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A factorization property for BMAP/G/1 vacation queues under variable service speed

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Abstract This paper proposes a simple factorization principle that can be used efficiently and effectively to derive the vector generating function of the queue length at an arbitrary time of the BMAP/G/1/ queueing systems under variable service speed. We first prove the factorization property. Then we provide moments formula. Lastly we present some applications of the factorization principle.

Keywords BMAP/G/1 · Variable service speed · Factorization

1 Introduction

This paper studies the BMAP/G/1 queue with variable service speed (VSS) and proposes a factorization principle that can be used efficiently to derive the queue length distribution. The queueing systems with VSS can be applied to communication systems and production/manu-facturing systems. For more studies on the queueing systems with VSS readers are referenced to Takine (2005), Ahn and Ramaswami (2004), Asmussen (1995), Eisen and Tainiter (1963), Halfin (1972), Purdue (1974), Ramaswami (1999), Tzenova et al. (2005), Yechiali (1973) and Yechiali and Naor (1971).

The queueing systems in which the workload increases or decreases linearly according to the phases of the underlying Markov chain (UMC) have been studied under the name

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of MMFF (Markov Modulated Fluid Flow) models. Readers are advised to see Ahn and Ramaswami (2004), Asmussen (1995), Ramaswami (1999), Takine (2005) and references therein.

The system we are studying is the BMAP/G/1/VSS system with generalized server vacations. Customers arrive according to the BMAP (Batch Markovian Arrival process) and the server provides services at variable speeds according to the phases of the UMC. Takine (2005) derived the queue length and waiting time distributions of the MMAP/G/1/VSS system. He constructed a modified system from the original VSS system such that the service rate becomes a constant. He analyzed the modified system and then converted the results to the original VSS system (more detailed explanation on the modified system will be given in Sect. 2). Even though his results can be adopted to an extensive classes of VSS systems, it is not easy to apply his method to vacation systems because it is necessary to build a separate modified system for each of the vacation systems. Here in this paper, we propose a factorization property that can be applied to a wide range of BMAP vacation systems under VSS. Our result is simple and elegant and is very easy to apply.

The term MAP (Markovian Arrival process) was coined by Lucantoni et al. (1990). It is known that the MAP includes the Poisson process, the IPP (Interrupted Poisson process), the MMPP (Markov modulated Poisson process) the phase-type (PH) renewal process and the suppositions of these. The BMAP (Batch Markovian Arrival process) was termed by Lucantoni (1993). It was found that the BMAP is equivalent to the “versatile Markovian point process” of Neuts (1979) and the “N-process” of Ramaswami (1980).

The studies on MAP(BMAP)/G/1 queues with vacations and/or control policies can be found in Chang et al. (2002), Lee and Ahn (2002), Lee et al. (2001, 2003, 2004), Lee and Baek (2005) and Lee and Song (2004). Lee et al. (2001) and Chang et al. (2002) showed that the vector generating functions (GF) of the queue length at an arbitrary time and at an arbitrary departure in the BMAP/G/1 queues with generalized vacations is decomposed into two parts, one of which is the queue length vector GF at an arbitrary point of time during the idle period. To be more specific they proved the following types of decompositions:

$$\mathbf{Y}(z) = \mathbf{p}_{idle}(z) \boldsymbol{\chi}_Y(z), \quad (1)$$

$$\mathbf{X}(z) = \mathbf{p}_{idle}(z) \boldsymbol{\chi}_X(z), \quad (2)$$

where

$$\boldsymbol{\chi}_Y(z) = (1 - \rho)(z - 1)\mathbf{A}(z)[z\mathbf{I} - \mathbf{A}(z)]^{-1}, \quad (3)$$

$$\boldsymbol{\chi}_X(z) = \frac{1}{\lambda}(1 - \rho)\mathbf{D}(z)\mathbf{A}(z)[z\mathbf{I} - \mathbf{A}(z)]^{-1}. \quad (4)$$

In (1) and (2) $\mathbf{p}_{idle}(z)$ is the conditional vector GF of the queue length at an arbitrary time during the idle period and $\boldsymbol{\chi}_Y(z)$ and $\boldsymbol{\chi}_X(z)$ are matrix GFs common to all generalized vacation systems. $\mathbf{D}(z)$ is the matrix GF of the parameter matrices $\{\mathbf{D}_1, \mathbf{D}_2, \dots\}$ of the BMAP and $\mathbf{A}(z)$ is the matrix GF of the number of customers that arrive during a service time.

The objective of this paper is to find a decomposition form like (1) that can be applied to an extensive classes of BMAP queueing systems with vacations under variable service speed.

2 The system

In our system, customers arrive according the BMAP with parameter matrices $\{\mathbf{D}_1, \mathbf{D}_2, \dots\}$ and the matrix GF $\mathbf{D}(z) = \sum_{n=0}^{\infty} \mathbf{D}_n z^n$. If we let $\boldsymbol{\pi}$ be the stationary probability vector of the

Fig. 1 The sample paths of the original VSS system and the modified system

UMC, the arrival rate becomes

$$\lambda = \sum_{n=1}^{\infty} \pi \mathbf{D}_n \mathbf{e}, \quad (5)$$

where $\mathbf{e}$ is the column vector of ones.

The service requirements (i.e., the amount of work that is offered to the system) of customers are independent and identically distributed (iid) with distribution function (DF) $S(x)$. The service is rendered at variable speeds that depend on the phases of the UMC. When the UMC phase is i , the service speed is r_i . In ordinary queueing systems the service speed is $r_i = 1$ for all i .

The upper part of Fig. 1 shows the sample path example of the queue length and the workload with service speeds $r_1 = 1$ and $r_2 = 2$. This means that the customer is served at normal speed when the UMC phase is 1 and at double speed when the UMC phase is 2. $N(t)$, $U(t)$ and $J(t)$ represent the queue length, workload and the UMC phase at time t .

If we define the matrix

$$\mathbf{R} = \text{diag}(r_1, r_2, \dots, r_m) = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_m \end{pmatrix},$$

the amount γ of work that can be processed per unit time becomes

$$\gamma = \pi \mathbf{R} \mathbf{e}. \quad (6)$$

Then the probability ρ_{vss} that the server is busy in the original VSS system becomes

$$\rho_{vss} = \gamma^{-1} \lambda E(S) < 1. \quad (7)$$

Since the service speed is variable, it is important to differentiate the “service time” (i.e., the time that it takes from the beginning of a service to its end) from the “service requirement”.

Now, we define the “modified system”. We first analyze the modified system and then convert the results to the original VSS system.

Definition 1.1 (The modified system) The system in which the time axis is extended by r_i times when the UMC phase is i is called the modified system (see Fig. 1).

For example, if the service speed is 2 when the UMC phase is in 2 (see Fig. 1), then the time flows at the slower rate of 1/2 which means that a service that is processed in an hour in the original VSS system now takes two hours in the modified system. In this way, the service speed becomes a constant in the modified system for all UMC phases. The lower part of Fig. 1 shows the sample path of the workload in the modified system. $U_{mod}(t)$ and $J_{mod}(t)$ shows the workload and the UMC phase in the modified system. What needs to be noticed here is that the length of the vacations needs to be extended proportionally to the service speed.

The sample path of the modified system can be seen as that of the ordinary BMAP/G/1 system with iid service time $S(x)$ and UMC parameter matrices $\{\mathbf{D}_{mod,1}, \mathbf{D}_{mod,2}, \dots\} = \{\mathbf{R}^{-1}\mathbf{D}_1, \mathbf{R}^{-1}\mathbf{D}_2, \dots\}$ with the matrix GF

$$\mathbf{D}_{mod}(z) = \sum_{n=0}^{\infty} \mathbf{D}_{mod,n} z^n = \sum_{n=0}^{\infty} \mathbf{R}^{-1} \mathbf{D}_n z^n. \quad (8)$$

Since the service speed in the modified is always 1, the UMC phase probability vector $\boldsymbol{\pi}_{mod}$ of the modified system satisfies

$$\boldsymbol{\pi}_{mod} \mathbf{R}^{-1} \left(\mathbf{D}_0 + \sum_{n=1}^{\infty} \mathbf{D}_n \right) = \mathbf{0}, \quad (9)$$

which means that $\boldsymbol{\pi}_{mod} \mathbf{R}^{-1}$ and $\boldsymbol{\pi}$ are proportional. Thus after a normalization, we get

$$\boldsymbol{\pi}_{mod} = \frac{\boldsymbol{\pi} \mathbf{R}}{\boldsymbol{\pi} \mathbf{R} \mathbf{e}} = \gamma^{-1} \boldsymbol{\pi} \mathbf{R}. \quad (10)$$

From (10) and (5), the arrival rate in the modified system becomes

$$\lambda_{mod} = \boldsymbol{\pi}_{mod} \left(\sum_{n=1}^{\infty} \mathbf{R}^{-1} \mathbf{D}_n \right) \mathbf{e} = \gamma^{-1} \lambda. \quad (11)$$

As can be seen in Fig. 1, the queue length vector GFs at the departures are equal in both systems:

$$\mathbf{X}_{vss}(z) = \mathbf{X}_{mod}(z). \quad (12)$$

This is because the two quantities are number-average quantities. This fact was also mentioned in Takine (2005).

The vector GFs of the queue length at arbitrary times in both systems are related by Takine (2005).

$$\mathbf{Y}_{vss}(z) = \gamma \mathbf{Y}_{mod}(z) \mathbf{R}^{-1}, \quad (13)$$

where $\mathbf{Y}_{vss}(z)$ and $\mathbf{Y}_{mod}(z)$ are the vector GFs of the queue length in the original VSS system and in the modified system respectively.

Remark 2.1

- (i) We note again that when constructing the modified system, the length of the vacations should be modified accordingly.
- (ii) Equation (13) can be interpreted in the following way. The queue length probabilities at an arbitrary time in the original VSS system is a time-average probability. Also, we note that the staying time in a state is extended by r_i times in the modified system when the UMC phase is in i . Thus, if we divide the staying time in the modified system by r_i and multiply the normalization constant γ , we can get the queue length distribution (13) in the original VSS system.
- (iii) Many quantities of interest in the modified system are still the same as in the original VSS system. For example, $\mathbf{A}_{vss}(z)$ which is the number of customers that arrive during a service time and is defined by

$$\mathbf{A}_{vss}(z) = \int_0^\infty e^{\mathbf{D}_{mod}(z)x} dS(x) \quad (14)$$

is still the same in the modified system and we have

$$\mathbf{A}_{vss}(z) = \mathbf{A}_{mod}(z). \quad (15)$$

For the same reason, $\mathbf{K}_{vss}(z)$ which represents the number of customers that are served during a cycle, $\mathbf{G}_{vss}(z)$ which represents the number of customers that are served during a fundamental period, $\mathbf{K}_{vss} = \mathbf{K}_{vss}(1)$, $\mathbf{G}_{vss} = \mathbf{G}_{vss}(1)$, and the stationary vectors $\boldsymbol{\kappa}_{vss}$, $\mathbf{g}_{vss}$ do not change in the modified system:

$$\mathbf{K}_{vss}(z) = \mathbf{K}_{mod}(z), \quad \mathbf{G}_{vss}(z) = \mathbf{G}_{mod}(z), \quad \boldsymbol{\kappa}_{vss} = \boldsymbol{\kappa}_{mod}, \quad \mathbf{g}_{vss} = \mathbf{g}_{mod}. \quad (16)$$

3 The main result: Factorization of the queue length distribution in the VSS vacation systems

Before we prove the factorization principle, it is worthwhile to note that the BMAP/G/1 queues with VSS can not be properly analyzed just by employing the conventional methods due to the variable service speeds. Takine (2005) proposed a methodology that is based on the construction of a modified system, which had been the ideas used earlier in Ramaswami (1999) and Ahn and Ramaswami (2004). But his method requires to set up different modified systems for different VSS systems with vacations. The factorization property that is to be proposed in this paper patches up the inefficiency.

We consider the BMAP/G/1/VSS queueing system that satisfies the following assumption.

Assumption 3.1

- (i) The service requirements of arriving customers are iid and independent of the arrival process, vacation process and UMC phases.

- (ii) Service completions do not change the UMC phases (an assumption of Takine and Takahashi 1998).
- (iii) The system is work-conserving and satisfies the stability condition.
- (iv) The services are service-time independent and are rendered according to FCFS.
- (v) The arrival process and vacation processes are independent.

Remark 3.1 The D -policy queueing system violates assumption-(i). In a D -policy queueing system, the idle server begins to serve the customers only when the total workload of the customers exceeds the threshold D . For example, if D is crossed by the third customer, it is obvious that the service times of the first two customers are less than D and they are conditionally-dependent on each other because one large service time would mean smaller the other. Thus we can not apply the factorization property of this paper to the D -policy systems.

Theorem 3.1 *Let $\mathbf{Y}_{vss}(z)$ and $\mathbf{X}_{vss}(z)$ be the vector GFs of the queue length at an arbitrary time and at a departure in the BMAP/G/1/VSS system with vacations that satisfies Assumption 3.1. Then, we have*

$$\mathbf{Y}_{vss}(z)\mathbf{D}(z) = \lambda(z-1)\mathbf{X}_{vss}(z). \quad (17)$$

Proof Let $\mathbf{Y}_{mod}(z)$ be the vector GF of the queue length at an arbitrary time in the modified system. Since the workload is processed at the speed of 1 in the modified system, we can apply the results of Takine and Takahashi (1998) and we have

$$\mathbf{Y}_{mod}(z)\mathbf{D}_{mod}(z) = \lambda_{mod}(z-1)\mathbf{X}_{mod}(z), \quad (18)$$

where $\mathbf{X}_{mod}(z)$ is the vector GF of the queue length at an arbitrary departure in the modified system. Now, using (18) in (13), we get

$$\mathbf{Y}_{vss}(z) = \gamma \lambda_{mod}(z-1)\mathbf{X}_{mod}(z)[\mathbf{D}_{mod}(z)]^{-1}\mathbf{R}^{-1}. \quad (19)$$

Applying (8, 11, 12) and (15) in (19) yields (17). $\square$

Remark 3.2 Equation (17) shows that the same rule of Takine and Takahashi (1998) applies to the VSS systems.

Theorem 3.2 *For the BMAP/G/1/VSS systems with vacations, we have the following factorization property*

$$\mathbf{Y}_{vss}(z) = \mathbf{p}_{idle}^{vss}(z) \cdot \chi_Y^{vss}(z), \quad (20)$$

where

$$\chi_Y^{vss}(z) = (1 - \rho_{vss})(z-1)\mathbf{R}\mathbf{A}_{vss}(z)[z\mathbf{I} - \mathbf{A}_{vss}(z)]^{-1}\mathbf{R}^{-1}. \quad (21)$$

Proof Since the work is processed at the rate of 1, the modified system can be considered as the simple BMAP/G/1 queue, which means that we can use the result of Lee et al. (2001) and Chang et al. (2002) and we have

$$\mathbf{Y}_{mod}(z) = \mathbf{p}_{idle}^{mod}(z)\chi_Y^{mod}(z), \quad (22)$$

where

$$\chi_Y^{mod}(z) = (1 - \rho_{mod})(z - 1)\mathbf{A}_{mod}(z)[z\mathbf{I} - \mathbf{A}_{mod}(z)]^{-1}. \quad (23)$$

Defining $\mathbf{y}_{idle,k}^{vss}$ and $\mathbf{y}_{idle,k}^{mod}$ as the joint vector probabilities that an arbitrary time belongs to an idle period and there are k customers in the system in the VSS system and in the modified system respectively, we have from (13)

$$\mathbf{y}_{idle,k}^{vss} = \gamma \cdot \mathbf{y}_{idle,k}^{mod} \cdot \mathbf{R}^{-1}, \quad (24)$$

which leads to

$$(1 - \rho_{vss})\mathbf{p}_{idle}^{vss}(z) = \gamma(1 - \rho_{mod})\mathbf{p}_{idle}^{mod}(z)\mathbf{R}^{-1}. \quad (25)$$

Now, we use (22) in (23). Then using (15) and arranging terms with respect to $\mathbf{Y}_{mod}(z)$, we get

$$\mathbf{Y}_{mod}(z) = \mathbf{p}_{idle}^{mod}(z)(1 - \rho_{mod})(z - 1)\mathbf{A}_{vss}(z)[z\mathbf{I} - \mathbf{A}_{vss}(z)]^{-1}. \quad (26)$$

Now using (26) in (25) yields

$$\mathbf{Y}_{mod}(z) = \gamma^{-1}\mathbf{p}_{idle}^{vss}(z)\mathbf{R}(1 - \rho_{vss})(z - 1)\mathbf{A}_{vss}(z)[z\mathbf{I} - \mathbf{A}_{vss}(z)]^{-1}. \quad (27)$$

Postmultiplying both sides of (27) by $\mathbf{R}^{-1}$ and using (13) yields (20) and (21). $\square$

Remark 3.3 ρ_{mod} that was used in (23) and (26) is the probability that the server is busy at an arbitrary point of time in the modified system. From (25), this quantity is related to ρ_{vss} in the following way,

$$\rho_{mod} = 1 - \frac{1}{\gamma}(1 - \rho_{vss})[\mathbf{p}_{idle}^{vss}(z)|_{z=1}]\mathbf{Re}. \quad (28)$$

Since the work is not processed during the idle period, we note that the conditional vector GF $\mathbf{p}_{idle}^{vss}(z)$ is exactly equal to that in the ordinary BMAP/G/1 queue with constant service speed.

4 Moments

In this section, we derive the recursive formula for the moments from (20) and (21). For notational simplicity, let us define, for a matrix (or vector) GF $\mathbf{H}(z)$,

$$\mathbf{H}^{(n)} = \left. \frac{d^n}{dz^n} \mathbf{H}(z) \right|_{z=1}, \quad \mathbf{H}^{(0)} = \mathbf{H} = \mathbf{H}(z)|_{z=1}.$$

Then we have the following theorem.

Theorem 4.1 *Let us define $\mathbf{f}_{vss}(z)$ and $\mathbf{f}_{mod}(z)$ as follows,*

$$\mathbf{f}_{vss}(z) = \mathbf{p}_{idle}^{vss}(z)\mathbf{R}\mathbf{A}_{vss}(z), \quad (29)$$

$$\mathbf{f}_{mod}(z) = \mathbf{p}_{idle}^{mod}(z)\mathbf{A}_{mod}(z). \quad (30)$$

Then, we have the recursive moments formula as follows:

$$\mathbf{Y}^{(n)} \mathbf{e} = \left[(1 - \rho_{vss}) n \mathbf{f}_{vss}^{(n-1)} - n \mathbf{Y}_{vss}^{(n-1)} \mathbf{R} (\mathbf{I} - \mathbf{A}_{vss}^{(1)}) + \sum_{k=2}^n \binom{n}{k} \mathbf{Y}_{vss}^{(n-k)} \mathbf{R} \mathbf{A}_{vss}^{(k)} \right] \times (\mathbf{I} - \mathbf{A}_{vss} + \mathbf{e} \boldsymbol{\pi}_{mod})^{-1} \mathbf{R}^{-1} \mathbf{e} + \mathbf{Y}_{mod}^{(n)} \mathbf{e}, \quad (31)$$

where

$$\mathbf{Y}_{mod}^{(n)} \mathbf{e} = \frac{1}{(n+1)(1 - \rho_{mod})} \left[(n+1) \theta_n + \frac{(n+1)(1 - \rho)}{\gamma} \mathbf{f}_{vss}^{(n)} \mathbf{e} + \frac{1}{\gamma} \sum_{k=2}^{n+1} \binom{n+1}{k} \mathbf{Y}_{vss}^{(n+1-k)} \mathbf{R} \mathbf{A}_{vss}^{(k)} \mathbf{e} \right], \quad (32)$$

in which

$$\theta_n = \left[(1 - \rho_{vss}) \frac{1}{\gamma} n \mathbf{f}_{vss}^{(n-1)} - \frac{n}{\gamma} \mathbf{Y}_{vss}^{(n-1)} \mathbf{R} (\mathbf{I} - \mathbf{A}_{vss}^{(1)}) + \frac{1}{\gamma} \sum_{k=2}^n \binom{n}{k} \mathbf{Y}_{vss}^{(n-k)} \mathbf{R} \mathbf{A}_{vss}^{(k)} \right] \times (\mathbf{I} - \mathbf{A}_{vss} + \mathbf{e} \boldsymbol{\pi}_{mod})^{-1} \mathbf{A}_{vss}^{(1)} \mathbf{e}. \quad (33)$$

Proof Equations (29) and (30) are related by

$$\begin{aligned} (1 - \rho_{vss}) \mathbf{f}_{vss}(z) &= (1 - \rho_{vss}) \mathbf{p}_{idle}^{vss}(z) \mathbf{R} \mathbf{A}_{vss}(z) \\ &= \gamma \cdot \frac{1}{\gamma} (1 - \rho_{vss}) \mathbf{p}_{idle}^{vss}(z) \mathbf{R} \mathbf{A}_{vss}(z) \\ &= \gamma (1 - \rho_{mod}) \mathbf{p}_{idle}^{mod}(z) \mathbf{A}_{vss}(z) \\ &= \gamma (1 - \rho_{mod}) \mathbf{f}_{mod}(z) \end{aligned} \quad (34)$$

in which the third equality came from (25).

Since the modified system is a BMAP system with parameter matrices $\mathbf{R}^{-1}(\mathbf{D}_0, \mathbf{D}_1, \dots)$ with constant service speed, we can use the moments result of Chang et al. (2002) and we get

$$\mathbf{Y}_{mod}^{(n)} = \left[(1 - \rho_{mod}) n \mathbf{f}_{mod}^{(n-1)} - n \mathbf{Y}_{mod}^{(n-1)} (\mathbf{I} - \mathbf{A}_{vss}^{(1)}) + \sum_{k=2}^n \binom{n}{k} \mathbf{Y}_{mod}^{(n-k)} \mathbf{A}_{vss}^{(k)} \right] \times (\mathbf{I} - \mathbf{A}_{vss} + \mathbf{e} \boldsymbol{\pi}_{mod})^{-1} + \mathbf{Y}_{mod}^{(n)} \mathbf{e} \boldsymbol{\pi}_{mod}, \quad (35)$$

where

$$\mathbf{Y}_{mod}^{(n)} \mathbf{e} = \frac{1}{(n+1)(1 - \rho_{mod})} \times \left[(n+1) \theta_n + (n+1)(1 - \rho_{mod}) \mathbf{f}_{mod}^{(n)} \mathbf{e} + \sum_{k=2}^{n+1} \binom{n+1}{k} \mathbf{Y}_{mod}^{(n+1-k)} \mathbf{A}_{vss}^{(k)} \mathbf{e} \right], \quad (36)$$

in which

$$\theta_n = \left[(1 - \rho_{mod}) n \mathbf{f}_{mod}^{(n-1)} - n \mathbf{Y}_{mod}^{(n-1)} (\mathbf{I} - \mathbf{A}_{vss}^{(1)}) + \sum_{k=2}^n \binom{n}{k} \mathbf{Y}_{mod}^{(n-k)} \mathbf{A}_{vss}^{(k)} \right] \times (\mathbf{I} - \mathbf{A}_{vss} + \mathbf{e} \boldsymbol{\pi}_{mod})^{-1} \mathbf{A}_{vss}^{(1)} \mathbf{e}. \quad (37)$$

Also, we have the following relationship from (13),

$$\frac{1}{\gamma} \mathbf{Y}_{vss}^{(n)} \mathbf{R} = \mathbf{Y}_{mod}^{(n)}, \quad (38)$$

which means

$$\mathbf{Y}_{vss}^{(n)} \mathbf{e} = \gamma \mathbf{Y}_{mod}^{(n)} \mathbf{R}^{-1} \mathbf{e}. \quad (39)$$

Rearranging (35–37) by using (34) and (28) yields (31–33). $\square$

5 Application examples

In this section, we consider three BMAP/G/1/VSS systems which have different types of vacations and use (20) to derive the factorized form of the queue length GFs. We note that our factorization formula works for much wider classes of vacation systems with BMAP arrivals and VSS. The three systems we will consider are as follows:

- (i) The system under the N -policy: The idle server begins to serve the customers only when the queue length becomes N .
- (ii) The multiple vacation system: The server takes repeated vacations until there are at least one customer in the system at the end of a vacation.
- (iii) The single vacation system: The server takes a vacation of random length V . At the end of the vacation, if there is one or more customers, the server begins to serve the customers. If not, the server waits in the system until a customer arrives.

(1) *The BMAP/G/1/VSS system under the N -policy* In this system, if we apply (20) to the results of Lee et al. (2001), we get

$$\begin{aligned} \mathbf{Y}_{vss}(z) &= \mathbf{p}_{idle}^{vss}(z) \boldsymbol{\chi}_Y^{vss}(z) \\ &= \frac{\boldsymbol{\kappa} \sum_{n=0}^{N-1} \mathbf{D}_n^* (-\mathbf{D}_0)^{-1} z^n}{\boldsymbol{\kappa} \sum_{n=0}^{N-1} \mathbf{D}_n^* (-\mathbf{D}_0)^{-1} \mathbf{e}} \cdot \mathbf{R} (1 - \rho_{vss})(z - 1) \mathbf{A}_{vss}(z) [z \mathbf{I} - \mathbf{A}_{vss}(z)]^{-1} \mathbf{R}^{-1}, \quad (40) \end{aligned}$$

where $\mathbf{D}_n^*$ which is given by $\mathbf{D}_n^* = \sum_{k=1}^n (-\mathbf{D}_0)^{-1} \mathbf{D}_k \mathbf{D}_{n-k}^*$ ($\mathbf{D}_0^* = \mathbf{I}$) represents the sum of all possible probabilities that lead to the visit of the level (queue length) k during the idle period (note that due to batch arrivals some levels may not be visited during the idle period).

In (40), $\boldsymbol{\kappa}$ is the stationary vector of the probability matrix $\mathbf{K}$ that represents the phase change of the UMC during a cycle. We can obtain $\mathbf{K}$ in the following way. Let $\boldsymbol{\Phi}(z)$ be the matrix GF of the queue length at the start of the busy period. Then, we get $\mathbf{K} = \boldsymbol{\Phi}(z)|_{z=\mathbf{G}_{mod}}$ where $\mathbf{G}_{mod}$ is the phase shift probability matrix during a fundamental period which can be computed from the algorithm of, for example, Lucantoni (1991). According to (16), we may use $\mathbf{G}_{vss}$ in place of $\mathbf{G}_{mod}$.

Once $\mathbf{K}$ is obtained, we can compute $\boldsymbol{\kappa}$ in the following way,

$$\boldsymbol{\kappa} = \boldsymbol{\kappa} \mathbf{K}, \quad \boldsymbol{\kappa} \mathbf{e} = 1. \quad (41)$$

(2) *The BMAP/G/1/VSS system under the multiple vacations* Using (20) to the results of Lee et al. (2001), we get

$$\begin{aligned} \mathbf{Y}_{vss}(z) &= \mathbf{p}_{idle}^{vss}(z) \boldsymbol{\chi}_Y^{vss}(z) \\ &= \frac{\boldsymbol{\kappa}(\mathbf{I} - \mathbf{V}_0)^{-1} [\mathbf{V}(z) - \mathbf{I}] [\mathbf{D}(z)]^{-1}}{E(V) \cdot \boldsymbol{\kappa}(\mathbf{I} - \mathbf{V}_0)^{-1} \mathbf{e}} \\ &\quad \times \mathbf{R}(1 - \rho_{vss})(z - 1) \mathbf{A}_{vss}(z) [z\mathbf{I} - \mathbf{A}_{vss}(z)]^{-1} \mathbf{R}^{-1}, \end{aligned} \quad (42)$$

where $\mathbf{V}(z)$ is the matrix GF of the number of customers that arrive during the idle period which is given by (with $\mathbf{V}_0 = \mathbf{V}(z)|_{z=0}$)

$$\mathbf{V}(z) = \int_0^\infty e^{\mathbf{D}(z)x} dV(x). \quad (43)$$

(3) *The BMAP/G/1/VSS system under the single vacation* Applying (20) to the results of Lee et al. (2001) again, we get

$$\begin{aligned} \mathbf{Y}_{vss}(z) &= \mathbf{p}_{idle}^{vss}(z) \boldsymbol{\chi}_Y^{vss}(z) \\ &= \frac{\boldsymbol{\kappa} \{ \mathbf{V}_0(-\mathbf{D}_0)^{-1} + [\mathbf{V}(z) - \mathbf{I}] \cdot [\mathbf{D}(z)]^{-1} \}}{\boldsymbol{\kappa} \mathbf{V}_0(-\mathbf{D}_0)^{-1} \mathbf{e} + E(V)} \\ &\quad \times \mathbf{R}(1 - \rho_{vss})(z - 1) \mathbf{A}_{vss}(z) [z\mathbf{I} - \mathbf{A}_{vss}(z)]^{-1} \mathbf{R}^{-1}. \end{aligned} \quad (44)$$

6 Research summary

In this paper, we presented a factorization formula that can be applied to a variety of BMAP systems with vacations under variable service speed. Our result is very simple and elegant, and is very easy to apply. For the proof, we constructed a modified system as demonstrated in Takine (2005) and applied the factorization principle of Lee et al. (2001) and Chang et al. (2002) for the generalized BMAP/G/1 queueing system under constant speed.

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