Stable formulation of FADI-FDTD method for multi-term, doubly, second-order dispersive media

Tan, Eng Leong; Heh, Ding Yu

2013

Heh, D. Y., & Tan, E. L. (2013). Stable Formulation of FADI-FDTD Method for Multi-Term, Doubly, Second-Order Dispersive Media. IEEE Transactions on Antennas and Propagation, 61(8), 4167-4175.

<https://hdl.handle.net/10356/99568>

<https://doi.org/10.1109/TAP.2013.2259458>

© 2013 IEEE. Personal use of this material is permitted. Permission from IEEE must be obtained for all other uses, in any current or future media, including reprinting/republishing this material for advertising or promotional purposes, creating new collective works, for resale or redistribution to servers or lists, or reuse of any copyrighted component of this work in other works. The published version is available at: [<http://dx.doi.org/10.1109/TAP.2013.2259458>].

Downloaded on 26 Apr 2023 12:21:28 SGT

Stable Formulation of FADI-FDTD Method for Multi-Term, Doubly, Second-Order Dispersive Media

Ding Yu Heh, *Member, IEEE* and Eng Leong Tan, *Senior Member, IEEE*

Abstract—This paper presents stable formulation of fundamental alternating-direction-implicit finite-difference time-domain (FADI-FDTD) method for multi-term, doubly, second-order dispersive media. The formulation is achieved by decomposing the electric and magnetic susceptibility functions into first-order poles and it is derived based on currents constitutive relations. It is conveniently applicable for most commonly used second-order dispersive models, such as Lorentz and Drude models, and equally applicable to first-order Debye model. The extension for full 3D doubly dispersive media using FADI-FDTD method makes the resultant update equations much more concise and simpler than using conventional ADI-FDTD method. The number of field variables and update coefficients are greatly reduced at the right-hand sides, with only first-order spatial differencing and no dispersive terms and magnetic update coefficients in the implicit E update equations. All these contribute to conciseness and programming simplicity, as well as leading to higher efficiency due to much less memory indexing overhead and fewer arithmetic operations. Most importantly, our present formulation is stable, while the contemporary second-order differential equation formulation has potential instability as to be demonstrated. The stability analysis is performed using Fourier method by examining the eigenvalues of the Fourier amplification matrix numerically.

Index Terms—Doubly dispersive media, FADI-FDTD method, stability.

I. INTRODUCTION

The unconditionally stable alternating-direction-implicit finite-difference time-domain (ADI-FDTD) method [1], [2] has been introduced to overcome the Courant-Friedrich-Lewy (CFL) constraint of explicit FDTD method [3]. Since then, there have been many extensions of the ADI-FDTD method for various dispersive media. In [4], an extension of the ADI-FDTD method to analyze Debye media has been described, which considers single Debye term only. In [5], general formulation of ADI-FDTD method for dispersive media has been presented using the Z-transform method. The formulation is limited to 2D TE only and involves flux (DE) constitutive relation that is cumbersome along with implicit D updates that are inconvenient for boundary condition implementations. In [6], formulation of ADI-FDTD is done for only 2D TE and Drude media based on piecewise linear recursive convolution method. The derivation is laborious with the use of recursive accumulator and it is not conveniently applicable for different

dispersive media. This is because the method involves discrete convolution operations which are cumbersome and not straight forward to derive for different dispersive media. In [7], an ADI-FDTD formulation has been presented for modeling dispersive media without splitting the constitutive relation. The formulation considers 1D Debye media only and it is not certain that such non-splitting formulation may still be applicable to other 3D dispersive media (see potential instability below). In [8], general formulation of ADI-FDTD method is presented for dispersive media using second-order model. The formulation considers only electrically dispersive model and calls for combination of current (JE) and polarization (PE) constitutive relations, which reduces to JE description only for Debye model.

All formulations mentioned so far have been using the conventional ADI-FDTD method whose update equations, when presented for full 3D doubly dispersive media having both electric and magnetic dispersive terms, would turn out to be extremely lengthy and unwieldy. This is because the conventional ADI-FDTD method has right-hand sides (RHS) complicated by matrix operators that would operate on the dispersive terms and get worsen in doubly dispersive case. The FADI-FDTD method, on the other hand, can successfully circumvent all the aforementioned complications by means of matrix-operator-free RHS [9]. Indeed, as to be compared later, the extension for full 3D doubly dispersive media using FADI-FDTD method makes the resultant update equations much more concise and simpler than using conventional ADI-FDTD method. Some previous formulations even invoke combination of both JE and PE descriptions [8], [10], perhaps due to synchronization of three time-levels not conveniently achievable based on two time-stencils in ADI-FDTD. Moreover, the potential instability of second-order dispersive models such as Lorentz media for explicit FDTD method at courant stability limit has been described in [11], [12]. Hence, there is no guarantee that implementing second-order dispersive models in ADI-FDTD method will result in unconditional stability. As we shall demonstrate later, improper formulation of second-order dispersive models may lead to potential instability for certain dispersive parameters.

Our previous works have formulated the FADI-FDTD method for biomedical applications [13], [14]. However, the application examples therein are only meant for electrically dispersive media, as most biomedical related materials are nonmagnetic. In addition, the formulations therein have been limited to dispersive models with first-order pole only (e.g. De-

Manuscript received

The authors are with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798.
(e-mail: dyheh@ntu.edu.sg; eeltan@ntu.edu.sg)

bye) and exclude those with second-order pole (e.g. Lorentz). It is also not clear on how to properly implement those dispersive models with second-order pole in two time-stencils of ADI-FDTD method. The authors of [10] used the second-order differential equation formulation which may result in potential instability. The second-order differential equation formulation is also commonly used in the explicit FDTD, c.f., eq. (9.47) in [3], eq. (7) in [15] and eq. (8) in [16]. For the explicit FDTD method, this may result in more stringent CFL condition for the Lorentz media compared to lossless media. This does not pose serious limitation because the explicit FDTD method is not unconditionally stable anyway. However, since the ADI-FDTD method aims to achieve unconditional stability, the implementation of wrong time update for second-order equation is not tolerable.

In this paper, we shall present stable formulation of FADI-FDTD method for multi-term, doubly, second-order dispersive media. The formulation is achieved by decomposing the electric and magnetic susceptibility functions into first-order poles. In Section II, the formulation of FADI-FDTD method for doubly dispersive media based on electric and magnetic currents (*JE-MH*) constitutive relations will be shown. The formulation is conveniently applicable for most commonly used second-order dispersive models, such as Lorentz and Drude models, and equally applicable to first-order Debye model. Since all second-order poles are decomposed into first-order poles, it only calls for two time-levels and complications would not arise in synchronizing with two time-stencils of ADI-FDTD and FADI-FDTD methods. The extension for full 3D doubly dispersive media using FADI-FDTD method makes the resultant update equations much more concise and simpler than using conventional ADI-FDTD method. Thorough discussions on the advantages and simplicity of update equations for the FADI-FDTD method will be provided in Section III along with some numerical results. Most importantly, our present formulation is stable, while the contemporary second-order differential equation formulation has potential instability as to be demonstrated. For stability analysis, the Fourier method will be utilized to examine the eigenvalues of the Fourier amplification matrix numerically. In the present scope, we shall exclude the consideration of other unconditionally stable methods, such as Split-Step FDTD (SS-FDTD), Locally-One-Dimensional FDTD (LOD-FDTD), Crank-Nicolson Direct-Splitting FDTD (CNDS-FDTD), etc.

II. FORMULATION BASED ON ELECTRIC AND MAGNETIC CURRENTS (*JE-MH*) CONSTITUTIVE RELATIONS

For a multi-term doubly dispersive medium, the complex relative permittivity and permeability are usually described as

$$\tilde{\epsilon}(\omega) = \epsilon_\infty + \sum_p \chi_{e,p}(\omega) \quad (1a)$$

$$\tilde{\mu}(\omega) = \mu_\infty + \sum_p \chi_{h,p}(\omega) \quad (1b)$$

where ϵ_∞ and μ_∞ are relative permittivity and permeability at infinite frequencies, respectively, and $\chi_{e,p}$ and $\chi_{h,p}$ are the p -th term electric and magnetic susceptibilities, respectively. The

formulation can be done by means of electric and magnetic current densities, $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{M}}$, respectively. Each susceptibility term will relate $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{M}}$ to their respective electric and magnetic fields as

$$\tilde{\mathbf{J}}_p(\omega) = j\omega\epsilon_0\chi_{e,p}(\omega)\tilde{\mathbf{E}}(\omega) \quad (2a)$$

$$\tilde{\mathbf{M}}_p(\omega) = j\omega\mu_0\chi_{h,p}(\omega)\tilde{\mathbf{H}}(\omega) \quad (2b)$$

where ϵ_0 and μ_0 are permittivity and permeability at free space, respectively.

The current densities are then incorporated into the time-domain Maxwell's curl equations as follows:

$$\nabla \times \mathbf{H}(t) = \epsilon_0\epsilon_\infty \frac{\partial}{\partial t} \mathbf{E}(t) + \sigma \mathbf{E}(t) + \sum_p \mathbf{J}_p(t) \quad (3a)$$

$$-\nabla \times \mathbf{E}(t) = \mu_0\mu_\infty \frac{\partial}{\partial t} \mathbf{H}(t) + \sigma^* \mathbf{H}(t) + \sum_p \mathbf{M}_p(t) \quad (3b)$$

where σ and σ^* are electric and magnetic conductivities, respectively. We first derive the update equations for conventional ADI-FDTD method. If the electric and magnetic susceptibility functions can be decomposed such that the p -th pole is only first-order, $\mathbf{J}$ and $\mathbf{M}$ can be updated in the following manner:

$$\mathbf{J}_p^{n+\frac{1}{2}} = c_{4,p}\mathbf{J}_p^n + c_{5,p}\mathbf{E}^{n+\frac{1}{2}} + c_{6,p}\mathbf{E}^n \quad (4a)$$

$$\mathbf{M}_p^{n+\frac{1}{2}} = d_{4,p}\mathbf{M}_p^n + d_{5,p}\mathbf{H}^{n+\frac{1}{2}} + d_{6,p}\mathbf{H}^n \quad (4b)$$

We shall let the coefficients c and d be general for now. These coefficients are dependent on different types of doubly dispersive media and shall be derived later based on the Auxiliary Differential Equation (ADE) method. Using semi-implicit scheme, we obtain

$$\frac{1}{2} (\mathbf{J}_p^n + \mathbf{J}_p^{n+\frac{1}{2}}) = \frac{1}{2} [(1 + c_{4,p}) \mathbf{J}_p^n + c_{5,p}\mathbf{E}^{n+\frac{1}{2}} + c_{6,p}\mathbf{E}^n] \quad (5a)$$

$$\frac{1}{2} (\mathbf{M}_p^n + \mathbf{M}_p^{n+\frac{1}{2}}) = \frac{1}{2} [(1 + d_{4,p}) \mathbf{M}_p^n + d_{5,p}\mathbf{H}^{n+\frac{1}{2}} + d_{6,p}\mathbf{H}^n] \quad (5b)$$

The semi-implicit scheme is employed to avoid any potential instability caused by the backward (explicit) scheme. Upon some manipulations, the update equations of conventional ADI-FDTD method for doubly dispersive media for the first substep (n to $n + \frac{1}{2}$) read:

$$E_x^{n+\frac{1}{2}} - c_2 \frac{\partial}{\partial y} H_z^{n+\frac{1}{2}} = c_1 E_x^n - c_2 \frac{\partial}{\partial z} H_y^n - \sum_p c_{3,p} J_{x,p}^n \quad (6a)$$

$$H_z^{n+\frac{1}{2}} - d_2 \frac{\partial}{\partial y} E_x^{n+\frac{1}{2}} = d_1 H_z^n - d_2 \frac{\partial}{\partial x} E_y^n - \sum_p d_{3,p} M_{z,p}^n \quad (6b)$$

$$J_{\xi,p}^{n+\frac{1}{2}} = c_{4,p} J_{\xi,p}^n + c_{5,p} E_\xi^{n+\frac{1}{2}} + c_{6,p} E_\xi^n \quad \forall p, \quad \xi = x, y, z \quad (6c)$$

$$M_{\xi,p}^{n+\frac{1}{2}} = d_{4,p} M_{\xi,p}^n + d_{5,p} H_\xi^{n+\frac{1}{2}} + d_{6,p} H_\xi^n \quad \forall p, \quad \xi = x, y, z \quad (6d)$$

where

$$c_1 = \frac{1 - \frac{\sigma\Delta t}{4\epsilon_0\epsilon_\infty} - \frac{\Delta t}{4\epsilon_0\epsilon_\infty} \sum_p c_{6,p}}{1 + \frac{\sigma\Delta t}{4\epsilon_0\epsilon_\infty} + \frac{\Delta t}{4\epsilon_0\epsilon_\infty} \sum_p c_{5,p}}, \quad (7a)$$

$$d_1 = \frac{1 - \frac{\sigma^*\Delta t}{4\mu_0\mu_\infty} - \frac{\Delta t}{4\mu_0\mu_\infty} \sum_p d_{6,p}}{1 + \frac{\sigma^*\Delta t}{4\mu_0\mu_\infty} + \frac{\Delta t}{4\mu_0\mu_\infty} \sum_p d_{5,p}}, \quad (7b)$$

$$c_2 = \frac{\frac{\Delta t}{2\epsilon_0\epsilon_\infty}}{1 + \frac{\sigma\Delta t}{4\epsilon_0\epsilon_\infty} + \frac{\Delta t}{4\epsilon_0\epsilon_\infty} \sum_p c_{5,p}}, \quad (7c)$$

$$d_2 = \frac{\frac{\Delta t}{2\mu_0\mu_\infty}}{1 + \frac{\sigma^*\Delta t}{4\mu_0\mu_\infty} + \frac{\Delta t}{4\mu_0\mu_\infty} \sum_p d_{5,p}}, \quad (7d)$$

$$c_{3,p} = \frac{1}{2}c_2(1 + c_{4,p}), \quad d_{3,p} = \frac{1}{2}d_2(1 + d_{4,p}). \quad (7e)$$

Other field components can be systematically written down. Second substep can be obtained by incrementing all time indices by $\frac{1}{2}$ and interchanging the direction of spatial operators and the operated field components. For instance, the terms $\frac{\partial}{\partial y} H_z^{n+\frac{1}{2}}$ and $\frac{\partial}{\partial z} H_y^n$ in (6a) are replaced with $\frac{\partial}{\partial z} H_y^{n+1}$ and $\frac{\partial}{\partial y} H_z^{n+\frac{1}{2}}$, respectively. To solve for E_x in the first substep, we need to substitute (6b) into (6a) to obtain the implicit E_x update equation as

$$\begin{aligned} & E_x^{n+\frac{1}{2}} - c_2 \frac{\partial}{\partial y} (d_2 \frac{\partial}{\partial y} E_x^{n+\frac{1}{2}}) \\ &= c_1 E_x^n - c_2 \frac{\partial}{\partial y} (d_2 \frac{\partial}{\partial x} E_y^n) + c_2 \frac{\partial}{\partial y} (d_1 H_z^n) - c_2 \frac{\partial}{\partial z} H_y^n \\ & - \sum_p c_{3,p} J_{x,p}^n - c_2 \frac{\partial}{\partial y} \sum_p d_{3,p} M_{z,p}^n. \end{aligned} \quad (8)$$

When the spatial differential operators are approximated using the second-order central differencing operated on a standard Yee's cell [3], the electric field components can be obtained by solving a set of tridiagonal matrices. The implementation of the implicit E update equation shall be discussed in more detail in Section III. After the E fields are solved, H , J and M fields can be obtained explicitly. The procedures are then repeated for the second substep.

From (8), the implicit E update equation involves many spatial operators (some up to second-order derivatives) and there exist many field variables and update coefficients. Furthermore, the spatial operators are even applied on certain dispersive terms ($M_{z,p}$). These are caused by the presence of spatial operators on the RHS of the conventional ADI-FDTD method update equations (6).

To eliminate these spatial operators, we next proceed to formulate the FADI-FDTD method. The conventional ADI-FDTD update equations are first cast into compact matrix form as

First procedure:

$$(\mathbf{I} - \mathbf{D}_2 \mathbf{A}) \mathbf{u}^{n+\frac{1}{2}} = (\mathbf{D}_1 + \mathbf{D}_2 \mathbf{B}) \mathbf{u}^n - \sum_p \mathbf{D}_{3,p} \mathbf{f}_p^n \quad (9a)$$

$$\mathbf{f}_p^{n+\frac{1}{2}} - \mathbf{D}_{5,p} \mathbf{u}^{n+\frac{1}{2}} = \mathbf{D}_{4,p} \mathbf{f}_p^n + \mathbf{D}_{6,p} \mathbf{u}^n \quad (9b)$$

Second procedure:

$$(\mathbf{I} - \mathbf{D}_2 \mathbf{B}) \mathbf{u}^{n+1} = (\mathbf{D}_1 + \mathbf{D}_2 \mathbf{A}) \mathbf{u}^{n+\frac{1}{2}} - \sum_p \mathbf{D}_{3,p} \mathbf{f}_p^{n+\frac{1}{2}} \quad (9c)$$

$$\mathbf{f}_p^{n+1} - \mathbf{D}_{5,p} \mathbf{u}^{n+1} = \mathbf{D}_{4,p} \mathbf{f}_p^{n+\frac{1}{2}} + \mathbf{D}_{6,p} \mathbf{u}^{n+\frac{1}{2}} \quad (9d)$$

where

$$\mathbf{u} = [E_x, E_y, E_z, H_x, H_y, H_z]^T \quad (10a)$$

$$\mathbf{f}_p = [J_{x,p}, J_{y,p}, J_{z,p}, M_{x,p}, M_{y,p}, M_{z,p}]^T \quad (10b)$$

$$\mathbf{D}_1 = \text{diag}(c_1, c_1, c_1, d_1, d_1, d_1) \quad (10c)$$

$$\mathbf{D}_2 = \text{diag}(c_2, c_2, c_2, d_2, d_2, d_2) \quad (10d)$$

$$\mathbf{D}_{3,p} = \text{diag}(c_{3,p}, c_{3,p}, c_{3,p}, d_{3,p}, d_{3,p}, d_{3,p}) \quad (10e)$$

$$\mathbf{D}_{4,p} = \text{diag}(c_{4,p}, c_{4,p}, c_{4,p}, d_{4,p}, d_{4,p}, d_{4,p}) \quad (10f)$$

$$\mathbf{D}_{5,p} = \text{diag}(c_{5,p}, c_{5,p}, c_{5,p}, d_{5,p}, d_{5,p}, d_{5,p}) \quad (10g)$$

$$\mathbf{D}_{6,p} = \text{diag}(c_{6,p}, c_{6,p}, c_{6,p}, d_{6,p}, d_{6,p}, d_{6,p}) \quad (10h)$$

$\mathbf{A}$ and $\mathbf{B}$ are matrix operators which consist of spatial operators. We can see now that the reason why there exist many spatial operators in the preceeding update equations is mainly due to the matrix operators $\mathbf{A}$ and $\mathbf{B}$ on the RHS. It can be shown that we can reduce (9) into fundamental form with the following update procedures:

First procedure:

$$\hat{\mathbf{v}}^n = \mathbf{D}'_1 \mathbf{u}^n - \hat{\mathbf{v}}^{n-\frac{1}{2}} - \sum_p \mathbf{D}_{3,p} \hat{\mathbf{g}}_p^{n-\frac{1}{2}} \quad (11a)$$

$$\hat{\mathbf{g}}_p^n = \mathbf{D}_{4,p} \hat{\mathbf{g}}_p^{n-\frac{1}{2}} + \mathbf{D}'_{5,p} \mathbf{u}^n \quad (11b)$$

$$(\mathbf{I} - \mathbf{D}_2 \mathbf{A}) \mathbf{u}^{n+\frac{1}{2}} = \hat{\mathbf{v}}^n \quad (11c)$$

Second procedure:

$$\hat{\mathbf{v}}^{n+\frac{1}{2}} = \mathbf{D}'_1 \mathbf{u}^{n+\frac{1}{2}} - \hat{\mathbf{v}}^n - \sum_p \mathbf{D}_{3,p} \hat{\mathbf{g}}_p^n \quad (11d)$$

$$\hat{\mathbf{g}}_p^{n+\frac{1}{2}} = \mathbf{D}_{4,p} \hat{\mathbf{g}}_p^n + \mathbf{D}'_{5,p} \mathbf{u}^{n+\frac{1}{2}} \quad (11e)$$

$$(\mathbf{I} - \mathbf{D}_2 \mathbf{B}) \mathbf{u}^{n+1} = \hat{\mathbf{v}}^{n+\frac{1}{2}} \quad (11f)$$

where

$$\mathbf{D}'_1 = \mathbf{I} + \mathbf{D}_1 - \sum_p \mathbf{D}_{3,p} \mathbf{D}_{5,p}, \quad \mathbf{D}'_{5,p} = (\mathbf{D}_{4,p} \mathbf{D}_{5,p} + \mathbf{D}_{6,p}). \quad (12)$$

Note that $\mathbf{f}_p$ is no longer needed and is replaced by auxiliary $\hat{\mathbf{g}}_p$.

Defining the auxiliary fields as

$$\hat{\mathbf{v}} = [\hat{e}_x, \hat{e}_y, \hat{e}_z, \hat{h}_x, \hat{h}_y, \hat{h}_z]^T \quad (13a)$$

$$\hat{\mathbf{g}}_p = [\hat{j}_{x,p}, \hat{j}_{y,p}, \hat{j}_{z,p}, \hat{m}_{x,p}, \hat{m}_{y,p}, \hat{m}_{z,p}]^T, \quad (13b)$$

the update equations of FADI-FDTD method for doubly dispersive media for the first procedure read:

$$\hat{e}_\xi^n = c'_1 E_\xi^n - \hat{e}_\xi^{n-\frac{1}{2}} - \sum_p c_{3,p} \hat{j}_{\xi,p}^{n-\frac{1}{2}}, \quad \xi = x, y, z \quad (14a)$$

$$\hat{h}_\xi^n = d'_1 H_\xi^n - \hat{h}_\xi^{n-\frac{1}{2}} - \sum_p d_{3,p} \hat{m}_{\xi,p}^{n-\frac{1}{2}}, \quad \xi = x, y, z \quad (14b)$$

$$\hat{j}_{\xi,p}^n = c_{4,p} \hat{j}_{\xi,p}^{n-\frac{1}{2}} + c'_{5,p} E_\xi^n, \quad \forall p, \quad \xi = x, y, z \quad (14c)$$

$$\hat{m}_{\xi,p}^n = d_{4,p} \hat{m}_{\xi,p}^{n-\frac{1}{2}} + d'_{5,p} H_\xi^n, \quad \forall p, \quad \xi = x, y, z \quad (14d)$$

$$E_x^{n+\frac{1}{2}} - c_2 \frac{\partial}{\partial y} H_z^{n+\frac{1}{2}} = \hat{e}_x^n \quad (14e)$$

$$H_z^{n+\frac{1}{2}} - d_2 \frac{\partial}{\partial y} E_x^{n+\frac{1}{2}} = \hat{h}_z^n \quad (14f)$$

where

$$c'_1 = 1 + c_1 - \sum_p c_{3,p} c_{5,p}, \quad c'_{5,p} = c_{4,p} c_{5,p} + c_{6,p}$$

$$d'_1 = 1 + d_1 - \sum_p d_{3,p} d_{5,p}, \quad d'_{5,p} = d_{4,p} d_{5,p} + d_{6,p} \quad (15)$$

Other field components can be systematically written down. Second substep can be similarly obtained by incrementing all time indices by $\frac{1}{2}$ and interchanging the direction of spatial operators and the operated field components.

The implicit E_x update equation is obtained by substituting (14f) into (14e) as follows:

$$E_x^{n+\frac{1}{2}} - c_2 \frac{\partial}{\partial y} (d_2 \frac{\partial}{\partial y} E_x^{n+\frac{1}{2}}) = \hat{e}_x^n + c_2 \frac{\partial}{\partial y} \hat{h}_z^n \quad (16)$$

More detailed comparisons shall be discussed in Section III. The above formulation of FADI-FDTD can then be applied for various doubly dispersive media shown below.

A. Lorentz Media

For Lorentz media, the frequency domain electric and magnetic susceptibility functions of the p -th pole are given as

$$\chi_{e,p} = \frac{\Delta\epsilon_p \omega_{e,p}^2}{\omega_{e,p}^2 + 2j\omega\delta_{e,p} - \omega^2}, \quad \chi_{h,p} = \frac{\Delta\mu_p \omega_{h,p}^2}{\omega_{h,p}^2 + 2j\omega\delta_{h,p} - \omega^2} \quad (17)$$

where $\Delta\epsilon_p$ and $\Delta\mu_p$ are change in relative permittivity and permeability respectively due to the p -th Lorentz pole, $\omega_{e,p}$ and $\omega_{h,p}$ are electric and magnetic resonant frequencies of the p -th pole respectively, $\delta_{e,p}$ and $\delta_{h,p}$ are electric and magnetic damping coefficients of the p -th pole respectively.

Contemporary formulation for Lorentz media is making use of second-order differential equation formulation. Such formulation is potentially susceptible to instability problem for large resonant frequency (as will be demonstrated later). Furthermore, the second-order differential equation formulation is difficult to synchronize with the Maxwell's curl equation in ADI-FDTD due to the presence of three-time levels not conveniently achievable based on two time stencils in ADI-FDTD method. We now propose a stable formulation involving

partial fraction expansion of second-order Lorentz model as follows:

$$\begin{aligned} & \frac{\Delta\epsilon_p \omega_{e,p}^2}{\omega_{e,p}^2 + 2j\omega\delta_{e,p} - \omega^2} \\ &= \frac{j\gamma_{e,p}/2}{j\omega + \delta_{e,p} + j\beta_{e,p}} + \frac{-j\gamma_{e,p}/2}{j\omega + \delta_{e,p} - j\beta_{e,p}} \end{aligned} \quad (18a)$$

$$\begin{aligned} & \frac{\Delta\mu_p \omega_{h,p}^2}{\omega_{h,p}^2 + 2j\omega\delta_{h,p} - \omega^2} \\ &= \frac{j\gamma_{h,p}/2}{j\omega + \delta_{h,p} + j\beta_{h,p}} + \frac{-j\gamma_{h,p}/2}{j\omega + \delta_{h,p} - j\beta_{h,p}} \end{aligned} \quad (18b)$$

where

$$\beta_{e,p} = \sqrt{\omega_{e,p}^2 - \delta_{e,p}^2}; \quad \gamma_{e,p} = \frac{\Delta\epsilon_p \omega_{e,p}^2}{\beta_{e,p}}; \quad (19a)$$

$$\beta_{h,p} = \sqrt{\omega_{h,p}^2 - \delta_{h,p}^2}; \quad \gamma_{h,p} = \frac{\Delta\mu_p \omega_{h,p}^2}{\beta_{h,p}}. \quad (19b)$$

The splitting is based on complex-conjugate pole-residue pair model in [17] which has been formulated for explicit FDTD method. The Lorentz medium here has only purely imaginary residue pairs. Since it is complex-conjugate pole-residue pair, we only need to update one first-order term. From (2), we have

$$\tilde{\mathbf{J}}_p(\omega) = \frac{j\epsilon_0 \gamma_{e,p}/2}{j\omega + \delta_{e,p} + j\beta_{e,p}} j\omega \tilde{\mathbf{E}}(\omega) \quad (20a)$$

$$\tilde{\mathbf{M}}_p(\omega) = \frac{j\mu_0 \gamma_{h,p}/2}{j\omega + \delta_{h,p} + j\beta_{h,p}} j\omega \tilde{\mathbf{H}}(\omega) \quad (20b)$$

(20) can be transformed into time domain as

$$\frac{\partial}{\partial t} \mathbf{J}_p(t) + (\delta_{e,p} + j\beta_{e,p}) \mathbf{J}_p(t) = j \frac{\epsilon_0 \gamma_{e,p}}{2} \frac{\partial}{\partial t} \mathbf{E}(t) \quad (21a)$$

$$\frac{\partial}{\partial t} \mathbf{M}_p(t) + (\delta_{h,p} + j\beta_{h,p}) \mathbf{M}_p(t) = j \frac{\mu_0 \gamma_{h,p}}{2} \frac{\partial}{\partial t} \mathbf{H}(t) \quad (21b)$$

Discretizing (21) using central difference approximation, the update coefficients in (4) are found to be

$$c_{4,p} = \frac{1 - (\delta_{e,p} + j\beta_{e,p}) \frac{\Delta t}{4}}{1 + (\delta_{e,p} + j\beta_{e,p}) \frac{\Delta t}{4}} = c_{4\text{Re},p} + jc_{4\text{Im},p}, \quad (22a)$$

$$d_{4,p} = \frac{1 - (\delta_{h,p} + j\beta_{h,p}) \frac{\Delta t}{4}}{1 + (\delta_{h,p} + j\beta_{h,p}) \frac{\Delta t}{4}} = d_{4\text{Re},p} + jd_{4\text{Im},p}, \quad (22b)$$

$$c_{5,p} = \frac{j\epsilon_0 \gamma_{e,p}/2}{1 + (\delta_{e,p} + j\beta_{e,p}) \frac{\Delta t}{4}} = c_{5\text{Re},p} + jc_{5\text{Im},p}, \quad (22c)$$

$$d_{5,p} = \frac{j\mu_0 \gamma_{h,p}/2}{1 + (\delta_{h,p} + j\beta_{h,p}) \frac{\Delta t}{4}} = d_{5\text{Re},p} + jd_{5\text{Im},p}, \quad (22d)$$

$$c_{6,p} = -c_{5,p}, \quad d_{6,p} = -d_{5,p}. \quad (22e)$$

Noting that $\mathbf{J} + \mathbf{J}^* = 2 \text{Re } \mathbf{J}$ and $\mathbf{M} + \mathbf{M}^* = 2 \text{Re } \mathbf{M}$, we shall replace $\sum_p c_{3,p} \hat{j}_{\xi,p}$ in (14a) and $\sum_p d_{3,p} \hat{m}_{\xi,p}$ in (14b) with $2 \text{Re } \sum_p c_{3,p} \hat{j}_{\xi,p}$ and $2 \text{Re } \sum_p d_{3,p} \hat{m}_{\xi,p}$, respectively. The update coefficients in (7) and (15) should also be modified as

$$c_1 = \frac{1 - \frac{\sigma \Delta t}{4\epsilon_0 \epsilon_\infty} - \frac{\Delta t}{2\epsilon_0 \epsilon_\infty} \operatorname{Re} \sum_p c_{6,p}}{1 + \frac{\sigma \Delta t}{4\epsilon_0 \epsilon_\infty} + \frac{\Delta t}{2\epsilon_0 \epsilon_\infty} \operatorname{Re} \sum_p c_{5,p}}, \quad (23a)$$

$$d_1 = \frac{1 - \frac{\sigma^* \Delta t}{4\mu_0 \mu_\infty} - \frac{\Delta t}{2\mu_0 \mu_\infty} \operatorname{Re} \sum_p d_{6,p}}{1 + \frac{\sigma^* \Delta t}{4\mu_0 \mu_\infty} + \frac{\Delta t}{2\mu_0 \mu_\infty} \operatorname{Re} \sum_p d_{5,p}}, \quad (23b)$$

$$c_2 = \frac{\frac{\Delta t}{2\epsilon_0 \epsilon_\infty}}{1 + \frac{\sigma \Delta t}{4\epsilon_0 \epsilon_\infty} + \frac{\Delta t}{2\epsilon_0 \epsilon_\infty} \operatorname{Re} \sum_p c_{5,p}}, \quad (23c)$$

$$d_2 = \frac{\frac{\Delta t}{2\mu_0 \mu_\infty}}{1 + \frac{\sigma^* \Delta t}{4\mu_0 \mu_\infty} + \frac{\Delta t}{2\mu_0 \mu_\infty} \operatorname{Re} \sum_p d_{5,p}}, \quad (23d)$$

$$c'_1 = 1 + c_1 - 2 \operatorname{Re} \sum_p c_{3,p} c_{5,p} \quad (23e)$$

$$d'_1 = 1 + d_1 - 2 \operatorname{Re} \sum_p d_{3,p} d_{5,p} \quad (23f)$$

For real number realizations, the update equations of complex $\hat{j}_\xi = \hat{j}_{\xi \operatorname{Re},p} + j \hat{j}_{\xi \operatorname{Im},p}$ in (14c) can be further decomposed into two separate ones involving only real number operations. The final JE update equations are given as

$$\hat{j}_{\xi \operatorname{Re},p}^n = c_{4 \operatorname{Re},p} \cdot \hat{j}_{\xi \operatorname{Re},p}^{n-\frac{1}{2}} - c_{4 \operatorname{Im},p} \cdot \hat{j}_{\xi \operatorname{Im},p}^{n-\frac{1}{2}} + c'_{5 \operatorname{Re},p} \cdot E_\xi^n \quad (24a)$$

$$\hat{j}_{\xi \operatorname{Im},p}^n = c_{4 \operatorname{Im},p} \cdot \hat{j}_{\xi \operatorname{Re},p}^{n-\frac{1}{2}} + c_{4 \operatorname{Re},p} \cdot \hat{j}_{\xi \operatorname{Im},p}^{n-\frac{1}{2}} + c'_{5 \operatorname{Im},p} \cdot E_\xi^n \quad (24b)$$

Similar decomposition of $\hat{m}_{\xi,p}$ can be done for MH update equations in (14d).

B. Drude

For Drude model, the frequency domain electric and magnetic susceptibility functions of the p -th pole are given as

$$\chi_{e,p} = \frac{-\omega_{e,p}^2}{\omega^2 - j\omega\nu_{e,p}}, \quad \chi_{h,p} = \frac{-\omega_{h,p}^2}{\omega^2 - j\omega\nu_{h,p}} \quad (25)$$

where $\omega_{e,p}$ and $\omega_{h,p}$ are the p -th Drude pole electric and magnetic plasma frequencies respectively. $\nu_{e,p}$ and $\nu_{h,p}$ are the p -th pole collision frequencies respectively.

Although the poles for Drude media are of second-order, it will lead to cancellation of one pole by one zero at the origin, resulting in only first-order pole. From (2), we have

$$\tilde{\mathbf{J}}_p(\omega) = \frac{\epsilon_0 \omega_{e,p}^2}{j\omega + \nu_{e,p}} \tilde{\mathbf{E}}(\omega) \quad (26a)$$

$$\tilde{\mathbf{M}}_p(\omega) = \frac{\epsilon_0 \omega_{h,p}^2}{j\omega + \nu_{h,p}} \tilde{\mathbf{H}}(\omega) \quad (26b)$$

(26) can be transformed into time domain as

$$\frac{\partial}{\partial t} \mathbf{J}_p(t) + \nu_{e,p} \mathbf{J}_p(t) = \epsilon_0 \omega_{e,p}^2 \mathbf{E}(t) \quad (27a)$$

$$\frac{\partial}{\partial t} \mathbf{M}_p(t) + \nu_{h,p} \mathbf{M}_p(t) = \mu_0 \omega_{h,p}^2 \mathbf{H}(t) \quad (27b)$$

Discretizing (27) using central difference approximation, the update coefficients in (4) are found to be

$$c_{4,p} = \frac{1 - \nu_{e,p} \frac{\Delta t}{4}}{1 + \nu_{e,p} \frac{\Delta t}{4}}, \quad c_{5,p} = \frac{\epsilon_0 \omega_{e,p}^2 \frac{\Delta t}{4}}{1 + \nu_{e,p} \frac{\Delta t}{4}}, \quad c_{6,p} = c_{5,p},$$

$$d_{4,p} = \frac{1 - \nu_{h,p} \frac{\Delta t}{4}}{1 + \nu_{h,p} \frac{\Delta t}{4}}, \quad d_{5,p} = \frac{\mu_0 \omega_{h,p}^2 \frac{\Delta t}{4}}{1 + \nu_{h,p} \frac{\Delta t}{4}}, \quad d_{6,p} = d_{5,p} \quad (28)$$

All other update coefficients remain unchanged as (7) and (15).

C. Debye

For completeness, we also include the formulation for first-order Debye model. For Debye model, the frequency domain electric and magnetic susceptibility functions of the p -th pole are given as

$$\chi_{e,p} = \frac{\Delta \epsilon_p}{1 + j\omega \tau_{e,p}}, \quad \chi_{h,p} = \frac{\Delta \mu_p}{1 + j\omega \tau_{h,p}} \quad (29)$$

where $\Delta \epsilon_p$ and $\Delta \mu_p$ are change in relative permittivity and permeability respectively due to the p -th Debye pole, $\tau_{e,p}$ and $\tau_{h,p}$ are the p -th pole electric and magnetic relaxation times respectively.

Since Debye model is of first-order pole, no decomposition is required. From (2), we have

$$\tilde{\mathbf{J}}_p(\omega) = \frac{j\omega \epsilon_0 \Delta \epsilon_p}{1 + j\omega \tau_{e,p}} \tilde{\mathbf{E}}(\omega) \quad (30a)$$

$$\tilde{\mathbf{M}}_p(\omega) = \frac{j\omega \mu_0 \Delta \mu_p}{1 + j\omega \tau_{h,p}} \tilde{\mathbf{H}}(\omega) \quad (30b)$$

(30) can be transformed into time domain as

$$\tau_{e,p} \frac{\partial}{\partial t} \mathbf{J}_p(t) + \mathbf{J}_p(t) = \epsilon_0 \Delta \epsilon_p \frac{\partial}{\partial t} \mathbf{E}(t) \quad (31a)$$

$$\tau_{h,p} \frac{\partial}{\partial t} \mathbf{M}_p(t) + \mathbf{M}_p(t) = \mu_0 \Delta \mu_p \frac{\partial}{\partial t} \mathbf{H}(t) \quad (31b)$$

Discretizing (31) using central difference approximation, the update coefficients in (4) are found to be

$$c_{4,p} = \frac{1 - \frac{\Delta t}{4\tau_{e,p}}}{1 + \frac{\Delta t}{4\tau_{e,p}}}, \quad c_{5,p} = \frac{\epsilon_0 \Delta \epsilon_p / \tau_{e,p}}{1 + \frac{\Delta t}{4\tau_{e,p}}}, \quad c_{6,p} = -c_{5,p},$$

$$d_{4,p} = \frac{1 - \frac{\Delta t}{4\tau_{h,p}}}{1 + \frac{\Delta t}{4\tau_{h,p}}}, \quad d_{5,p} = \frac{\mu_0 \Delta \mu_p / \tau_{h,p}}{1 + \frac{\Delta t}{4\tau_{h,p}}}, \quad d_{6,p} = -d_{5,p} \quad (32)$$

All other update coefficients remain unchanged as (7) and (15).

III. DISCUSSIONS AND NUMERICAL RESULTS

For subsequent discussions, let us expand fully (8) by applying central difference approximation to all spatial derivatives where subscript indices are included for the field variables. Also, we include the general inhomogeneity in the medium, which is indicated by update coefficients with subscript indices (for different materials at different locations). Here, the subscript indices of update coefficients c and d shall follow those of $E_x|_{i+\frac{1}{2},j,k}$ and $H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}$ respectively. The implicit E_x update equation of conventional ADI-FDTD (8) reads:

$$\begin{aligned} & \frac{-c_2|_{i+\frac{1}{2},j,k} d_2|_{i+\frac{1}{2},j-\frac{1}{2},k}}{\Delta y^2} E_x|_{i+\frac{1}{2},j-1,k}^{n+\frac{1}{2}} \\ & + \left(1 + \frac{c_2|_{i+\frac{1}{2},j,k} d_2|_{i+\frac{1}{2},j+\frac{1}{2},k}}{\Delta y^2} \right. \\ & \quad \left. + \frac{c_2|_{i+\frac{1}{2},j,k} d_2|_{i+\frac{1}{2},j-\frac{1}{2},k}}{\Delta y^2} \right) E_x|_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}} \\ & - \frac{c_2|_{i+\frac{1}{2},j,k} d_2|_{i+\frac{1}{2},j+\frac{1}{2},k}}{\Delta y^2} E_x|_{i+\frac{1}{2},j+1,k}^{n+\frac{1}{2}} \\ & = c_1|_{i+\frac{1}{2},j,k} E_x|_{i+\frac{1}{2},j,k}^n \end{aligned}$$

TABLE I
COMPARISONS OF NUMBER OF RHS TERMS AND FLOPS COUNT
BETWEEN CONVENTIONAL ADI-FDTD AND FADI-FDTD METHODS

Scheme	Update Equations		No. of Terms		No. of Flops	
			Field	Coeff.	M/D	A/S
ADI	Implicit E	NDP	54	36	36	48
		EDP	6	6	6	6
		MDP	12	12	12	12
	Explicit H	NDP	30	18	18	24
		MDP	6	6	6	6
	Auxiliary J	EDP	18	18	18	12
	Auxiliary M	MDP	18	18	18	12
	Total		144	114	114	120

FADI	Implicit E	NDP	30	12	12	18
	Auxiliary $\hat{e}$	EDP	6	6	6	6
	Explicit H	NDP	24	12	12	18
	Auxiliary $\hat{h}$	MDP	6	6	6	6
	Auxiliary $\hat{j}$	EDP	12	12	12	6
	Auxiliary $\hat{m}$	MDP	6	12	12	6
	Total		84	60	60	60

NDP: nondispersive part
EDP: electric dispersive part
MDP: magnetic dispersive part

$$\begin{aligned}
& - \frac{c_2|_{i+\frac{1}{2},j,k} d_2|_{i+\frac{1}{2},j+\frac{1}{2},k}}{\Delta x \Delta y} \left(E_y|_{i+1,j+\frac{1}{2},k}^n - E_y|_{i,j+\frac{1}{2},k}^n \right) \\
& + \frac{c_2|_{i+\frac{1}{2},j,k} d_2|_{i+\frac{1}{2},j-\frac{1}{2},k}}{\Delta x \Delta y} \left(E_y|_{i+1,j-\frac{1}{2},k}^n - E_y|_{i,j-\frac{1}{2},k}^n \right) \\
& + \frac{c_2|_{i+\frac{1}{2},j,k} d_1|_{i+\frac{1}{2},j+\frac{1}{2},k}}{\Delta y} H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^n \\
& - \frac{c_2|_{i+\frac{1}{2},j,k} d_1|_{i+\frac{1}{2},j-\frac{1}{2},k}}{\Delta y} H_z|_{i+\frac{1}{2},j-\frac{1}{2},k}^n \\
& - \frac{c_2|_{i+\frac{1}{2},j,k}}{\Delta z} \left(H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^n - H_y|_{i+\frac{1}{2},j,k-\frac{1}{2}}^n \right) \\
& - \sum_p c_{3,p}|_{i+\frac{1}{2},j,k} J_{x,p}|_{i+\frac{1}{2},j,k}^n \\
& - \frac{c_2|_{i+\frac{1}{2},j,k}}{\Delta y} \sum_p d_{3,p}|_{i+\frac{1}{2},j+\frac{1}{2},k} M_{z,p}|_{i+\frac{1}{2},j+\frac{1}{2},k}^n \\
& + \frac{c_2|_{i+\frac{1}{2},j,k}}{\Delta y} \sum_p d_{3,p}|_{i+\frac{1}{2},j-\frac{1}{2},k} M_{z,p}|_{i+\frac{1}{2},j-\frac{1}{2},k}^n \quad (33)
\end{aligned}$$

One would be daunted by the tremendous length of the equation, where the RHS involves many field variables and update coefficients. Noticeably, there exists spatial differencing of up to second-order for E field terms (involving Δx and Δy). Besides $J_{x,p}$, $M_{z,p}$ also appears in electric field update equation, in which spatial differencing (Δy) is further required. Not only the field terms, update coefficients d_1 , d_2 and $d_{3,p}$ also require spatial differencing at the RHS.

On the other hand, the implicit E_x update equation of FADI-FDTD (16) reads:

$$\begin{aligned}
& - \frac{c_2|_{i+\frac{1}{2},j,k} d_2|_{i+\frac{1}{2},j-\frac{1}{2},k}}{\Delta y^2} E_x|_{i+\frac{1}{2},j-1,k}^{n+\frac{1}{2}} \\
& + \left(1 + \frac{c_2|_{i+\frac{1}{2},j,k} d_2|_{i+\frac{1}{2},j+\frac{1}{2},k}}{\Delta y^2} \right. \\
& \quad \left. + \frac{c_2|_{i+\frac{1}{2},j,k} d_2|_{i+\frac{1}{2},j-\frac{1}{2},k}}{\Delta y^2} \right) E_x|_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
& - \frac{c_2|_{i+\frac{1}{2},j,k} d_2|_{i+\frac{1}{2},j+\frac{1}{2},k}}{\Delta y^2} E_x|_{i+\frac{1}{2},j+1,k}^{n+\frac{1}{2}} \\
& = \hat{e}_x|_{i+\frac{1}{2},j,k}^n + \frac{c_2|_{i+\frac{1}{2},j,k}}{\Delta y} \left(\hat{h}_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^n - \hat{h}_z|_{i+\frac{1}{2},j-\frac{1}{2},k}^n \right) \quad (34)
\end{aligned}$$

For completeness, we also expand (14a) to obtain

$$\begin{aligned}
\hat{e}_x|_{i+\frac{1}{2},j,k}^n & = c'_1|_{i+\frac{1}{2},j,k} E_x|_{i+\frac{1}{2},j,k}^n - \hat{e}_x|_{i+\frac{1}{2},j,k}^{n-\frac{1}{2}} \\
& - \sum_p c_{3,p}|_{i+\frac{1}{2},j,k} \hat{j}_{x,p}|_{i+\frac{1}{2},j,k}^{n-\frac{1}{2}}. \quad (35)
\end{aligned}$$

Note that (34) and (35) form the update equation of FADI-FDTD involving E_x field component. Compared to (33) of conventional ADI-FDTD, the combined number of field variables and update coefficients of (34) and (35) are greatly reduced at the RHS. In particular, only first-order spatial differencing is involved. Dispersive term $\hat{j}_{x,p}$ is now only present in the auxiliary $\hat{e}_x$ update equation which is explicit and not in the implicit E_x update equation. Also, no $\hat{m}_{z,p}$ and magnetic update coefficients d 's are involved here in electric field update equation. Furthermore, we can place (14f) in the same loop with z -component of (14b) and (14d) at next half time step ($n + \frac{1}{2}$) to yield

$$H_{reg} = \hat{h}_z^n + d_2 \frac{\partial}{\partial y} E_x^{n+\frac{1}{2}} \quad (36a)$$

$$\hat{h}_z^{n+\frac{1}{2}} = d'_1 H_{reg} - \hat{h}_z^n - \sum_p d_{3,p} m_{z,p}^n \quad (36b)$$

$$\hat{m}_{z,p}^{n+\frac{1}{2}} = d_{4,p} \hat{m}_{z,p}^n + d'_{5,p} H_{reg}, \quad \forall p \quad (36c)$$

where H_{reg} is preferably stored in register of 4 or 8 bytes for float or double data types to be reused subsequently.

Table I gives more detailed comparisons of number of different RHS terms (with different indices) and flops count between conventional ADI-FDTD and FADI-FDTD methods. The number of terms include field variables and update coefficients, while the flops count include multiplications/divisions (M/D) and additions/subtractions (A/S) operations. The terms within the update equations include those associated with nondispersive part (NDP), electric (EDP) and magnetic dispersive parts (MDP) per pole. From the table, a substantially high number of RHS terms and flops count result from implicit E of ADI, as evident from (33). Note that magnetic dispersive part is also present here, which incurs additional number of RHS terms and flops count due to spatial differencing. For FADI, although implicit E is grouped together with auxiliary $\hat{e}$, the combined number of RHS terms and flops count are still much fewer than ADI. Another important aspect to note is that the electric and magnetic dispersive parts of FADI are well decoupled in the update equations, i.e., electric dispersive part appears only in E update equation while its magnetic counterpart only appears in H update equation. Overall, it can be observed that the number of RHS terms and flops count of FADI are fewer than ADI in every category of update equations. In total, there are around 120 fewer field variables and update coefficients terms per iteration for FADI compared to ADI. The total flops count of FADI is also around 120 fewer

Fig. 1. Scatter plot of the eigenvalues of Fourier amplification matrix for (a) second-order formulation at CFLN=1, (b) second-order formulation at CFLN=3.7, (c) our formulation at CFLN=100, and (d) our formulation at CFLN=400.

than that of ADI. Such simplicity of the FADI-FDTD method is mainly due to its matrix-operator-free RHS.

For memory storage requirement, we note that the field memory arrays needed by the FADI-FDTD method are

$$\begin{aligned}\hat{\mathbf{v}}_e &\leftarrow (\hat{e}^{n-\frac{1}{2}}, \hat{e}^n, \hat{e}^{n+\frac{1}{2}}), \mathbf{u}_e \leftarrow (E^n, E^{n+\frac{1}{2}}, E^{n+1}), \\ \hat{\mathbf{v}}_h &\leftarrow (\hat{h}^{n-\frac{1}{2}}, \hat{h}^n, \hat{h}^{n+\frac{1}{2}}), \\ \hat{\mathbf{g}}_j &\leftarrow (\hat{j}^{n-\frac{1}{2}}, \hat{j}^n, \hat{j}^{n+\frac{1}{2}}), \hat{\mathbf{g}}_m \leftarrow (\hat{m}^{n-\frac{1}{2}}, \hat{m}^n, \hat{m}^{n+\frac{1}{2}}).\end{aligned}\quad (37)$$

The variables within the same bracket can be reused in the same memory array. On the other hand, the field memory arrays needed by the conventional ADI-FDTD method are

$$\begin{aligned}\hat{\mathbf{v}}_e &\leftarrow (E^{n+\frac{1}{2}}), \mathbf{u}_e \leftarrow (E^n, E^{n+1}), \\ \hat{\mathbf{v}}_h &\leftarrow (H^n, H^{n+\frac{1}{2}}, H^{n+1}), \\ \hat{\mathbf{g}}_j &\leftarrow (J^n, J^{n+\frac{1}{2}}, J^{n+1}), \hat{\mathbf{g}}_m \leftarrow (M^n, M^{n+\frac{1}{2}}, M^{n+1}),\end{aligned}\quad (38)$$

which are just the same as the FADI-FDTD method.

We shall now demonstrate the potential instability encountered by the contemporary second-order differential equation formulation of ADI-FDTD. The update equation of polarization $\mathbf{P}$ using second-order differential equation formulation is given by [10]

$$\begin{aligned}\mathbf{P}_p^{n+\frac{1}{2}} &= \left(\frac{2 - \frac{\Delta t^2}{4} \omega_{e,p}^2}{1 + \delta_{e,p} \frac{\Delta t}{2}} \right) \mathbf{P}_p^n + \left(\frac{\delta_{e,p} \frac{\Delta t}{2} - 1}{\delta_{e,p} \frac{\Delta t}{2} + 1} \right) \mathbf{P}_p^{n-\frac{1}{2}} \\ &\quad + \left(\frac{\epsilon_0 \Delta \epsilon_p \frac{\Delta t^2}{4} \omega_{e,p}^2}{1 + \delta_{e,p} \frac{\Delta t}{2}} \right) \mathbf{E}^n.\end{aligned}\quad (39)$$

Note that the second-order differential equation formulations are also usually employed in explicit FDTD method [3], [15], [16].

To ascertain the instability of the previous second-order formulation, we utilize Fourier method to investigate the eigenvalues of the Fourier amplification matrix numerically. Fig. 1 shows the scatter plot of the eigenvalues of second-order formulation Fourier amplification matrix within a unit semi circle for 100 random wave numbers along x , y and z directions.

Fig. 2. Electric field recorded at centre of doubly Lorentz media slab computed using our formulation, c.f. (24).

TABLE II
EFFICIENCY GAIN AND NORMALIZED RMS ERROR WITH RESPECT TO EXPLICIT FDTD METHOD

CFLN	4	6	8	10
ADI-FDTD	0.99	1.49	1.98	2.48
FADI-FDTD	3.18	4.78	6.37	8.00
RMS error	0.33%	0.46%	0.71 %	1.13%

The medium is described by doubly Lorentz media having the following parameters: $\epsilon_\infty = \mu_\infty = 1$, $\Delta\epsilon = \Delta\mu = 1$, $\omega_{e,1} = \omega_{h,1} = 4 \times 10^{12}$ rad/s and $\delta_{e,1} = \delta_{h,1} = 4 \times 10^{11}$ rad/s. While the second-order formulation is still stable at CFLN=1 (CFLN = $\Delta t / \Delta t_{CFL}$, Δt_{CFL} is the CFL time step in explicit FDTD method), its eigenvalues start to deviate outside the unit semi circle around CFLN = 3.7, indicating instability. On the contrary, we also show scatter plot of the eigenvalues of our present formulation. Since the rigorous analytical proof of unconditional stability is difficult, we provide the numerical eigenvalues for very large CFLNs. It can be observed that at a larger CFLN=100 and even CFLN=400, our formulation still maintains stability with its eigenvalues all lying within the unit semi circle.

Consider a 3D slab having the dimension of 30 mm $\times$ 5 mm $\times$ 1 mm. The dispersive parameters are taken to be the same as above. Cell size is chosen as 0.1 mm and time step is specified at CFLN = 4. A continuous wave excitation is generated with frequency 30 GHz impinging upon the 3D slab. Fig. 2 plots the steady state electric field recorded at the centre and computed using our present formulation. The electric field solution remains bounded after $t = 1.5$ ns for CFLN=4. The time step can be further increased to CFLN=10 and the electric field solution is still stable and remains bounded. The electric field computed using the explicit FDTD method is also shown (CFLN=1), and validates our FADI-FDTD method. It can also be seen that the FADI-FDTD and the conventional ADI-FDTD methods yield the same computation, due to the fact that they are equivalent to each other.

To demonstrate the efficiency of the FADI-FDTD method,

Fig. 3. Magnitude of reflection coefficient computed at various CFLN versus frequency.

Table II shows the efficiency gain in terms of CPU run time and normalized root mean square (RMS) error with respect to explicit FDTD method. It can be seen that at CFLN=4, the efficiency gain of conventional ADI-FDTD method is only 1 compared to explicit FDTD method. To achieve higher efficiency gain, CFLN has to be selected larger than 4 at the expense of larger RMS error. The inefficiency of the conventional ADI-FDTD method is unsurprising due to the complicated update equation, c.f. (33), which involves many memory indexing (for field variables and coefficients) and arithmetic operations. On the other hand, the efficiency gain of FADI-FDTD method is 3.18 at CFLN=4 and can be as efficient as 8 at CFLN=10 with respect to explicit FDTD method. Comparing between FADI-FDTD and conventional ADI-FDTD methods, the FADI-FDTD method runs around three times faster. Such efficiency gain of the FADI-FDTD method appears higher than expected in Table I, which may be attributed to less memory indexing overhead required for fewer terms. Still, there exist other factors that may affect the actual computation efficiency, such as computer platform, hardware configuration, operating system, compiler software, program code arrangement, etc.

We now simulate a plane wave propagation in air impinging upon a doubly dispersive slab. The thickness of the slab is $20 \mu\text{m}$ and is modeled by doubly Lorentz media having the following parameters: $\epsilon_\infty = \mu_\infty = 1$, $\Delta\epsilon = \Delta\mu = 1$, $\omega_{e,1} = 1.885 \times 10^{11}$ rad/s, $\omega_{h,1} = 1.26 \times 10^{11}$ rad/s, $\delta_{e,1} = 1.885 \times 10^{10}$ rad/s, and $\delta_{h,1} = 1.26 \times 10^{10}$ rad/s. The cell size chosen for the air region is 0.1 mm while cell size is $5 \mu\text{m}$ for the doubly dispersive slab and interface regions (for finer discretization). Fig. 3 plots the magnitude of reflection coefficient of the doubly Lorentz slab computed at various CFLN versus frequency. Two resonance frequencies at 20 GHz and 30 GHz are clearly visible. For high CFLN values of 50, 100 and 200, the results are in good agreement to the analytical curve. This shows the advantage of FADI-FDTD method in fine mesh structures where high CFLN can be used. It can also be seen that the results computed using

Fig. 4. Snapshot of electric field depicting wave propagation through (a) doubly Lorentz medium, (b) doubly Drude medium. Black lines indicate slab interfaces.

both FADI-FDTD and conventional ADI-FDTD methods are the same.

After the validation, we proceed to use our FADI-FDTD to demonstrate electromagnetic wave propagation through a doubly dispersive medium. A 3D slab with different dimension of $30 \text{ mm} \times 10 \text{ mm} \times 1 \text{ mm}$ is now considered. The slab is chosen to be doubly Lorentz media having the parameters: $\epsilon_\infty = \mu_\infty = 1$, $\Delta\epsilon = \Delta\mu = 1$, $\omega_{e,1} = \omega_{h,1} = 1.54 \times 10^{11}$ rad/s and $\delta_{e,1} = \delta_{h,1} = 1.54 \times 10^8$ rad/s. Such parameters are chosen such that the slab resembles a double negative medium having permittivity and permeability close to -1 at 30 GHz with some losses. Same cell size and source excitation are employed and time step is at CFLN=4. Fig. 4a shows the snapshot of electric field after $t = 1.86 \text{ ns}$ at z -cut xy plane. The width of the slab corresponds to $1\lambda_0$ (wavelength at free space) and the source is located $0.5\lambda_0$ from the interface of the slab. It is observed that the wave can be focused at two locations, one at the centre of the slab and the other at the opposite end outside the slab, also $0.5\lambda_0$ away. These two focuses have smaller amplitudes compared to that of the source point due to the fact that the double negative slab is also lossy in nature. The negative index of refraction had been demonstrated experimentally in [18], [19] and its realistic applications had been mentioned in [20]. Our numerical experiment above has demonstrated that such negative index of refraction can also be described using our doubly Lorentz media.

We next demonstrate propagation of electromagnetic wave through a doubly Drude medium using the FADI-FDTD method. A 3D slab with dimension of $30 \text{ mm} \times 0.2 \text{ mm} \times 1 \text{ mm}$ is considered. The slab is chosen to be doubly Drude media having the parameters: $\epsilon_\infty = \mu_\infty = 1$, $\omega_{e,1} = \omega_{h,1} = 1.89 \times 10^{11}$ rad/s and $\nu_{e,1} = \nu_{h,1} = 1.89 \times 10^6$ rad/s [21]. Such parameters are chosen to obtain a near zero refractive index at 30 GHz with some losses. In this example, the width of the slab is only 0.2 mm . Such thin slab is chosen to reduce

the attenuation through otherwise thick slab. We shall employ a cell size of 0.01 mm for the width of the slab, 10 times smaller than the cell size for the remaining domain. Due to smaller cell size, the CFL time step Δt_{CFL} of the explicit FDTD method is greatly restricted, but the time step for FADI-FDTD method remains unaffected. Fig. 4b shows the snapshot of electric field after $t = 1.92$ ns at z -cut xy plane. With the aid of FADI-FDTD method, it is demonstrated that the electric field transmitted across the slab can be shaped into near plane-wave behaviour. The CPU time taken is only 5 percent that of explicit FDTD method. This further demonstrates the strength of our FADI-FDTD method in nonuniform cell size scenario. For realistic application, the zero-index material demonstrated above can be utilized to enhance the gain of an antenna [22].

IV. CONCLUSION

This paper has presented stable formulation of FADI-FDTD method for multi-term, doubly, second-order dispersive media. The formulation has been achieved by decomposing the electric and magnetic susceptibility functions into first-order poles and it has been derived based on currents constitutive relations. It is conveniently applicable for most commonly used second-order dispersive models, such as Lorentz and Drude models, and equally applicable to first-order Debye model. The extension for full 3D doubly dispersive media using FADI-FDTD method has made the resultant update equations much more concise and simpler than using conventional ADI-FDTD method. The number of field variables and update coefficients have been greatly reduced at the right-hand sides, with only first-order spatial differencing and no dispersive terms and magnetic update coefficients in the implicit E update equations. All these contribute to conciseness and programming simplicity, as well as leading to higher efficiency due to much less memory indexing overhead and fewer arithmetic operations. Most importantly, our present formulation is stable, while the contemporary second-order differential equation formulation has potential instability as has been demonstrated. The stability analysis has been performed using Fourier method by examining the eigenvalues of the Fourier amplification matrix numerically.

REFERENCES

- [1] F. Zheng, Z. Chen, and J. Zhang, "Toward the development of a three-dimensional unconditionally stable finite-difference time-domain method," *IEEE Trans. Microw. Theory Tech.*, vol. 48, no. 9, pp. 1550–1558, Sep. 2000.
- [2] T. Namiki, "3-D ADI-FDTD method: unconditionally stable time-domain algorithm for solving full vector maxwell's equations," *IEEE Trans. Microw. Theory Tech.*, vol. 48, no. 9, pp. 1743–1748, Oct. 2000.
- [3] A. Taflov and S. C. Hagness, *Computational Electrodynamics: The Finite-Difference Time-Domain Method* Boston, M. A.: Artech House, 2005.
- [4] S. G. Garcia, R. G. Rubio, A. R. Bretones and R. G. Martin, "Extension of the ADI-FDTD method to debye media," *IEEE Trans. Antennas Propag.*, vol. 51, no. 11, pp. 3183–3186, Nov. 2003.
- [5] X. T. Dong, N. V. Venkatarayalu, B. Guo, W. Y. Yin and Y. B. Gan, "General formulation of unconditionally stable ADI-FDTD method in linear dispersive media," *IEEE Trans. Microw. Theory Tech.*, vol. 52, no. 1, pp. 170–174, Jan. 2004.
- [6] L. Xu and N. Yuan, "PLJERC-ADI-FDTD method for isotropic plasma," *IEEE Microw. Wireless Compon. Lett.*, vol. 15, no. 4, pp. 277–279, Apr. 2005.

- [7] J. A. Pereda, O. Gonzalez, A. Grande and A. Vegas, "An alternating-direction implicit FDTD modeling of dispersive media without constitutive relation splitting," *IEEE Microw. Wireless Compon. Lett.*, vol. 18, no. 11, pp.719–721, Nov. 2008.
- [8] O. Ramadan, "General ADI-FDTD formulations for multi-term dispersive electromagnetic applications," *IEEE Microw. Wireless Compon. Lett.*, vol. 21, no. 10, pp. 513–515, Oct. 2011.
- [9] E. L. Tan, "Fundamental schemes for efficient unconditionally stable implicit finite-difference time-domain methods," *IEEE Trans. Antennas Propag.*, vol. 56, no. 1, pp. 170–177, Jan. 2008.
- [10] K. -Y. Jung and F. L. Teixeira, "Multispecies ADI-FDTD algorithm for nanoscale three-dimensional photonic metallic structures," *IEEE Photonics Tech. Lett.*, vol. 19, no. 8, pp.586–588, Apr. 2007.
- [11] Z. Lin and L. Thylen, "On the accuracy and stability of several widely used FDTD approaches for modeling Lorentz dielectrics," *IEEE Trans. Antennas Propag.*, vol. 57, no. 10, pp. 3378–3381, Oct. 2009.
- [12] A. H. Panaretos and R. E. Diaz "On the stability of the finite-difference time-domain modeling of Lorentz media," *IEEE Microw. Wireless Compon. Lett.*, vol.21, no. 6, pp. 283–285, Jun. 2011.
- [13] D. Y. Heh and E. L. Tan, "Modeling hemoglobin at optical frequency using the unconditionally stable fundamental ADI-FDTD method," *Biomed. Opt. Express*, vol. 2, no. 5, pp. 1169–1183, May 2011.
- [14] D. Y. Heh and E. L. Tan, "Modeling the interaction of terahertz pulse with healthy skin and basal cell carcinoma using the unconditionally stable fundamental ADI-FDTD method," *Progress In Electromagnetics Research B*, vol. 37, pp. 365–386, 2012.
- [15] Y. Takayama and W. Klaus, "Reinterpretation of the auxiliary differential equation method for FDTD," *IEEE Microw. Wireless Compon. Lett.*, vol. 12, no. 3, pp. 102–104, Mar. 2002.
- [16] M. Okoniewski, M. Mrozowski and M.A. Stuchly, "Simple treatment of multi-term dispersion in FDTD," *IEEE Microw. Guided Wave Lett.*, vol. 7, no. 5, pp. 121–123, May 1997.
- [17] M. Han, R. W. Dutton and S. Fan, "Model dispersive media in finite-difference time-domain method with complex-conjugate pole-residue pairs," *IEEE Microw. Wireless Compon. Lett.*, vol. 16, no. 3, pp. 119–121, Mar. 2006.
- [18] R. A. Shelby, D. R. Smith and S. Schultz, "Experimental verification of a negative index of refraction," *Science*, vol. 292, pp. 77–79, Apr. 2001.
- [19] S. Wang, F. Garet, K. Blary, E. Lheurette and J. L. Coutaz, "Experimental verification of negative refraction for a wedge-type negative index metamaterial operating at terahertz," *Appl. Phys. Lett.*, vol. 97, pp. 181902-1–181902-3, 2010.
- [20] N. Engheta and R. W. Ziolkowski, "A positive future for double-negative metamaterials," *IEEE Trans. Microw. Theory Tech.*, vol. 53, no. 4, pp. 1535–1556, Apr. 2005.
- [21] R. W. Ziolkowski, "Propagation in and scattering from a matched metamaterial having a zero index of refraction," *Phys. Rev. E*, vol. 70, pp. 046608, 2004.
- [22] M. Sun, Z. N. Chen and X. Qing, "Gain enhancement of antipodal tapered slot antenna using zero-index metamaterial," *2012 IEEE Asia-Pacific Conf. on Antennas and Propag.*, art. no. 6333152, pp. 70–71.