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Radially graded index whispering gallery mode resonator for penetration enhancement

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Abstract: This paper theoretically analyzes a hollow cylindrical whispering gallery mode resonator with radially inhomogeneous cladding. We propose an index profile of \( n(r) = \frac{b}{r} \) to enhance field penetration towards the resonator core. With such index profile, externally coupled evanescent wave can easily penetrate the resonator cladding without any potential barrier.

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References and links

1. Introduction

Whispering gallery mode (WGM) resonators [1, 2], due to their small mode volume and high Q factor, have been applied in various applications such as narrowband filtering [3], microlasing [4] and highly sensitive sensor [5–8]. Presently, most WGM devices rely on solid material such as silica, chalcogenide and CaF2. Liquids, on the other hand, are usually avoided because of the difficulty in manipulation, short evaporative lifetime, and mechanical instability [1]. This actually limits the advance of WGM applications because many liquid materials are associated with attractive characteristics that are unique for novel WGM devices, e.g. liquid crystal resonators [9] and liquid dye lasers [10]. Moreover, WGM can also be utilized to characterize liquid material properties, e.g. the liquid core ring sensor [8, 11, 12].

A common approach to implement liquid resonator is to encapsulate the liquid with solid claddings – in the shape of tube, bubble (bottle) or hollow sphere. Figure 1(a) shows a schematic diagram of a typical liquid core optical ring resonator (LCORR). In this resonator, liquid fills a capillary tube and a fiber taper is used as the coupler. Evanescent wave coupled from the taper circulates in the resonator and form WGM. As light interacts with the core medium, transmission spectra monitored from the output of the fiber taper reflects the properties of the target medium, such as refractive index, gain/attenuation or microstructures attached at the interior wall. Usually, if the core consists of high index materials, WGM could be excited in the core; if it consists of low index materials, WGMs are confined in the cladding and we rely on the evanescent tail extended from the inner surface of the cladding to interact with the target medium. However, since lower radial order WGMs tend to stay close to the surface of the cavity, externally coupled WGM can hardly penetrate the cladding to reach the core. To counteract this problem, thin wall must be used [13] and recent applications have achieved fabrication of submicron thin wall capillary tubes [8, 11]. It is also worth mentioning that a non-evanescent sensing mode has been demonstrated in prism-coupled microtube sensors [14]; in this particular case, strong field penetration towards the core is achieved and it leads to a significant improvement of the sensitivity.

In this work, we propose an alternative solution to enable externally coupled light to reach the core by penetrating the claddings. The general idea is to adopt a radially graded index profile in the cladding, i.e. let the inner part have higher refractive than that of the outer part (see Fig. 1(c)). As light tends to enter higher index regions, it will bend towards the center of the cavity. To fulfill piecewise angular momentum conservation, we put $r \times k_\phi$ to be a constant, and hence yields $n(r) = b/r$, where $n(r)$ is the proposed index profile for the cladding.

Fig. 1. (a) A typical capillary tube based cylindrical WGM refractometer; (b) cross section of a step index cavity; (c) graded index cavity with index profile $b/r$ in the cladding

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r is the radial position in the cavity and b is a constant with unit RIU-µm. Under this profile, photons are free to orbit inside the cladding without any potential well traps, and penetration is therefore ensured. Such design principle is also applicable in construction of optical attractor or “optical black hole” [15, 16].

2. Theoretical derivation

Here, we present a theoretical analysis, including the calculation of the field distribution and eigenfrequencies, for a WGM cavity with graded-index cladding (n(r) = b/r) shown in Fig. 1. Starting from Maxwell’s equation, the electromagnetic field in the cylindrical microcavity follows

\begin{align*}
\nabla^2 \mathbf{H} + \frac{\nabla \mathbf{E} \times \nabla \times \mathbf{H}}{\varepsilon} = -k^2 n^2(r) \mathbf{H} \\
\nabla^2 \mathbf{E} + \nabla \left( \frac{\mathbf{E} \cdot \nabla \mathbf{E}}{\varepsilon} \right) = -k^2 n^2(r) \mathbf{E}
\end{align*}

where \( k = \omega \sqrt{\mu_\varepsilon} = \omega c \), \( \varepsilon \) is only radially inhomogeneous and \( \varepsilon(r) = n^2(r) \). The solution falls into two modes: transverse electric (TE) mode and transverse magnetic (TM) mode. For TE mode, magnetic field only has z-direction (axial direction) component; and TM mode has electric field only in z-direction. Assume \( \mathbf{H} = H(r,z)\exp(-im\phi) \) and \( \mathbf{E} = E(r,z)\exp(-im\phi) \), where \( m \) is the azimuthal quantum number. By separation of variable in cylindrical coordinates, we get the radial equation for TE and TM mode respectively [17]:

\begin{align*}
\frac{d^2 H_z (r)}{dr^2} + \left( 1 - \frac{2}{n(r)} \frac{dn(r)}{dr} \right) \frac{d}{dr} H_z (r) + \left( k^2 n^2(r) - \frac{m^2}{r^2} \right) H_z (r) &= 0 \\
\frac{d^2 E_z (r)}{dr^2} + \frac{1}{r} \frac{d}{dr} E_z (r) + \left( k^2 n^2(r) - \frac{m^2}{r^2} \right) E_z (r) &= 0
\end{align*}

For the sake of simplicity and clarity, we focus on the demonstration of TM mode only. The solutions of Eq. (3) are divided into three regions: the core region (\( r \leq R_1 \)), the cladding region (\( R_1 < r < R_2 \)), and the outer region (\( r > R_2 \)). \( R_1 \) and \( R_2 \) are the inner and outer radius of cladding, respectively.

In the core region, \( n(r) = n_1 \), the solution is

\[ E_z (r) = E_v(r) = a_1 J_m(n_1 kr) \] (4)

In the outer region, \( n(r) = n_3 \), the solution is

\[ E_z (r) = E_v(r) = a_2 H_m^{(1)}(n_3 kr) \] (5)

In the cladding region, \( n(r) = n_2(r) = b/r \), the solution is

\[ E_z (r) = E_v(r) = a_2 \cos \left( \ln(r) \Delta \right) + a_{22} \sin \left( \ln(r) \Delta \right) \] (6)

\( J_m(z) \) and \( H_m^{(1)}(z) \) are the Bessel and Hankel function of the first kind, respectively; \( \Delta = \sqrt{(b^2 k^2 - m^2)} \); \( a_1, a_{21}, a_{22} \) and \( a_3 \) are constants. Applying boundary conditions at all interfaces, we obtain the characteristic equation for \( k \):

\[ \frac{J_m(kn_1R_1)H_m^{(1)}(kn_2R_2)}{J_m(kn_1R_1)H_m^{(1)}(kn_2R_2)} \tan(\ln(R_2)\Delta) \tan(\ln(R_1)\Delta) = \frac{R_1n_1}{R_2n_3} \] (7)
3. Results and discussion

Here, we discuss two different cases: high index core \((n_1>b/R_1)\) and low index core \((n_1<b/R_1)\). In both cases, externally coupled light is able to penetrate the cladding regardless of cladding thickness and mode excited.

3.1 High index core

When \(n_1>b/R_1\), both the core and cladding are able to confine WGM. We use \(q\) (radial quantum number) to denote the number of field maxima along the radial axis of the cavity.

For lower \(q\) orders, field is confined near the surface of the core. We can estimate the relationship between \(k\) and \(m\) as \(m = kn_eR_e\), where \(n_e\) and \(R_e\) are the effective radius and refractive index at the position where WGM is localized. In this case, \(n_e = n_1\) and \(R_e = R_1\). Therefore \(k^2b^2-m^2 = k^2(b-n_1R_1)^2 < 0\) and \(\Delta\) is imaginary. From Eq. (6), we can see that cosine and sine terms become hyper-cosine and hyper-sine, corresponding to fast decaying field near the inner interface. Figure 2(a) shows the field distribution along radial axis for the first two radial order modes with \(m = 110\). In this case, however, evanescent tail does not leak to the outer region, which makes evanescent coupling from outside impossible. Therefore, lower \(q\) order modes are not able to be excited from external coupling.

For higher radial order modes, where \(k^2b^2 < m^2\), \(\Delta\) becomes real; \(E_z(r)\) now exhibits sinusoidal behavior, and the field oscillates along radial direction with an evanescent tail extended to the outer region. This is the intended scenario in which field penetrates deep into the core, and yet allows easy external coupling. Here, if the core-cladding boundary cuts the resonance from the cladding at its tail (after the resonance peak), we identify this situation as full resonances (see Fig. 2(b)); on the other hand, if the boundary cuts the resonance before or at the resonance peak, we identify this as partial resonances (see Fig. 2(c)). When full resonances are formed in claddings, cladding retains a significant portion of the energy. Figure 2(b) shows the field distribution along radial axis for \(m = 110, q = 12\) and 14, where full resonances are formed in the cladding. On the contrary, it is observed that a partial resonance in the cladding allows a larger portion of the field energy to enter the core. In this case, resonances in cladding serve as the tail of the core resonances (see Fig. 2(c) \(q = 13\) and 15). It is worth noting that, if lower order modes can be generated through nonlinear processes in the core, extremely high quality factor may be achieved.

![Fig. 2. Field distribution in the cavity. (a) For lower q orders, field is strictly confined inside the core and fast decays in the cladding; while for higher q orders, field in cladding exhibits sinusoidal behavior across the cladding and leaks an evanescent tail to the outside for external coupling. (b) When full resonance happens in the cladding, cladding retains more field energy; (c) partial resonances in cladding results in strong resonances in core. \(n_i = 3, n_s = 1, b = 15\) RIU-\(\mu\)m, \(R_i = 10\mu\)m, \(R_2 = 14\mu\)m, \(m = 110\).](image-url)
To compare the penetration differences between step index and graded index cavities with high index cores, we performed a 2D Finite-Difference Time-Domain (FDTD) simulation. The simulation models the configuration in Fig. 1, which consists of the cavity (\(n_1 = 3, n_3 = 1, \ n_2 = 15 \ \text{RIU} - \ \mu\text{m/r}) for graded index and 1.5 for step index, \(R_1 = 10\ \mu\text{m}, R_2 = 14\ \mu\text{m}) and a taper placed to its left for coupling. A mode source with frequency range between 1.54 and 1.57 \(\mu\text{m}) is placed at one end of the taper, and the transmission spectrum is monitored at the other end. Figure 3(a) shows the transmission spectrum for step index cavity, and Fig. 3(b) and 3(c) show the field patterns at resonance wavelength 1.543 and 1.554 \(\mu\text{m}) respectively. Figure 3(d) is the transmission spectrum for graded index cavity, and (e) and (f) shows the field pattern at 1.544 \(\mu\text{m}) (TM_{13,77}) and 1.559 \(\mu\text{m}) (TM_{13,78}) respectively. The results well match our theoretical derivation. It can be seen that graded index cladding always allows evanescent field from external coupler to penetrate through and form high radial order WGM inside the core. Step index cavity, on the other hand, either does not allow penetration or retain most of the field energy inside the cladding.

3.2 Low index core

For low index cores, i.e. \(n_1 < b/R_3\), the modes are confined in the cladding, which resembles a ring resonator [18, 19]. This configuration is widely adopted as low index refractometer, biosensing [20] and liquid dye lasers [10]. They require core material to interact with evanescent tail extended from the inner interface of the cladding. It is mentioned that extremely thin wall and high \(l\) order is required to achieve penetration [19], where \(l\) denotes the number of field maxima in the cladding; however, our design is robust – no matter how thick the cladding is, no matter which \(l\) order mode is considered, evanescent wave will always penetrate into the core, since field is evenly distributed across the cladding. Figure 4 shows a comparison between the step index and graded index liquid-core ring resonator based on the settings of White, et al. [19]
3.3 Effective potential pattern

The proposed index profile \((n = b/r)\) allows photon angular momentum conservation anywhere inside the cladding, and therefore allows light passing through freely. To verify this, we apply quantum analogy to demonstrate the effective potential \(U_{\text{eff}}\) in the cavity [21]. For cylindrical WGM cavities, the effective potential is the sum of the attractive well of depth \((n'(r)-1)k^2\) and the centrifugal potential \((m^2-1/4)/r^2\) [17, 22]. Figure 5 shows the effective potential for our cavity under different index profiles. Graded index designs effectively flatten the potential barrier originally associated with the step index cavities, and \(n_2(r)\sim r^{-1}\) is found to be the critical changing rate as predicted. This can be used as a general criterion: at any point, if index gradient \((n'(r))\) is greater than that of the critical rate, photons will be pulled towards the core; otherwise, it will be pushed outwards. This conclusion is consistent with that in [16], which proves that \(\alpha = 2\) is the critical value for the profile \(n(r)\sim 1/r^{\alpha/2}\).

![Effective potential pattern](image)

Fig. 5. Effective potential \(U_{\text{eff}}\) for a cylindrical tube cavity with air surrounded and high index material inside. Graded index designs flatten the potential barriers in the cladding, and \(n_2(r)\sim r^{-1}\) is the critical changing rate.

For fabrication, our index profile may be implemented using multilayer In\(_x\)Ga\(_{1-x}\)As alloys or dopant diffusion from inner core. Note that if multilayer approach is adopted, methods in [23] may be particularly useful for theoretical analysis. The requirement for index differences depends on the ratio of the inner and outer radius, i.e. given a fixed index changing range, larger wall thickness is allowed for larger cavities. For example, at 1660nm, the refractive indices of In\(_x\)Ga\(_{1-x}\)As are 3.372 and 3.619 for \(x = 0\) and 0.5 respectively. This 7.3% index difference allows the wall thickness to be 7.3% of the inner radius. In addition, even if the
gradient of b/r cannot be satisfied, lower gradient may also have enhancement effect on penetration, though less effective.

4. Conclusion

In conclusion, we proposed and analyzed, for the first time, a graded index profile in the form of $n = b/r$ for liquid-core cylindrical WGM cavities. Cladding with this profile imposes no potential barrier and allows light to penetrate freely. Field distribution exhibits sinusoidal behavior and evenly distributes across the cladding. Not only cylindrical cavities, but spherical cavities, bottle cavities and micro disk cavities can also adopt the same cladding index profile. With such design, more liquid based WGM resonators may be implemented. In addition, this configuration enables isolation between target medium and the coupling region, which prevents contamination and promises high-Q cavity potential.

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