M}$ and $\boldsymbol{M}^{(n)}$ to denote $\boldsymbol{M} = \boldsymbol{M}(z)|_{z=1}$, $\boldsymbol{M}^{(n)} = \frac{d^n}{dz^n}\boldsymbol{M}(z)|_{z=1}$. The mean queue length L_d at an arbitrary departure epoch can be obtained from (3.21) which becomes

$$L_d = \boldsymbol{X}^{(1)}\mathbf{e} = (\boldsymbol{X}_{SC}^{(1)} + \boldsymbol{X}_{OC}^{(1)})\mathbf{e}. \tag{5.1}$$

From (3.15), we get

$$\begin{aligned}\boldsymbol{X}_{SC}^{(1)}\mathbf{e} = (1-\rho)\Bigg\{ &\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} k\left[\boldsymbol{u}(0) + \int_{y=0+}^{D}\boldsymbol{u}(y)\,dy\right]\frac{\boldsymbol{D}_{i+k}}{\lambda}\mathbf{e} \\ &+ \int_{x=0}^{D}\int_{w=0}^{D-x}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\left[\boldsymbol{u}(0)s^{(i)}(x) + \int_{y=0+}^{x}\boldsymbol{u}(y)s^{(i)}(x-y)\,dy\right] \\ &\times \frac{\boldsymbol{D}_{i+k}}{\lambda}s^{(k)}(w)\boldsymbol{\Psi}_{D-x-w}^{(1)} \cdot \mathbf{e}\,dw\,dx\end{aligned}$$

$$+\int_{x=0}^{D}\int_{w=0}^{D-x}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\left[\boldsymbol{u}(0)s^{(i)}(x)+\int_{y=0+}^{x}\boldsymbol{u}(y)s^{(i)}(x-y)\,dy\right]$$

$$\times\frac{\boldsymbol{D}_{i+k}}{\lambda}s^{(k)}(w)\boldsymbol{\Psi}_{D-x-w}\cdot\left[\frac{d}{dz}e^{\boldsymbol{D}(z)x}\right]_{z=1}\mathbf{e}\,dw\,dx$$

$$+\int_{x=D}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{\infty}\left[\boldsymbol{u}(0)s^{(i)}(x)+\int_{y=0+}^{x}\boldsymbol{u}(y)s^{(i)}(x-y)\,dy\right]$$

$$\times\frac{\boldsymbol{D}_{i+n}}{\lambda}\left[\frac{d}{dz}e^{\boldsymbol{D}(z)x}\right]_{z=1}\mathbf{e}\,dx$$

$$+\int_{x=0}^{D}\int_{w=D-x}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\left[\boldsymbol{u}(0)s^{(i)}(x)+\int_{y=0+}^{x}\boldsymbol{u}(y)s^{(i)}(x-y)\,dy\right]$$

$$\times\frac{\boldsymbol{D}_{i+n}}{\lambda}s^{(n)}(w)\left[\frac{d}{dz}e^{\boldsymbol{D}(z)x}\right]_{z=1}\mathbf{e}\,dw\,dx\Bigg\} \tag{5.2}$$

To derive $\boldsymbol{X}_{OC}^{(1)}\mathbf{e}$, let us write (3.19) as follows,

$$\boldsymbol{X}_{OC}(z)[z\boldsymbol{I}-\boldsymbol{A}(z)]=\boldsymbol{T}(z) \tag{5.3}$$

where

$$\boldsymbol{T}(z)=[\boldsymbol{X}_{SC}^{last}(z)-\boldsymbol{x}_0]\boldsymbol{A}(z), \tag{5.4}$$

and $\boldsymbol{A}$(z) was given in (3.18). Taking derivatives of both sides of (5.3) and using $z=1$ yields

$$\boldsymbol{X}_{OC}^{(1)}(\boldsymbol{I}-\boldsymbol{A})+\boldsymbol{X}_{OC}(\boldsymbol{I}-\boldsymbol{A}^{(1)})=\boldsymbol{T}^{(1)}. \tag{5.5}$$

Subtracting $\boldsymbol{X}_{OC}^{(1)}\mathbf{e}\boldsymbol{\pi}$ from both sides, we get

$$\boldsymbol{X}_{OC}^{(1)}=\boldsymbol{X}_{OC}^{(1)}\mathbf{e}\boldsymbol{\pi}+\left[\boldsymbol{T}^{(1)}-\boldsymbol{X}_{OC}(\boldsymbol{I}-\boldsymbol{A}^{(1)})\right](\boldsymbol{I}-\boldsymbol{A}+\mathbf{e}\boldsymbol{\pi})^{-1}. \tag{5.6}$$

Taking the second derivative of (5.3) and using $z=1$ yields

$$\boldsymbol{X}_{OC}^{(1)}\boldsymbol{\beta}=\boldsymbol{X}_{OC}^{(1)}\mathbf{e}-\frac{1}{2}\left[\boldsymbol{T}^{(2)}\mathbf{e}+\boldsymbol{X}_{OC}\boldsymbol{A}^{(2)}\mathbf{e}\right]. \tag{5.7}$$

Multiplying both sides of (5.6) by $\boldsymbol{\beta}$ and adding (5.7) yields

$$\boldsymbol{X}_{OC}^{(1)}\mathbf{e}-\boldsymbol{X}_{OC}^{(1)}\mathbf{e}\boldsymbol{\pi}\boldsymbol{\beta}=\left[\boldsymbol{T}^{(1)}-\boldsymbol{X}_{OC}(\boldsymbol{I}-\boldsymbol{A}^{(1)})\right](\boldsymbol{I}-\boldsymbol{A}+\mathbf{e}\boldsymbol{\pi})^{-1}\boldsymbol{\beta}$$
$$+\frac{1}{2}\left[\boldsymbol{T}^{(2)}\mathbf{e}+\boldsymbol{X}_{OC}\boldsymbol{A}^{(2)}\mathbf{e}\right]. \tag{5.8}$$

Using $\pi\beta = \rho$ (Neuts[20]), we finally get

$$X_{OC}^{(1)}\mathbf{e} = \frac{1}{2(1-\rho)}\left\{T^{(2)}\mathbf{e} + X_{OC}A^{(2)}\mathbf{e} + 2\left[T^{(1)} - X_{OC}(I - A^{(1)})\right](I - A + \mathbf{e}\pi)^{-1}\beta\right\}. \tag{5.9}$$

To derive the mean queue length at an arbitrary time, we can use (4.1). Taking derivatives of (4.1) and using $z = 1$, we get

$$Y^{(1)}D = \lambda X - \pi D^{(1)}. \tag{5.10}$$

Taking the second derivative of (4.1), we get

$$Y^{(1)}D^{(1)}\mathbf{e} = \lambda X^{(1)}\mathbf{e} - \frac{1}{2}\pi D^{(2)}\mathbf{e}. \tag{5.11}$$

Adding $Y^{(1)}\mathbf{e}\pi$ to (5.10) and arranging terms yields

$$Y^{(1)} = Y^{(1)}\mathbf{e}\pi + (\lambda X - \pi D^{(1)})(\mathbf{e}\pi + D)^{-1}. \tag{5.12}$$

Postmultiplying (5.12) by $D^{(1)}\mathbf{e}$ and using $\pi D^{(1)}\mathbf{e} = \lambda$ yields

$$\lambda Y^{(1)}\mathbf{e} = Y^{(1)}D^{(1)}\mathbf{e} - (\lambda X - \pi D^{(1)})(\mathbf{e}\pi + D)^{-1}D^{(1)}\mathbf{e}. \tag{5.13}$$

Using (5.11) in (5.13), we get

$$Y^{(1)}\mathbf{e} = X^{(1)}\mathbf{e} - \frac{\lambda^{-1}}{2}\pi D^{(2)}\mathbf{e} + \left(\lambda^{-1}\pi D^{(1)} - X\right)(\mathbf{e}\pi + D)^{-1}D^{(1)}\mathbf{e}. \tag{5.14}$$

6. NUMERICAL EXAMPLE (COMPARISON WITH M^X/G/1/D–POLICY QUEUE)

In this section, we compare the mean queue lengths of the three systems:

(System-1) BMAP/G/1/ queue under *D*-policy,
(System-2) M^X/G/1 queue under *D*-policy,
(System-3) M^X/G/1 queue without *D*-policy.

By comparing system-1 and system-2, we can see the effect of BMAP arrivals. By comparing system-2 and system-3, we can see the effect of the policy.

We use the following parameter matrices for BMAP queue:

$$D_0 = \begin{pmatrix} -10 & 1 \\ 0.4 & -0.8 \end{pmatrix}, \quad D_k = \begin{pmatrix} 9p(1-p)^{k-1} & 0 \\ 0 & 0.4r(1-r)^{k-1} \end{pmatrix}, \quad (k = 1, 2, \dots),$$

TABLE 1 Comparison of mean queue lengths for three systems ($D = 0.5$)

ρ	$L(BMAP)$	$L(Poisson)$	$L(Poisson)$ (without D-policy)	$\frac{L(BMAP)}{L(Poisson)}$
0.1	12.1912	13.8973	0.2168	0.8772
0.3	5.5084	5.1954	0.8363	1.0602
0.5	9.3791	4.3547	1.9513	2.1538
0.7	22.0477	6.0711	4.5530	3.6316
0.9	88.8252	18.5603	17.5617	4.7857

We use $p = 0.5$ and $r = 0.8$. Then, we get $\boldsymbol{D} = \sum_{n=0}^{\infty} \boldsymbol{D}_n = \begin{pmatrix} -1 & 1 \\ \frac{2}{5} & -\frac{2}{5} \end{pmatrix}$. From $\boldsymbol{\pi D} = 0$ and $\boldsymbol{\pi}\mathbf{e} = 0$, the steady-state phase probability vector becomes $\boldsymbol{\pi} = \{\pi_1, \pi_2\} = \left(\frac{2}{7}, \frac{5}{7}\right)$. Then, the mean group arrival rate λ_g and the mean customer arrival rate λ become $\lambda_g = \boldsymbol{\pi} \sum_{n=1}^{\infty} \boldsymbol{D}_n \mathbf{e} = \frac{20}{7}$ and $\lambda = \boldsymbol{\pi} \sum_{n=1}^{\infty} n\boldsymbol{D}_n \mathbf{e} = \frac{11}{2}$. An arrival group is of size k with probability $g_k = \frac{\boldsymbol{\pi D}_n \mathbf{e}}{\lambda_g} = \frac{9}{10} p(1-p)^{k-1} + \frac{1}{10} r(1-r)^{k-1} = \frac{9}{10}(\frac{1}{2})^k + \frac{4}{10}(\frac{1}{5})^k, (k \geq 1)$.

We will use the same λ, λ_g and g_k for the $M^X/G/1$ queues with and without D-policy. For the three systems, we use the exponential distribution as the common service time.

Table 1 shows the mean queue lengths of the three systems for different traffic intensities. The last column shows the ratio $\frac{L(BMAP)}{L(Poisson)}$ to see the BMAP effect over the simple compound Poisson arrivals.

From the table, we see that, in both BMAP/G/1 and $M^X/G/1$ queues under D-policy, higher traffic intensity ρ does not necessarily mean larger mean queue length (see the mean queue lengths when $\rho = 0.1$ and $\rho = 0.3$.). This is especially so when the threshold D is much larger than the mean service time. This is due to the fact that larger ρ means the larger mean idle period. For example, if $\rho = 0.1$, i.e, $E(S) = \frac{2}{110}$, it will take more than 27 customers (or equivalently 9 arrival groups) on the average until the server becomes busy, which means that the queue length is high on the average during the idle period.

It is observed that as ρ gets closer to 1, the relative difference between the BMAP queue and Poisson queue gets larger. This implies that any naive Poisson assumptions may result in severe underestimation of the mean queue lengths.

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