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## SHARP ERROR BOUNDS FOR JACOBI EXPANSIONS AND GEGENBAUER–GAUSS QUADRATURE OF ANALYTIC FUNCTIONS\*

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**Abstract.** This paper provides a rigorous and delicate analysis for exponential decay of Jacobi polynomial expansions of analytic functions associated with the Bernstein ellipse. Using an argument that can recover the best estimate for the Chebyshev expansion, we derive various new and sharp bounds of the expansion coefficients, which are featured with explicit dependence of all related parameters and valid for degree  $n \geq 1$ . We demonstrate the sharpness of the estimates by comparing with existing ones, in particular, the very recent results in *SIAM J. Numer. Anal.*, 50 (2012), pp. 1240–1263. We also extend this argument to estimate the Gegenbauer–Gauss quadrature remainder of analytic functions, which leads to some new tight bounds for quadrature errors.

**Key words.** Bernstein ellipse, exponential convergence, analytic functions, Jacobi polynomials, Gegenbauer–Gauss quadrature, error bounds, sharp estimate

**AMS subject classifications.** 65N35, 65E05, 65M70, 41A05, 41A10, 41A25

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**1. Introduction.** The spectral method employs global orthogonal polynomials or Fourier complex exponentials as basis functions, so it enjoys high-order accuracy (with only a few basis functions) if the underlying function is smooth (and periodic in the Fourier case). The convergence rate  $O(n^{-r})$ , where  $n$  is the number of basis functions involved in a spectral expansion and  $r$  is related to the Sobolev-regularity of the underlying function, is typically documented in various monographs on spectral methods [20, 17, 16, 4, 23, 38, 8, 9, 26, 35]. It is also widely appreciated that if the function under consideration is analytic, the convergence rate is of exponential order  $O(q^n)$  (for constant  $0 < q < 1$ ). However, there are few discussions of such error bounds (mostly mentioned, but not proved) [16, 38, 8]. Indeed, as commented by Hale and Trefethen [25], the general idea of such convergence goes back to Bernstein [5], but such results do not appear in many textbooks or monographs, and there is not much uniformity in the constants in the upper bounds.

An important result in Bernstein [6] (also see [30]) states that  $u$  is analytic on  $[-1, 1]$  if and only if

$$\sup_{N \rightarrow \infty} \lim_{N \rightarrow \infty} \sqrt[N]{E_N(u)} = \frac{1}{\rho}, \quad E_N(u) = \inf_{v \in \mathbb{P}_N} \|v - u\|_\infty,$$

where  $\mathbb{P}_N$  is the polynomial space of degree no more than  $N$  and  $\rho > 1$  is the sum of the semi-axes of the maximum ellipse  $\mathcal{E}_\rho$  with foci  $\pm 1$ , known as the *Bernstein ellipse*,

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on and within which  $u$  can be analytically extended to. One immediate implication is that the best polynomial approximation in the maximum norm enjoys exponential convergence. A more precise estimate for the Chebyshev expansion can be found in various approximation theory texts (see, e.g., [14], [34, Theorem 3.8], and [31, Theorem 5.16]):

$$(1.1) \quad |\hat{u}_n^C| \leq \frac{2M}{\rho^n} \quad \forall n \geq 0; \quad \|u - S_N^C u\|_\infty \leq \frac{M}{(\rho-1)\rho^N}; \quad M = \max_{z \in \mathcal{E}_\rho} |u(z)|,$$

where  $\{\hat{u}_n^C\}$  are the Chebyshev expansion coefficients of  $u$  and  $S_N^C u$  is the partial sum involving the first  $N+1$  terms. Also refer to [36, 13, 34, 7, 31, 39, 40] and the references therein for verification and descriptions of exponential convergence of Fourier, Chebyshev, or Legendre expansions. We remark that Gottlieb and Shu [22, 21] studied exponential convergence of Gegenbauer expansions (when the parameter grows linearly with  $n$ ) in the context of defeating the Gibbs phenomenon.

Here, we particularly highlight that a very recent paper of Xiang [42] provided a simple approach to obtain the bounds for Jacobi expansion coefficients of analytic functions on and within the Bernstein ellipse  $\mathcal{E}_\rho$ :

$$(1.2) \quad |\hat{u}_n^{\alpha,\beta}| \leq \frac{2M}{\rho^{n-1}(\rho-1)} \sqrt{\frac{\gamma_0^{\alpha,\beta}}{\gamma_n^{\alpha,\beta}}}, \quad \text{where } \hat{u}_n^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} \int_{-1}^1 u(x) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx.$$

Here,  $\{P_n^{(\alpha,\beta)}(x)\} (\alpha, \beta > -1)$  are Jacobi polynomials mutually orthogonal with the weight function  $\omega^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$  and with the normalization factor  $\gamma_n^{\alpha,\beta}$  (cf. (2.8)). The key step of Xiang's approach is to insert the Chebyshev expansion  $u(x) = \sum_{j=0}^{\infty} \hat{u}_j^C T_j(x)$  into the Jacobi expansion coefficients and rewrite

$$\hat{u}_n^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} \sum_{j=n}^{\infty} \hat{u}_j^C \int_{-1}^1 T_j(x) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx,$$

so the bound for the Chebyshev coefficient in (1.1) could be used.

The first purpose of the paper is to take a different approach to derive sharp estimates for general Jacobi expansion of analytic functions. The assertion of sharpness is in the following sense:

- (i) The bound for the general Jacobi case is tighter than (1.2) (see Remark 2.3).
- (ii) Refined estimates can be obtained for Gegenbauer expansion ( $\alpha = \beta > -1$ ), Chebyshev-type expansion ( $\alpha = k - 1/2, \beta = l - 1/2$  for nonnegative integers  $k, l$ ), and Legendre-type expansion ( $\alpha = k, \beta = l$  for nonnegative integers  $k, l$ ). The argument can recover the bounds known to be the sharpest (e.g., the Chebyshev case), and some obtained estimates are new and significantly improve the existing ones (see, e.g., Remark 2.5).

A second purpose of this work is to extend the argument to analyze the Gegenbauer–Gauss quadrature of analytical functions. Recall that the remainder of the Gauss quadrature with the nodes and weights  $\{x_j, \omega_j\}_{j=1}^n$ , takes the form (see, e.g., [14])

$$(1.3) \quad E_n[u] = \int_{-1}^1 u(x) \omega(x) dx - \sum_{j=1}^n u(x_j) \omega_j = \frac{1}{\pi i} \oint_{\mathcal{E}_\rho} \frac{q_n(z)}{p_n(z)} u(z) dz,$$

where  $\{x_j\}_{j=1}^n$  are the zeros of  $p_n(x)$ , orthogonal with respect to the weight function  $\omega(x)$ , and

$$(1.4) \quad q_n(z) = \frac{1}{2} \int_{-1}^1 \frac{p_n(x)\omega(x)}{z-x} dx.$$

The estimate of quadrature errors has attracted much attention (see, e.g., [12, 11, 3, 19, 14, 18, 27, 28]). Among these results, intensive discussions have been centered around the Chebyshev case and its family, e.g., Chebyshev of the second kind, but with very limited results even for the Legendre–Gauss quadrature (see, e.g., [10, 29]). In fact, the analysis heavily relies on the availability of explicit expression of  $p_n(z)$  on  $\mathcal{E}_\rho$ . Armed with a delicate estimate of  $q_n(z)$  (in the first part of the paper) and the explicit formula of the Gegenbauer polynomial in [43], we are able to derive a sharp bound for the Gegenbauer–Gauss quadrature errors.

We remark that there has been much interest in estimating spectral differentiation errors of analytic functions. Tadmor [37] first attempted to estimate the aliasing errors to verify exponential convergence of Fourier and Chebyshev spectral differentiation with a different assumption on analyticity. The results for analyticity characterized by the Bernstein ellipse include Reddy and Weideman [33] for the Chebyshev case and Xie, Wang, and Zhao [43] for the Gegenbauer spectral differentiation. It is also interesting to point out that Zhang [44, 45, 46] studied superconvergence of spectral interpolation and differentiation. We stress that the analysis tools and arguments in the above literature are different from these in this work.

The rest of this paper is organized as follows. In Section 2, we provide sharp bounds for general Jacobi expansions of analytic functions, followed by some refined results for Chebyshev-type and Legendre-type expansions. In Section 3, we extend the argument to analyze Gegenbauer–Gauss quadrature errors. In the final section, we provide results to show the sharpness of the bounds by comparing them with existing ones.

**2. Sharp bounds for Jacobi expansions.** We derive in this section sharp bounds for Jacobi expansions of functions analytic on and within the Bernstein ellipse  $\mathcal{E}_\rho$ .

**2.1. Preliminaries.** It is known (see, e.g., [13]) that the Bernstein ellipse is transformed from the circle

$$(2.1) \quad \mathcal{C}_\rho = \{w = \rho e^{i\theta} : \theta \in [0, 2\pi]\}, \quad \rho > 1,$$

via the conformal mapping:  $z = (w + w^{-1})/2$ , namely,

$$(2.2) \quad \mathcal{E}_\rho := \left\{ z \in \mathbb{C} : z = \frac{1}{2}(w + w^{-1}) \text{ with } w = \rho e^{i\theta}, \theta \in [0, 2\pi] \right\},$$

where  $\mathbb{C}$  is the set of all complex numbers and  $i = \sqrt{-1}$  is the complex unit. It has the foci at  $\pm 1$ , and the major and minor semi-axes are

$$(2.3) \quad a = \frac{1}{2}(\rho + \rho^{-1}), \quad b = \frac{1}{2}(\rho - \rho^{-1}),$$

respectively, so the sum of two semi-axes is  $\rho$ . The perimeter of  $\mathcal{E}_\rho$  can be approximated by (see Ramanujan [32]):  $\pi(3\rho - \sqrt{4\rho^2 - \rho^{-2}})$ . We also refer to the very recent result (see [41, (2.15)])

$$(2.4) \quad L(\mathcal{E}_\rho) \leq \frac{(24 - 5\sqrt{2}\pi)\rho - (8 - (1 + \sqrt{2})\pi)(\rho^2 - \rho^{-2})/\rho - (16 - 5\pi)\sqrt{\rho^2 + \rho^{-2}}}{2(3 - 2\sqrt{2})}.$$

The distance from  $\mathcal{E}_\rho$  to the interval  $[-1, 1]$  is defined as

$$(2.5) \quad d_\rho = \frac{1}{2}(\rho + \rho^{-1}) - 1.$$

We see that  $d_\rho$  increases with respect to  $\rho$ , and  $d_\rho \rightarrow 0^+$  as  $\rho \rightarrow 1^+$  (so the ellipse reduces to the interval  $[-1, 1]$ ). Thus, by the theory of analytic continuation, we have that for any analytic function  $u$  on  $[-1, 1]$ , there always exists a Bernstein ellipse  $\mathcal{E}_\rho$  with  $\rho > 1$  such that the continuation of  $u$  is analytic on and within  $\mathcal{E}_\rho$ . Hereafter, we denote by

$$(2.6) \quad \mathcal{A}_\rho := \{u : u \text{ is analytic on and within } \mathcal{E}_\rho\}, \quad 1 < \rho < \rho_{\max},$$

where  $\mathcal{E}_{\rho_{\max}}$  labels the largest ellipse within which  $u$  is analytic. In particular, if  $\rho_{\max} = \infty$ ,  $u$  is an entire function.

Throughout this paper, the Jacobi polynomials, denoted by  $P_n^{(\alpha, \beta)}(x)$  (with  $\alpha, \beta > -1$  and  $x \in I := (-1, 1)$ ), are normalized as in Szegő [36], i.e.,

$$(2.7) \quad \int_{-1}^1 P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) \omega^{\alpha, \beta}(x) dx = \gamma_n^{\alpha, \beta} \delta_{m, n},$$

where  $\omega^{\alpha, \beta}(x) = (1-x)^\alpha(1+x)^\beta$ ,  $\delta_{m, n}$ , is the Kronecker delta and

$$(2.8) \quad \gamma_n^{\alpha, \beta} = \frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n! \Gamma(n+\alpha+\beta+1)}.$$

In Appendix A, we collect the relevant properties of Jacobi polynomials.

In the analysis, we also use the following property of the Gamma function (see [1, (6.1.38)]):

$$(2.9) \quad \Gamma(x+1) = \sqrt{2\pi} x^{x+1/2} \exp\left(-x + \frac{\theta}{12x}\right) \quad \forall x > 0, \quad 0 < \theta < 1.$$

LEMMA 2.1. For any constants  $a, b$ , we have that for  $n \geq 1$ ,  $n+a > 1$ , and  $n+b > 1$ ,

$$(2.10) \quad \frac{\Gamma(n+a)}{\Gamma(n+b)} \leq \Upsilon_n^{a,b} n^{a-b},$$

where

$$(2.11) \quad \Upsilon_n^{a,b} = \exp\left(\frac{a-b}{2(n+b-1)} + \frac{1}{12(n+a-1)} + \frac{(a-1)(a-b)}{n}\right).$$

*Proof.* Let  $\theta_1, \theta_2$  be two constants in  $(0, 1)$ . We find from (2.9) that

$$\begin{aligned} \frac{\Gamma(n+a)}{\Gamma(n+b)} &= \frac{(n+a-1)^{n+a-1/2}}{(n+b-1)^{n+b-1/2}} \exp\left(b-a + \frac{\theta_1}{12(n+a-1)} - \frac{\theta_2}{12(n+b-1)}\right) \\ &\leq (n+a-1)^{a-b} \left(1 + \frac{a-b}{n+b-1}\right)^{n+b-1/2} \exp\left(b-a + \frac{1}{12(n+a-1)}\right) \\ &\leq n^{a-b} \left(1 + \frac{a-1}{n}\right)^{a-b} \exp\left(b-a + \frac{(a-b)(n+b-1/2)}{n+b-1} + \frac{1}{12(n+a-1)}\right) \\ &\leq n^{a-b} \exp\left(\frac{a-b}{2(n+b-1)} + \frac{1}{12(n+a-1)} + \frac{(a-1)(a-b)}{n}\right) := \Upsilon_n^{a,b} n^{a-b}, \end{aligned}$$

where we used the fact that  $1+x \leq e^x$  for real  $x$ .  $\square$

*Remark 2.1.* Applying (2.11) to  $\gamma_n^{\alpha,\beta}$  leads to that for  $\alpha, \beta > -1$ ,  $n \geq 1$ , and  $n + \alpha + \beta > 0$ ,

$$(2.12) \quad \gamma_n^{\alpha,\beta} \leq \frac{2^{\alpha+\beta+1}}{2n + \alpha + \beta + 1} \Upsilon_n^{\alpha+1,1} \Upsilon_n^{\beta+1,\alpha+\beta+1} = C_n \frac{2^{\alpha+\beta}}{n},$$

where

$$(2.13) \quad C_n := C_n(\alpha, \beta) = \frac{2n \Upsilon_n^{\alpha+1,1} \Upsilon_n^{\beta+1,\alpha+\beta+1}}{2n + \alpha + \beta + 1}.$$

Note that for fixed  $\alpha$  and  $\beta$ ,

$$(2.14) \quad \Upsilon_n^{\alpha,\beta} = 1 + O(n^{-1}), \quad C_n = 1 + O(n^{-1}).$$

Hereafter, we do not specify the dependence of the constants (e.g.,  $C_n$ ) on  $\alpha, \beta$ .

**2.2. Main tools.** Our starting point is the following important representation

LEMMA 2.2. *Let  $\{\hat{u}_n^{\alpha,\beta}\}$  be the Jacobi polynomial expansion coefficients given by*

$$(2.15) \quad \hat{u}_n^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} \int_{-1}^1 u(x) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx, \quad \alpha, \beta > -1, \quad n \geq 0.$$

If  $u \in \mathcal{A}_\rho$  with  $\rho > 1$ , we have the representation

$$(2.16) \quad \hat{u}_n^{\alpha,\beta} = \frac{1}{\pi i} \sum_{j=0}^{\infty} \sigma_{n,j}^{\alpha,\beta} \oint_{\mathcal{E}_\rho} \frac{u(z)}{w^{n+j+1}} dz, \quad n \geq 0,$$

where  $z = (w + w^{-1})/2$  with  $w = \rho e^{i\theta}$ ,  $\theta \in [0, 2\pi]$ , and

$$(2.17) \quad \sigma_{n,j}^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} \int_{-1}^1 U_{n+j}(x) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx, \quad n, j \geq 0.$$

Here,  $U_{n+j}(x)$  is the Chebyshev polynomial of the second kind of degree  $n + j$  (cf. (A.5)).

Actually, the formula (2.16)–(2.17) can be obtained by assembling several formulas in Szegő [36] and then using the generating function of  $U_k(x)$  (cf. [1]). For the readers' reference, we sketch its derivation in Appendix B. Note that the contour integral expressions of  $\hat{u}_n^{\alpha,\beta}$  in (B.1) and (B.2) are special cases of the general formulas in [15, (2.2)–(2.3)] (which were derived by a quite different approach).

The establishment of sharp bounds heavily relies on estimating  $\sigma_{n,j}^{\alpha,\beta}$ . The following explicit formulas follow from (2.17) and some properties of Jacobi polynomials listed in Appendix A. We remark that the formula (2.20) can be found in various books, e.g., [13, 31], while the formula (2.21) is due to Heine (see [12]). We also highlight that the formula (2.22) for the general Jacobi case seems new.

COROLLARY 2.3. *Let  $n \geq 0$ .*

(i) *For  $\alpha = \beta > -1$  (ultraspherical/Gegenbauer polynomial),<sup>1</sup>*

$$(2.18) \quad \sigma_{n,j}^{\alpha,\alpha} = 0 \quad \text{for odd } j$$

and  $\sigma_{n,j}^{\alpha,\alpha}$  for even  $j$  are computed by (2.22) below.

(ii) *For  $\alpha = \beta = 1/2$  (Chebyshev polynomial of the second kind),*

<sup>1</sup>In this paper, we do not distinguish between ultraspherical and Gegenbauer polynomials.

$$(2.19) \quad \sigma_{n,0}^{1/2,1/2} = \frac{\sqrt{\pi}}{2} \frac{(n+1)!}{\Gamma(n+3/2)}; \quad \sigma_{n,j}^{1/2,1/2} = 0 \quad \text{for } j \geq 1.$$

(iii) For  $\alpha = \beta = -1/2$  (Chebyshev polynomial),

$$(2.20) \quad \sigma_{n,j}^{-1/2,-1/2} = \begin{cases} \frac{2\sqrt{\pi}\Gamma(n+1)}{\Gamma(n+1/2)} & \text{for even } j, \\ 0 & \text{for odd } j. \end{cases}$$

(iv) For  $\alpha = \beta = 0$  (Legendre polynomial),

$$(2.21) \quad \sigma_{n,j}^{0,0} = \begin{cases} \frac{2n+1}{2} \frac{\Gamma(l+1/2)}{\Gamma(l+1)} \frac{\Gamma(n+l+1)}{\Gamma(n+l+3/2)} & \text{for even } j = 2l, \\ 0 & \text{for odd } j. \end{cases}$$

(v) For general  $\alpha, \beta > -1$  (Jacobi polynomial),

$$(2.22) \quad \sigma_{n,j}^{\alpha,\beta} = \frac{\sqrt{\pi}(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{2\Gamma(n+\alpha+1)} \times \sum_{m=0}^j \frac{(-1)^m \Gamma(2n+j+m+2)\Gamma(n+m+\alpha+1)}{m!(j-m)!\Gamma(n+m+3/2)\Gamma(2n+m+\alpha+\beta+2)}.$$

*Proof.* (i) The property (2.18) is a direct consequence of the parity of ultraspherical polynomials.

(ii) For  $\alpha = \beta = 1/2$ , we find from (A.5) and the orthogonality (2.7)–(2.8) that

$$\begin{aligned} \sigma_{n,j}^{1/2,1/2} &= \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{\gamma_{n+j}^{1/2,1/2}}} \frac{1}{\gamma_n^{1/2,1/2}} \int_{-1}^1 P_{n+j}^{(1/2,1/2)}(x) P_n^{(1/2,1/2)}(x) (1-x^2)^{1/2} dx \\ &= \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{\gamma_{n+j}^{1/2,1/2}}} \delta_{j,0}, \end{aligned}$$

where  $\delta_{j,0}$  is the Kronecker delta. Working out the constant leads to (2.19).

(iii) For  $\alpha = \beta = -1/2$ , if  $j = 2l$ , by (A.5), (A.6a), and (A.6b), we have

$$\begin{aligned} \sigma_{n,2l}^{-1/2,-1/2} &= \frac{1}{\gamma_n^{-1/2,-1/2}} \frac{1}{n+2l+1} \int_{-1}^1 T'_{n+2l+1}(x) P_n^{(-1/2,-1/2)}(x) (1-x^2)^{-1/2} dx \\ &= \frac{2}{\gamma_n^{-1/2,-1/2}} \int_{-1}^1 T_n(x) P_n^{(-1/2,-1/2)}(x) (1-x^2)^{-1/2} dx \\ &= \frac{2\sqrt{\pi}\Gamma(n+1)}{\Gamma(n+1/2)}, \end{aligned}$$

which, together with (2.18), implies (2.20).

(iv) For  $\alpha = \beta = 0$ , we derive from [36, (4.9.8)] that

$$\begin{aligned} \sigma_{n,2l}^{0,0} &= \frac{1}{\gamma_n^{0,0}} \int_{-1}^1 P_n(x) U_{n+2l}(x) dx = \frac{2n+1}{2} \int_0^\pi P_n(\cos \theta) \sin((n+2l+1)\theta) d\theta \\ &= \frac{2n+1}{2} \frac{\Gamma(l+1/2)}{\Gamma(l+1)} \frac{\Gamma(n+l+1)}{\Gamma(n+l+3/2)}, \quad l \geq 0. \end{aligned}$$

This yields (2.21).

(v) The formula (2.22) follows from a combination of (2.8), (A.4), and (A.5).  $\square$

With the aid of Lemma 2.2, we can derive the following estimate, from which our sharp bounds stem.

LEMMA 2.4. *For any  $u \in \mathcal{A}_\rho$  with  $\rho > 1$ , we have that for  $\alpha, \beta > -1$  and  $n \geq 0$ ,*

$$(2.23) \quad |\hat{u}_n^{\alpha,\beta}| \leq \frac{M}{\rho^n} \left( |\sigma_{n,0}^{\alpha,\beta}| + \frac{1}{\rho} |\sigma_{n,1}^{\alpha,\beta}| + \frac{1}{\rho^2} \sum_{j=0}^{\infty} |\sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta}| \frac{1}{\rho^j} \right),$$

where  $M = \max_{z \in \mathcal{E}_\rho} |u(z)|$  and  $\{\sigma_{n,j}^{\alpha,\beta}\}$  are given by (2.17).

*Proof.* Since  $z = (w + w^{-1})/2 \in \mathcal{E}_\rho$  with  $w \in \mathcal{C}_\rho$  (cf. (2.1)–(2.2)), we can rewrite  $\hat{u}_n^{\alpha,\beta}$  in (2.16) as

$$(2.24) \quad \begin{aligned} \hat{u}_n^{\alpha,\beta} &= \frac{1}{2\pi i} \sum_{j=0}^{\infty} \sigma_{n,j}^{\alpha,\beta} \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+j+1}} \left(1 - \frac{1}{w^2}\right) dw \\ &= \frac{1}{2\pi i} \sum_{j=0}^{\infty} \sigma_{n,j}^{\alpha,\beta} \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+j+1}} dw - \frac{1}{2\pi i} \sum_{j=0}^{\infty} \sigma_{n,j}^{\alpha,\beta} \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+j+3}} dw \\ &= \frac{1}{2\pi i} \sigma_{n,0}^{\alpha,\beta} \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+1}} dw + \frac{1}{2\pi i} \sigma_{n,1}^{\alpha,\beta} \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+2}} dw \\ &\quad + \frac{1}{2\pi i} \sum_{j=0}^{\infty} (\sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta}) \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+j+3}} dw. \end{aligned}$$

Hence, we arrive at

$$(2.25) \quad \begin{aligned} |\hat{u}_n^{\alpha,\beta}| &\leq \frac{M}{2\pi} \frac{2\pi\rho}{\rho^{n+1}} |\sigma_{n,0}^{\alpha,\beta}| + \frac{M}{2\pi} \frac{2\pi\rho}{\rho^{n+2}} |\sigma_{n,1}^{\alpha,\beta}| + \frac{M}{2\pi} \frac{2\pi\rho}{\rho^{n+3}} \sum_{j=0}^{\infty} |\sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta}| \frac{1}{\rho^j} \\ &= \frac{M}{\rho^n} |\sigma_{n,0}^{\alpha,\beta}| + \frac{M}{\rho^{n+1}} |\sigma_{n,1}^{\alpha,\beta}| + \frac{M}{\rho^{n+2}} \sum_{j=0}^{\infty} |\sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta}| \frac{1}{\rho^j}. \end{aligned}$$

This ends the proof.  $\square$

Observe from the proof that we split the contour integral on  $\mathcal{E}_\rho$  into two parts on  $\mathcal{C}_\rho$ , which actually allows us to take advantage of cancelation of  $\sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta}$ . Indeed, the bound (2.23) is tight, as we will see shortly that this argument can recover the best estimate for the Chebyshev case (see [34, Theorem 3.8] and (1.1)) and improve the bounds in [42] (see (1.2)).

**2.3. Main results.** For clarity of exposition, we first present the result on the general Jacobi polynomial expansions, followed by the refined results on the Chebyshev-type expansions ( $\alpha = k - 1/2, \beta = l - 1/2$  with  $k, l \in \mathbb{N} := \{0, 1, 2, \dots\}$ ) and Legendre-type expansions ( $\alpha = k, \beta = l$  with  $k, l \in \mathbb{N}$ ).

**2.3.1. General Jacobi expansions ( $\alpha, \beta > -1$ ).**

THEOREM 2.5. *For any  $u \in \mathcal{A}_\rho$  (with  $\rho > 1$ ),  $\alpha, \beta > -1$ , and  $n \geq 0$ , we have*

$$(2.26) \quad |\hat{u}_n^{\alpha,\beta}| \leq \frac{M}{\rho^n} \left[ |\sigma_{n,0}^{\alpha,\beta}| + \frac{|\sigma_{n,1}^{\alpha,\beta}|}{\rho} + \frac{2}{\rho(\rho-1)} \sqrt{\frac{\gamma_0^{\alpha,\beta}}{\gamma_n^{\alpha,\beta}}} \right],$$



where

$$(2.27) \quad \sigma_{n,0}^{\alpha,\beta} = \frac{\sqrt{\pi}}{2} \frac{(2n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+3/2)\Gamma(2n+\alpha+\beta+1)}, \quad \sigma_{n,1}^{\alpha,\beta} = \frac{(\beta-\alpha)(2n+2)}{2n+\alpha+\beta+2} \sigma_{n,0}^{\alpha,\beta},$$

and  $\gamma_n^{\alpha,\beta}$  is defined in (2.8).

In particular, if  $\alpha = \beta$ , we have

$$(2.28) \quad |\hat{u}_n^{\alpha,\alpha}| \leq \frac{M}{\rho^n} \left[ |\sigma_{n,0}^{\alpha,\alpha}| + \frac{2}{\rho^2-1} \sqrt{\frac{\gamma_0^{\alpha,\alpha}}{\gamma_n^{\alpha,\alpha}}} \right].$$

*Proof.* By (2.23),

$$(2.29) \quad |\hat{u}_n^{\alpha,\beta}| \leq \frac{M}{\rho^n} |\sigma_{n,0}^{\alpha,\beta}| + \frac{M}{\rho^{n+1}} |\sigma_{n,1}^{\alpha,\beta}| + \frac{M}{\rho^{n+2}} \sum_{j=0}^{\infty} |\sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta}| \frac{1}{\rho^j}.$$

The factors  $\sigma_{n,0}^{\alpha,\beta}$  and  $\sigma_{n,1}^{\alpha,\beta}$  in (2.27) are computed from (2.22) directly, so it suffices to estimate the infinite sum in (2.29). Recall the identity (cf. [31])

$$(2.30) \quad U_k(x) - U_{k-2}(x) = 2T_k(x), \quad k \geq 2.$$

Then we infer from (2.17) that

$$(2.31) \quad \begin{aligned} \sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta} &= \frac{1}{\gamma_n^{\alpha,\beta}} \int_{-1}^1 (U_{n+j+2}(x) - U_{n+j}(x)) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx \\ &= \frac{2}{\gamma_n^{\alpha,\beta}} \int_{-1}^1 T_{n+j+2}(x) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx, \quad n, j \geq 0. \end{aligned}$$

Thus, using the Cauchy–Schwartz inequality, the orthogonality (2.7), and the fact  $|T_k(x)| \leq 1$  leads to

$$(2.32) \quad |\sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta}| \leq \frac{2}{\sqrt{\gamma_n^{\alpha,\beta}}} \left( \int_{-1}^1 T_{n+j+2}^2(x) \omega^{\alpha,\beta}(x) dx \right)^{1/2} \leq 2 \sqrt{\frac{\gamma_0^{\alpha,\beta}}{\gamma_n^{\alpha,\beta}}}.$$

Therefore, the bound (2.26) follows from  $\sum_{j=0}^{\infty} \rho^{-j} = 1/(1-\rho^{-1})$ , as  $\rho > 1$ .

For  $\alpha = \beta$ , since  $|\sigma_{n,2l+1}^{\alpha,\alpha}| = 0$ , for all  $l \geq 0$  (cf. Corollary 2.3(i)), we have

$$\sum_{j=0}^{\infty} |\sigma_{n,j+2}^{\alpha,\alpha} - \sigma_{n,j}^{\alpha,\alpha}| \frac{1}{\rho^j} = \sum_{l=0}^{\infty} |\sigma_{n,2l+2}^{\alpha,\alpha} - \sigma_{n,2l}^{\alpha,\alpha}| \frac{1}{\rho^{2l}} \leq 2 \sqrt{\frac{\gamma_0^{\alpha,\alpha}}{\gamma_n^{\alpha,\alpha}}} \frac{1}{1-\rho^{-2}}.$$

This yields the refined bound in (2.28).  $\square$

*Remark 2.2.* Using Lemma 2.1, we can characterize the explicit dependence of the upper bounds in (2.26) and (2.28) on  $n, \alpha$ , and  $\beta$ . Indeed, we derive from (2.10) and (2.27) that for  $\alpha, \beta > -1, n \geq 1$ , and  $n + \alpha + \beta > 0$ ,

$$\begin{aligned} \sigma_{n,0}^{\alpha,\beta} &= \frac{\sqrt{\pi}}{2} \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+3/2)} \frac{\Gamma(2n+2)}{\Gamma(2n+\alpha+\beta+1)} \\ &\leq \frac{\sqrt{\pi}}{2} (\Upsilon_n^{\alpha+\beta+1,3/2} n^{\alpha+\beta+1-3/2}) (\Upsilon_{2n}^{2,\alpha+\beta+1} (2n)^{2-(\alpha+\beta+1)}) \\ &= \frac{\sqrt{\pi n}}{2^{\alpha+\beta}} \Upsilon_n^{\alpha+\beta+1,3/2} \Upsilon_{2n}^{2,\alpha+\beta+1}, \end{aligned}$$

which implies

$$|\sigma_{n,1}^{\alpha,\beta}| = \frac{|\alpha - \beta|(2n + 2)}{2n + \alpha + \beta + 2} \sigma_{n,0}^{\alpha,\beta} \leq \frac{|\alpha - \beta|(2n + 2)}{2n + \alpha + \beta + 2} \frac{\sqrt{\pi n}}{2^{\alpha+\beta}} \Upsilon_n^{\alpha+\beta+1,3/2} \Upsilon_{2n}^{2,\alpha+\beta+1}.$$

Similarly, by (2.8) and (2.10),

$$\begin{aligned} \frac{\gamma_0^{\alpha,\beta}}{\gamma_n^{\alpha,\beta}} &= (2n + \alpha + \beta + 1) \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} \frac{\Gamma(n + 1)\Gamma(n + \alpha + \beta + 1)}{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)} \\ &\leq (2n + \alpha + \beta + 1) \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} \Upsilon_n^{1,\alpha+1} \Upsilon_n^{\alpha+\beta+1,\beta+1}. \end{aligned}$$

Consequently, we infer from (2.26) that for  $\alpha, \beta > -1, n \geq 1$ , and  $n + \alpha + \beta > 0$ ,

$$\begin{aligned} (2.33) \quad |\hat{u}_n^{\alpha,\beta}| &\leq \frac{M}{\rho^n} \left[ \left( 1 + \frac{1}{\rho} \frac{|\alpha - \beta|(2n + 2)}{2n + \alpha + \beta + 2} \right) \frac{\sqrt{\pi n}}{2^{\alpha+\beta}} \Upsilon_n^{\alpha+\beta+1,3/2} \Upsilon_{2n}^{2,\alpha+\beta+1} \right. \\ &\quad \left. + \frac{2\sqrt{2n + \alpha + \beta + 1}}{\rho(\rho - 1)} \sqrt{\frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)}} \sqrt{\Upsilon_n^{1,\alpha+1} \Upsilon_n^{\alpha+\beta+1,\beta+1}} \right] \\ &\leq \hat{C}_n M \left( \frac{\sqrt{\pi}}{2^{\alpha+\beta}} \left( 1 + \frac{|\alpha - \beta|}{\rho} \right) + \sqrt{\frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)}} \frac{2\sqrt{2}}{\rho(\rho - 1)} \right) \frac{\sqrt{n}}{\rho^n}, \end{aligned}$$

where using the fact  $2n + \alpha + \beta + 2 > 2n$ , we denote

$$(2.34) \quad \hat{C}_n = \max \left\{ \frac{n + 1}{n} \Upsilon_n^{\alpha+\beta+1,3/2} \Upsilon_{2n}^{2,\alpha+\beta+1}, \sqrt{\frac{2n + \alpha + \beta + 1}{2n}} \sqrt{\Upsilon_n^{1,\alpha+1} \Upsilon_n^{\alpha+\beta+1,\beta+1}} \right\}.$$

Similarly, we find from (2.28) that for  $\alpha > -1, n \geq 1$ , and  $n + 2\alpha > 0$ ,

$$(2.35) \quad |\hat{u}_n^{\alpha,\alpha}| \leq \bar{C}_n M \left( \frac{\sqrt{\pi}}{2^{2\alpha}} + \frac{\Gamma(\alpha + 1)}{\sqrt{\Gamma(2\alpha + 2)}} \frac{2\sqrt{2}}{\rho^2 - 1} \right) \frac{\sqrt{n}}{\rho^n},$$

where

$$(2.36) \quad \bar{C}_n = \max \left\{ \Upsilon_n^{2\alpha+1,3/2} \Upsilon_{2n}^{2,2\alpha+1}, \sqrt{\frac{2n + 2\alpha + 1}{2n}} \sqrt{\Upsilon_n^{1,\alpha+1} \Upsilon_n^{2\alpha+1,\alpha+1}} \right\}.$$

By (2.14), we know that  $\hat{C}_n = 1 + O(n^{-1})$  and  $\bar{C}_n = 1 + O(n^{-1})$ .

*Remark 2.3.* It is worthwhile to show that the bound obtained from this way is tighter than (1.2) in [42]. Indeed, it follows from (A.5), (A.6b), and (2.7) that for  $n \geq 1$  and  $j = 0, 1$ ,

$$\begin{aligned} \sigma_{n,j}^{\alpha,\beta} &= \frac{1}{\gamma_n^{\alpha,\beta}} \frac{1}{n + j + 1} \int_{-1}^1 T'_{n+j+1}(x) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx \\ &= \frac{2}{\gamma_n^{\alpha,\beta}} \sum_{\substack{k=0 \\ k+n+j+1 \text{ odd}}}^{n+j} \frac{1}{c_k} \int_{-1}^1 T_k(x) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx \\ &= \frac{2}{\gamma_n^{\alpha,\beta}} \int_{-1}^1 T_{n+j}(x) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx, \end{aligned}$$

where  $c_0 = 2$  and  $c_k = 1$  for  $k \geq 1$ . Following (2.31)–(2.32), we have

$$|\sigma_{n,j}^{\alpha,\beta}| \leq 2\sqrt{\frac{\gamma_0^{\alpha,\beta}}{\gamma_n^{\alpha,\beta}}}, \quad n \geq 1, \quad j = 0, 1.$$

Finally, a straightforward calculation leads to

$$(2.37) \quad \frac{M}{\rho^n} \left[ |\sigma_{n,0}^{\alpha,\beta}| + \frac{|\sigma_{n,1}^{\alpha,\beta}|}{\rho} + \frac{2}{\rho(\rho-1)} \sqrt{\frac{\gamma_0^{\alpha,\beta}}{\gamma_n^{\alpha,\beta}}} \right] \leq \frac{2M}{\rho^{n-1}(\rho-1)} \sqrt{\frac{\gamma_0^{\alpha,\beta}}{\gamma_n^{\alpha,\beta}}}.$$

Moreover, we claim from (2.28) that the strict inequality holds, when  $\alpha = \beta > -1$ . One may refer to Section 4 for numerical evidence.

**2.3.2. Chebyshev-type expansions ( $\alpha = k - 1/2, \beta = l - 1/2$  with  $k, l \in \mathbb{N}$ ).** In view of (2.20), it follows from (2.24) that the Chebyshev coefficient takes the simplest form:

$$(2.38) \quad \hat{u}_n^{-1/2,-1/2} = \frac{\sigma_{n,0}^{-1/2,-1/2}}{2\pi i} \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+1}} dw.$$

Thus, using (2.20) and (2.23) leads to

$$(2.39) \quad |\hat{u}_n^{-1/2,-1/2}| \leq \frac{2\sqrt{\pi}\Gamma(n+1)M}{\Gamma(n+1/2)\rho^n}.$$

This leads to the estimate for the expansion coefficients, denoted by  $\{\hat{u}_n^C\}$  as before, in terms of  $\{T_n(x)\}$ ,

$$(2.40) \quad |\hat{u}_n^C| \leq \frac{2M}{\rho^n}, \quad n \geq 0,$$

as documented in, e.g., [34].

For the second kind of Chebyshev case, we find from (2.16) the closed-form formula like (2.38),

$$(2.41) \quad \hat{u}_n^{1/2,1/2} = \frac{\sigma_{n,0}^{1/2,1/2}}{\pi i} \oint_{\mathcal{E}_\rho} \frac{u(z)}{w^{n+1}} dz,$$

but the contour integration is on  $\mathcal{E}_\rho$ . It follows from (2.19) and (2.24) that

$$(2.42) \quad \begin{aligned} |\hat{u}_n^{1/2,1/2}| &\leq \frac{\sigma_{n,0}^{1/2,1/2}}{\pi} \left| \oint_{\mathcal{E}_\rho} \frac{u(z)}{w^{n+1}} dz \right| \\ &= \frac{1}{4\sqrt{\pi}} \frac{(n+1)!}{\Gamma(n+3/2)} \left| \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+1}} \left(1 - \frac{1}{w^2}\right) dw \right| \\ &\leq \frac{\sqrt{\pi}}{2} \frac{(n+1)!}{\Gamma(n+3/2)} \frac{M}{\rho^n} \left(1 + \frac{1}{\rho^2}\right). \end{aligned}$$

Like (2.40), if we rescale the expansion in terms of  $\{U_n\}$ , i.e.,

$$\hat{u}_n^U = \frac{2}{\pi} \int_{-1}^1 u(x) U_n(x) \sqrt{1-x^2} dx,$$

then we find from (A.5) and (2.42) that

$$(2.43) \quad |\hat{u}_n^U| = \frac{2}{\sqrt{\pi}} \frac{\Gamma(n+3/2)}{\Gamma(n+2)} |\hat{u}_n^{1/2,1/2}| \leq \frac{M}{\rho^n} \left(1 + \frac{1}{\rho^2}\right).$$

*Remark 2.4.* It is seen from (2.41) that the second kind of Chebyshev coefficient takes the simplest form on the contour  $\mathcal{E}_\rho$ . This motivates us to estimate the contour integral directly by

$$\left| \oint_{\mathcal{E}_\rho} \frac{u(z)}{w^{n+1}} dz \right| \leq \frac{M}{\rho^{n+1}} \oint_{\mathcal{E}_\rho} |dz| = \frac{M}{\rho^{n+1}} L(\mathcal{E}_\rho),$$

which implies

$$(2.44) \quad |\hat{u}_n^U| \leq \frac{M}{\rho^{n+1}} \frac{L(\mathcal{E}_\rho)}{\pi}.$$

Using the upper bound  $\pi\sqrt{\rho^2 + \rho^{-2}}$  of  $L(\mathcal{E}_\rho)$  [33] (note that it is larger than the upper bound in (2.4); see [41, Remark 3.5]), we derive

$$\frac{L(\mathcal{E}_\rho)}{\pi\rho} < \sqrt{1 + \frac{1}{\rho^4}} < 1 + \frac{1}{\rho^2}.$$

This shows the estimate (2.44) is sharper than (2.43).

Some refined results can also be derived for  $\alpha = k + 1/2, \beta = l + 1/2$  with  $k, l \in \mathbb{N}$ . Indeed, we find that  $\{\sigma_{n,j}^{k+1/2,l+1/2}\}$  can be computed explicitly by the following formula.

PROPOSITION 2.6. *For any  $k, l, n, j \in \mathbb{N}$ ,*

$$(2.45) \quad \sigma_{n,j}^{k+1/2,l+1/2} = \sqrt{\frac{\pi}{2}} \frac{1}{\gamma_n^{k+1/2,l+1/2}} \sum_{m=n}^{n+k+l} d_m^{k+1/2,l+1/2} \sqrt{\gamma_m^{1/2,1/2}} \delta_{m,n+j},$$

where  $\{d_m^{k+1/2,l+1/2}\}_{m=n}^{n+k+l}$  are given in (A.3) and  $\delta_{m,n+j}$  is the Kronecker delta.

*Proof.* Using (A.3) (with  $\alpha = \beta = 1/2$ ), (2.17), and the properties of Jacobi polynomials (cf. (2.7) and (A.5)) leads to

$$\begin{aligned} \sigma_{n,j}^{k+1/2,l+1/2} &= \frac{1}{\gamma_n^{k+1/2,l+1/2}} \sum_{m=n}^{n+k+l} d_m^{k+1/2,l+1/2} \int_{-1}^1 U_{n+j}(x) P_m^{(1/2,1/2)}(x) (1-x^2)^{1/2} dx \\ &= \frac{1}{\gamma_n^{k+1/2,l+1/2}} \sqrt{\frac{\pi}{2}} \sum_{m=n}^{n+k+l} d_m^{k+1/2,l+1/2} \sqrt{\gamma_m^{1/2,1/2}} \delta_{m,n+j}, \end{aligned}$$

This completes the proof.  $\square$

Equipped with (2.45), we can obtain the bound for Chebyshev-type expansion coefficients by computing  $\{d_m^{k+1/2,l+1/2}\}$  explicitly. To fix the idea, we just consider the case  $k = 1$  and  $l = 0$ . One finds

$$d_n^{3/2,1/2} = 1, \quad d_{n+1}^{3/2,1/2} = -\frac{2n+2}{2n+3},$$

and

$$\sigma_{n,0}^{3/2,1/2} = \frac{\sqrt{\pi}}{4} \frac{n!(n+2)}{\Gamma(n+3/2)}, \quad \sigma_{n,1}^{3/2,1/2} = -\frac{\sqrt{\pi}}{4} \frac{(n+1)!}{\Gamma(n+3/2)}, \quad \sigma_{n,j}^{3/2,1/2} = 0, \quad j \geq 2.$$

The estimate (2.15) reduces to

$$\begin{aligned}\hat{u}_n^{3/2,1/2} &\leq \frac{M}{\rho^n} \left[ \sigma_{n,0}^{3/2,1/2} + \frac{\sigma_{n,1}^{3/2,1/2}}{\rho} + \frac{\sigma_{n,0}^{3/2,1/2}}{\rho^2} + \frac{\sigma_{n,1}^{3/2,1/2}}{\rho^3} \right] \\ &= \frac{M}{\rho^n} \left( 1 + \frac{1}{\rho^2} \right) \left[ \sigma_{n,0}^{3/2,1/2} + \frac{\sigma_{n,1}^{3/2,1/2}}{\rho} \right].\end{aligned}$$

Thus, we have

$$(2.46) \quad |\hat{u}_n^{3/2,1/2}| \leq \frac{\sqrt{\pi}}{4} \frac{(n+1)!}{\Gamma(n+3/2)} \frac{M}{\rho^n} \left( 1 + \frac{1}{\rho^2} \right) \left( \frac{n+2}{n+1} + \frac{1}{\rho} \right),$$

and by (2.10), we have for  $n \geq 0$ ,

$$(2.47) \quad \frac{(n+1)!}{\Gamma(n+3/2)} \leq \sqrt{n} \exp \left( \frac{8n+7}{12(2n+1)(n+1)} + \frac{1}{4n} \right).$$

Actually, the infinite sum in (2.23) does not appear for the Chebyshev-type expansions, which allows us to derive very tight bounds. However, for the Legendre-type expansions, some care has to be taken to handle this sum.

**2.3.3. Legendre-type expansions ( $\alpha = k, \beta = l$  with  $k, l \in \mathbb{N}$ ).** We first consider the Legendre case. By (2.8) and (2.27),

$$\gamma_n^{0,0} = \frac{2}{2n+1}, \quad \frac{\gamma_0^{0,0}}{\gamma_n^{0,0}} = 2n+1, \quad \sigma_{n,0}^{0,0} = \frac{\sqrt{\pi}\Gamma(n+1)}{\Gamma(n+1/2)},$$

so the estimate (2.28) reduces to

$$(2.48) \quad |\hat{u}_n^{0,0}| \leq \frac{M}{\rho^n} \left[ \frac{\sqrt{\pi}\Gamma(n+1)}{\Gamma(n+1/2)} + \frac{2\sqrt{2n+1}}{\rho^2-1} \right].$$

In fact, we can improve this estimate, as highlighted in the following theorem, by using the explicit information of  $\sigma_{n,2l}^{0,0}$ .

**THEOREM 2.7.** *Let  $\{\hat{u}_n^{0,0}\}$  be the Legendre expansion coefficients of any  $u \in \mathcal{A}_\rho$  with  $\rho > 1$ . Then for any  $n \geq 1$ ,*

$$(2.49) \quad |\hat{u}_n^{0,0}| \leq \frac{M\sqrt{\pi n}}{\rho^n} \left( 1 + \frac{n+2}{2n+3} \frac{1}{\rho^2-1} \right) \exp \left( \frac{8n-1}{12n(2n-1)} \right).$$

*Proof.* A straightforward calculation from (2.21) yields

$$(2.50) \quad \sigma_{n,2l+2}^{0,0} - \sigma_{n,2l}^{0,0} = -\frac{n+2l+2}{2(l+1)(n+l+3/2)} \sigma_{n,2l}^{0,0}, \quad l \geq 0,$$

which implies  $\{\sigma_{n,2l}^{0,0}\}$  is strictly descending with respect to  $l$ . Hence, we have

$$(2.51) \quad |\sigma_{n,2l+2}^{0,0} - \sigma_{n,2l}^{0,0}| = \frac{n+2l+2}{2(l+1)(n+l+3/2)} \sigma_{n,2l}^{0,0} \leq \frac{n+2}{2n+3} \sigma_{n,0}^{0,0},$$

where we used the fact that  $n + 2l + 2/((l + 1)(n + l + 3/2))$  is strictly descending with respect to  $l$ . Then, we obtain the improved bound from (2.23):

$$(2.52) \quad \begin{aligned} |\hat{u}_n^{0,0}| &\leq \frac{M}{\rho^n} \sigma_{n,0}^{0,0} \left( 1 + \frac{n+2}{2n+3} \sum_{l=0}^{\infty} \frac{1}{\rho^{2l+2}} \right) \\ &= \frac{\sqrt{\pi}M}{\rho^n} \frac{\Gamma(n+1)}{\Gamma(n+1/2)} \left( 1 + \frac{n+2}{2n+3} \frac{1}{\rho^2-1} \right), \end{aligned}$$

and by (2.10),

$$(2.53) \quad \frac{\Gamma(n+1)}{\Gamma(n+1/2)} \leq \sqrt{n} \exp\left(\frac{8n-1}{12n(2n-1)}\right), \quad n \geq 1.$$

This completes the proof.  $\square$

*Remark 2.5.* We compare the bound in (2.49) with the existing ones. Davis [13, p. 313] stated the bound

$$|\hat{u}_n^{0,0}| \leq \frac{2n+1}{2} \frac{ML(\mathcal{E}_\rho)}{\rho^n(\rho-1)},$$

where clearly the algebraic order of  $n$  in the numerator is not optimal. The following asymptotic bound can be obtained from [29, (32), (38)] and [13, (12.4.25)]:

$$|\hat{u}_n^{0,0}| \leq \frac{M\sqrt{\pi n}}{\rho^n} \frac{\sqrt{\rho^4+1}}{\rho^2-1}, \quad n \gg 1;$$

the asymptotic estimate derived from (2.49) is

$$(2.54) \quad |\hat{u}_n^{0,0}| \leq \frac{M\sqrt{\pi n}}{\rho^n} \frac{\rho^2-1/2}{\rho^2-1}, \quad n \gg 1,$$

which is sharper. Another bound for comparison is obtained in the recent paper [42]:

$$(2.55) \quad |\hat{u}_n^{0,0}| \leq \frac{2\sqrt{n}M}{\rho^n} \left( 1 + \frac{1}{\rho^2-1} \right), \quad n \geq 1,$$

which is also inferior to our estimate (2.49). Some comparisons in numerical perspective are given in Section 4.

Like the Chebyshev case, we can derive similar refined estimates for Legendre-type expansions with  $\alpha = k, \beta = l$ , and  $k, l \in \mathbb{N}$ . The counterpart of Proposition 2.6 is stated as follows, which can be obtained by using (A.3) (with  $\alpha = \beta = 0$ ), (2.17), and the properties of Jacobi polynomials (e.g., (2.7)) as before.

**PROPOSITION 2.8.** *For any  $k, l, n, j \in \mathbb{N}$ ,*

$$(2.56) \quad \sigma_{n,j}^{k,l} = \frac{1}{\gamma_n^{k,l}} \sum_{m=n}^{n+k+l} d_m^{k,l} \gamma_m^{0,0} \sigma_{m,n+j-m}^{0,0},$$

where  $\{d_m^{k,l}\}_{m=n}^{n+k+l}$  are the same as in (A.3) and  $\{\sigma_{m,n+j-m}^{0,0}\}$  are computed by (2.21).

Once again, to fix the idea, we just consider the case  $k = 1$  and  $l = 0$ . One finds  $d_n^{1,0} = 1, d_{n+1}^{1,0} = -1$ , and

$$\sigma_{n,j}^{1,0} = \frac{1}{\gamma_n^{1,0}} (\gamma_n^{0,0} \sigma_{n,j}^{0,0} - \gamma_{n+1}^{0,0} \sigma_{n+1,j-1}^{0,0}) = \frac{n+1}{2n+1} \sigma_{n,j}^{0,0} - \frac{n+1}{2n+3} \sigma_{n+1,j-1}^{0,0}.$$

By (2.21),

$$\sigma_{n,2l}^{1,0} = \frac{n+1}{2n+1} \sigma_{n,2l}^{0,0}, \quad \sigma_{n,2l+1}^{1,0} = -\frac{n+1}{2n+3} \sigma_{n+1,2l}^{0,0}, \quad l \geq 0.$$

Therefore, with (2.50) and (2.51), the estimate (2.23) reduces to

$$\begin{aligned} |\hat{u}_n^{1,0}| &\leq \frac{M}{\rho^n} |\sigma_{n,0}^{1,0}| + \frac{M}{\rho^{n+1}} |\sigma_{n,1}^{1,0}| + \frac{M}{\rho^{n+2}} \sum_{j=0}^{\infty} |\sigma_{n,j+2}^{1,0} - \sigma_{n,j}^{1,0}| \frac{1}{\rho^j} \\ &= \frac{M}{\rho^n} \frac{n+1}{2n+1} \left( \sigma_{n,0}^{0,0} + \frac{1}{\rho^2} \sum_{l=0}^{\infty} |\sigma_{n,2l+2}^{0,0} - \sigma_{n,2l}^{0,0}| \frac{1}{\rho^{2l}} \right) \\ &\quad + \frac{M}{\rho^{n+1}} \frac{n+1}{2n+3} \left( \sigma_{n+1,0}^{0,0} + \frac{1}{\rho^2} \sum_{l=0}^{\infty} |\sigma_{n+1,2l+2}^{0,0} - \sigma_{n+1,2l}^{0,0}| \frac{1}{\rho^{2l}} \right) \\ &\leq \sigma_{n,0}^{0,0} \frac{n+1}{2n+1} \frac{M}{\rho^n} \left( 1 + \frac{n+2}{2n+3} \frac{1}{\rho^2 - 1} \right) + \sigma_{n+1,0}^{0,0} \frac{n+1}{2n+3} \frac{M}{\rho^{n+1}} \left( 1 + \frac{n+3}{2n+5} \frac{1}{\rho^2 - 1} \right). \end{aligned}$$

Working out the expressions of  $\sigma_{n,0}^{0,0}$  and  $\sigma_{n+1,0}^{0,0}$  by (2.27), we have

$$(2.57) \quad |\hat{u}_n^{1,0}| \leq \frac{M}{\rho^n} \frac{\sqrt{\pi} \Gamma(n+2)}{\Gamma(n+3/2)} \left\{ \frac{1}{2} + \frac{n+2}{2(2n+3)} \frac{1}{\rho^2 - 1} + \frac{1}{\rho} \frac{n+1}{2n+3} \left( 1 + \frac{n+3}{2n+5} \frac{1}{\rho^2 - 1} \right) \right\}.$$

Note that the ratio of the Gamma functions can be bounded as in (2.47).

The same process applies to other  $k, l \in \mathbb{N}$ , but the derivation seems tedious.

**2.4. Estimates for truncated Jacobi expansions.** Given a cut-off number  $N \geq 1$  and  $N \in \mathbb{N}$ , we define the partial sum

$$(2.58) \quad (\pi_N^{\alpha,\beta} u)(x) = \sum_{n=0}^{N-1} \hat{u}_n^{\alpha,\beta} P_n^{(\alpha,\beta)}(x),$$

where  $\{\hat{u}_n^{\alpha,\beta}\}$  are the Jacobi expansion coefficients defined in (2.15). To this end, let  $L^2_{\omega^{\alpha,\beta}}(I)$  be the weighted  $L^2$ -space on  $I = (-1, 1)$ , and its norm is denoted by  $\|\cdot\|_{\omega^{\alpha,\beta}}$ , where we drop the weight function if  $\alpha = \beta = 0$ .

Notice that  $\pi_N^{\alpha,\beta} u$  is the  $L^2_{\omega^{\alpha,\beta}}$ -projection of  $u$  upon  $\mathbb{P}_{N-1}$  (denoting the set of all algebraic polynomials of degree at most  $N-1$ ), that is,  $\pi_N^{\alpha,\beta} u$  is the best approximation to  $u$  in the norm  $\|\cdot\|_{\omega^{\alpha,\beta}}$ . With the previous bounds for the expansion coefficients, we can estimate the truncation error straightforwardly.

**THEOREM 2.9.** *For any  $u \in \mathcal{A}_\rho$  with  $\rho > 1$ ,  $\alpha, \beta > -1$ ,  $N \geq 1$ , and  $N + \alpha + \beta > 0$ , we have*

$$(2.59) \quad \|\pi_N^{\alpha,\beta} u - u\|_{\omega^{\alpha,\beta}} \leq \left( \sqrt{\frac{\pi}{2^{\alpha+\beta}}} \left( 1 + \frac{|\alpha - \beta|}{\rho} \right) + \frac{2\sqrt{\gamma_0^{\alpha,\beta}}}{\rho(\rho - 1)} \right) \frac{\tilde{C}_N M}{\rho^{N-1} \sqrt{\rho^2 - 1}},$$

where  $\gamma_0^{\alpha,\beta}$  is given in (2.8) and

$$\tilde{C}_N = \max \left\{ 1, \hat{C}_N \sqrt{C_N} \right\},$$

with  $C_N$  and  $\hat{C}_N$  being defined in (2.13) and (2.34), respectively.

*Proof.* By the orthogonality (cf. (2.7)–(2.8)) of Jacobi polynomials, we have

$$\|\pi_N^{\alpha,\beta} u - u\|_{\omega^{\alpha,\beta}}^2 = \sum_{n=N}^{\infty} |\hat{u}_n^{\alpha,\beta}|^2 \gamma_n^{\alpha,\beta}.$$

By (2.12) and (2.33), we have that for  $n \geq N \geq 1$  and  $n + \alpha + \beta > 0$ ,

$$\begin{aligned} |\hat{u}_n^{\alpha,\beta}| \sqrt{\gamma_n^{\alpha,\beta}} &\leq \hat{C}_n M \left( \frac{\sqrt{\pi}}{2^{\alpha+\beta}} \left( 1 + \frac{|\alpha - \beta|}{\rho} \right) \right. \\ &\quad \left. + \sqrt{\frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} \frac{2\sqrt{2}}{\rho(\rho - 1)}} \right) \frac{\sqrt{n}}{\rho^n} \sqrt{C_n \frac{2^{\alpha+\beta}}{n}} \\ &= M \hat{C}_n \sqrt{C_n} \left( \sqrt{\frac{\pi}{2^{\alpha+\beta}}} \left( 1 + \frac{|\alpha - \beta|}{\rho} \right) \right. \\ &\quad \left. + \sqrt{\frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} \frac{2\sqrt{2^{\alpha+\beta+1}}}{\rho(\rho - 1)}} \right) \frac{1}{\rho^n}. \end{aligned}$$

Therefore, by (2.8) and Remark 2.2, we have

$$\begin{aligned} \|\pi_N^{\alpha,\beta} u - u\|_{\omega^{\alpha,\beta}} &\leq \tilde{C}_N M \left( \sqrt{\frac{\pi}{2^{\alpha+\beta}}} \left( 1 + \frac{|\alpha - \beta|}{\rho} \right) + \frac{2\sqrt{\gamma_0^{\alpha,\beta}}}{\rho(\rho - 1)} \right) \left( \sum_{n=N}^{\infty} \frac{1}{\rho^{2n}} \right)^{1/2} \\ &= \left( \sqrt{\frac{\pi}{2^{\alpha+\beta}}} \left( 1 + \frac{|\alpha - \beta|}{\rho} \right) + \frac{2\sqrt{\gamma_0^{\alpha,\beta}}}{\rho(\rho - 1)} \right) \frac{\tilde{C}_N M}{\rho^{N-1} \sqrt{\rho^2 - 1}}. \end{aligned}$$

This ends the proof.  $\square$

*Remark 2.6.* Note that  $\{ \frac{d^l}{dx^l} P_n^{(\alpha,\beta)}(x) \}_{n \geq l}$  are mutually orthogonal with respect to  $\omega^{\alpha+l,\beta+l}$ , so we can estimate  $\|(\pi_N^{\alpha,\beta} u - u)^{(l)}\|_{\omega^{\alpha+l,\beta+l}}$  in a similar fashion.

*Remark 2.7.* Some refined estimates can be obtained from the refined bounds for special cases, e.g.,  $\alpha = \beta$  or  $\alpha = \beta = 0, -1/2$ . Here, we just state the result for the Legendre case:

$$(2.60) \quad \|\pi_N^{0,0} u - u\| \leq \left( 1 + \frac{N+2}{2N+3} \frac{1}{\rho^2 - 1} \right) \frac{\tilde{C}_N M \sqrt{\pi}}{\rho^{N-1} \sqrt{\rho^2 - 1}},$$

where  $\tilde{C}_N = \exp(\frac{8N-1}{12N(2N-1)})$ . It follows from Theorem 2.7 and the above process. Note that Xiang [42] derived the following estimate for the Legendre expansion:

$$(2.61) \quad \|\pi_N^{0,0} u - u\| \leq \frac{2\sqrt{2}M}{\rho^{N-2}(\rho - 1)^2}.$$

The estimate (2.60) is tighter.

### 3. Error estimates for the Gegenbauer–Gauss quadrature.

**3.1. Preliminaries.** The Gegenbauer–Gauss quadrature remainder (1.3)–(1.4) with the nodes being zeros of the Gegenbauer polynomial  $P_n^{(\alpha,\alpha)}(x)$  takes the form



$$(3.1) \quad E_n^{GG}[u] = \frac{\gamma_n^{\alpha,\alpha}}{\pi i} \oint_{\mathcal{E}_\rho} \frac{Q_n^{(\alpha,\alpha)}(z)}{P_n^{(\alpha,\alpha)}(z)} u(z) dz \quad \forall u \in \mathcal{A}_\rho,$$

where  $Q_n^{(\alpha,\alpha)}(z)$  is defined as in (B.2). By (B.7) and (2.18), we have

$$(3.2) \quad Q_n^{(\alpha,\alpha)}(z) = \frac{1}{2\gamma_n^{\alpha,\alpha}} \int_{-1}^1 \frac{P_n^{(\alpha,\alpha)}(x)\omega^{\alpha,\alpha}(x)}{z-x} dx = \sum_{j=0}^\infty \frac{\sigma_{n,j}^{\alpha,\alpha}}{w^{n+j+1}} = \sum_{l=0}^\infty \frac{\sigma_{n,2l}^{\alpha,\alpha}}{w^{n+2l+1}}.$$

*Remark 3.1.* We point out that (3.1)–(3.2) are also valid for the general Jacobi–Gauss quadrature. However, the error estimates heavily rely on the explicit formula of  $P_n^{(\alpha,\beta)}(z)$  with  $z \in \mathcal{E}_\rho$ , which is only available for  $\alpha = \beta$  (see Lemma 3.1 below). Hence, we merely consider the Gegenbauer–Gauss case.

As mentioned, the analysis of quadrature errors (even for the Chebyshev case) has attracted much attention (see, e.g., [12, 11, 3, 19, 14, 18, 27, 28]). Chawla and Jain [12, Theorem 5] obtained the estimate

$$(3.3) \quad |E_n^{CG}[u]| \leq \frac{2\pi M}{\rho^{2n}-1} \quad \forall u \in \mathcal{A}_\rho \quad \forall n \geq 1.$$

Hunter [27] derived the general bound

$$(3.4) \quad |E_n^{GG}[u]| \leq \frac{4 \int_{-1}^1 (1-x^2)^\alpha dx}{\rho^{2n-2}(\rho^2-1)}, \quad n \geq 1,$$

and some refined results for  $\alpha = \pm 1/2$  and  $\beta = \pm 1/2$  by expanding  $Q_n^{(\alpha,\alpha)}/P_n^{(\alpha,\alpha)}$  into the Laurent series of  $w$  in the disk enclosed by  $\mathcal{C}_\rho$  (defined in (2.1)) and manipulating the series. It is worthwhile to note that Gautschi and Varga [19] estimated the Jacobi–Gauss quadrature (with  $P_n^{(\alpha,\beta)}$  and  $Q_n^{(\alpha,\beta)}$  in place of  $P_n^{(\alpha,\alpha)}$  and  $Q_n^{(\alpha,\alpha)}$  in (3.1), respectively) by

$$(3.5) \quad |E_n^{JG}[u]| \leq \pi^{-1} \gamma_n^{\alpha,\beta} ML(\mathcal{E}_\rho) \max_{z \in \mathcal{E}_\rho} |Q_n^{(\alpha,\beta)}(z)/P_n^{(\alpha,\beta)}(z)|$$

and attempted to find the exact maximum value on the Bernstein ellipse, which was feasible for  $\alpha = \pm 1/2$  and  $\beta = \pm 1/2$  again. Some conjecture and empirical results were explored in [19] for the general Jacobi case.

Using the explicit expression of Legendre polynomials on the Bernstein ellipse (see, e.g., [13, Lemma 12.4.1]), Kambo [29] obtained the bound for the Legendre–Gauss quadrature:

$$(3.6) \quad |E_n^{LG}[u]| \leq \pi^{-1} \gamma_n^{0,0} ML(\mathcal{E}_\rho) \frac{\max_{z \in \mathcal{E}_\rho} |Q_n^{(0,0)}(z)|}{\min_{z \in \mathcal{E}_\rho} |P_n(z)|} \leq \frac{d_n M \rho^2 + 1}{\rho^{2n} \rho^2 - 2}, \quad \rho > \sqrt{2},$$

where  $0 < d_n \leq \pi$ . While this bound is only valid for  $\rho > \sqrt{2}$ , it holds for all  $n$  when compared with the asymptotic estimate (with  $n \gg 1$ ) for the Legendre–Gauss quadrature in [10].

In what follows, we aim to extend our analysis to estimate  $E_n^{GG}[u]$  in (3.1). The essential tools include the explicit formula for the Gegenbauer polynomial  $P_n^{(\alpha,\alpha)}(z)$  on  $\mathcal{E}_\rho$  derived in our recent paper [43] and the previous argument for estimating  $Q_n^{(\alpha,\alpha)}(z)$ . Let us recall the important formula stated in [43, Lemma 3.1].

LEMMA 3.1. *Let  $z = \frac{1}{2}(w + w^{-1})$ . Then we have*

$$(3.7) \quad P_n^{(\alpha,\alpha)}(z) = A_n^\alpha \sum_{k=0}^n g_k^\alpha g_{n-k}^\alpha w^{n-2k}, \quad n \geq 0, \quad \alpha > -1, \quad \alpha \neq -1/2,$$

where

$$(3.8) \quad g_0^\alpha = 1, \quad g_k^\alpha = \frac{\Gamma(k + \alpha + 1/2)}{k! \Gamma(\alpha + 1/2)}, \quad 1 \leq k \leq n, \quad \text{and} \quad A_n^\alpha = \frac{\Gamma(2\alpha + 1) \Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1) \Gamma(n + 2\alpha + 1)}.$$

Remark 3.2. This formula excludes the Chebyshev case. For  $\alpha = -1/2$ , we define

$$(3.9) \quad g_0^{-1/2} = g_n^{-1/2} = 1, \quad g_k^{-1/2} = 0, \quad 1 \leq k \leq n - 1, \quad \text{and} \quad A_n^{-1/2} = \frac{\Gamma(n + 1/2)}{2\sqrt{\pi}n!},$$

since (see, e.g., [13])

$$(3.10) \quad T_n(z) = \frac{1}{2}(w^n + w^{-n}) = \frac{1}{2A_n^{-1/2}} P_n^{(-1/2,-1/2)}(z).$$

Hence, we understand that (3.7) holds for  $\alpha = -1/2$  with the constants given by (3.9).

**3.2. Main results.** We adopt two approaches to estimate the quadrature remainder. The first one is to expand  $Q_n^{(\alpha,\alpha)}/P_n^{(\alpha,\alpha)}$  in Laurent series of  $w \in \mathcal{C}_\rho$ , and then we use an argument as for Theorem 2.4 to obtain the tight error bound. However, this situation is reminiscent of that in Gautschi and Varga [19], that is, computable bounds can be derived for general  $\alpha$ . We highlight that the computational part (see (3.11)) is independent of  $\rho$  and  $u$ .

The second approach is based on an important relation between the quadrature remainder and Gegenbauer expansion coefficient (see (3.22)).

The main estimate resulted from the first approach is stated as follows.

THEOREM 3.2. *For any  $u \in \mathcal{A}_\rho$  with  $\rho > 1$ , we have that for  $\alpha > -1$  and  $n \geq 1$ ,*

$$(3.11) \quad |E_n^{GG}[u]| \leq \gamma_n^{\alpha,\alpha} \left[ |\mu_{n,0}^{\alpha,\alpha}| + \max_{l \geq 0} |\mu_{n,2l+2}^{\alpha,\alpha} - \mu_{n,2l}^{\alpha,\alpha}| \frac{1}{\rho^2 - 1} \right] \frac{M}{\rho^{2n}},$$

where  $\{\mu_{n,2l}^{\alpha,\alpha}\}_{l \geq 0}$  are computed by the recursive formula

$$(3.12) \quad \mu_{n,2l}^{\alpha,\alpha} = \frac{1}{g_n^\alpha} \left( \frac{\sigma_{n,2l}^{\alpha,\alpha}}{A_n^\alpha} - \sum_{k=1}^{\min\{n,l\}} g_k^\alpha g_{n-k}^\alpha \mu_{n,2l-2k}^{\alpha,\alpha} \right), \quad l \geq 1, \quad \mu_{n,0}^{\alpha,\alpha} = \frac{\sigma_{n,0}^{\alpha,\alpha}}{A_n^\alpha g_n^\alpha}.$$

*Proof.* A straightforward calculation from (3.2) (note that  $\sigma_{n,2l+1}^{\alpha,\alpha} = 0$  for all  $l \geq 0$ ) and (3.7) leads to

$$(3.13) \quad \frac{Q_n^{(\alpha,\alpha)}(z)}{P_n^{(\alpha,\alpha)}(z)} = \sum_{l=0}^{\infty} \frac{\mu_{n,2l}^{\alpha,\alpha}}{w^{2n+2l+1}} \quad \text{with} \quad \sigma_{n,2l}^{\alpha,\alpha} = A_n^\alpha \sum_{k=0}^{\min\{n,l\}} g_k^\alpha g_{n-k}^\alpha \mu_{n,2l-2k}^{\alpha,\alpha},$$

so solving out  $\mu_{n,2l}^{\alpha,\alpha}$  yields (3.12).

Next, following the same lines as the derivation of (2.24), we infer from (3.1) and (3.13),

$$\begin{aligned}
 |E_n^{GG}[u]| &\leq \gamma_n^{\alpha,\alpha} \frac{M}{2\pi} \left| \sum_{l=0}^{\infty} \mu_{n,2l}^{\alpha,\alpha} \oint_{\mathcal{C}_\rho} \frac{1}{w^{2n+2l+1}} \left(1 - \frac{1}{w^2}\right) dw \right| \\
 &\leq \gamma_n^{\alpha,\alpha} \frac{M}{2\pi} \left[ \frac{2\pi\rho}{\rho^{2n+1}} |\mu_{n,0}^{\alpha,\alpha}| + \frac{2\pi\rho}{\rho^{2n+3}} \sum_{l=0}^{\infty} |\mu_{n,2l+2}^{\alpha,\alpha} - \mu_{n,2l}^{\alpha,\alpha}| \frac{1}{\rho^{2l}} \right] \\
 &\leq \gamma_n^{\alpha,\alpha} \frac{M}{\rho^{2n}} \left[ |\mu_{n,0}^{\alpha,\alpha}| + \max_{l \geq 0} |\mu_{n,2l+2}^{\alpha,\alpha} - \mu_{n,2l}^{\alpha,\alpha}| \sum_{l=0}^{\infty} \frac{1}{\rho^{2l+2}} \right] \\
 &= \gamma_n^{\alpha,\alpha} \frac{M}{\rho^{2n}} \left[ |\mu_{n,0}^{\alpha,\alpha}| + \max_{l \geq 0} |\mu_{n,2l+2}^{\alpha,\alpha} - \mu_{n,2l}^{\alpha,\alpha}| \frac{1}{\rho^2 - 1} \right].
 \end{aligned}
 \tag{3.14}$$

This completes the proof.  $\square$

*Remark 3.3.* We find from (3.12) that for  $\alpha = -1/2$ ,

$$\mu_{n,0}^{-1/2,-1/2} = \frac{2\pi}{\gamma_n^{-1/2,-1/2}}, \quad |\mu_{n,2l+2}^{-1/2,-1/2} - \mu_{n,2l}^{-1/2,-1/2}| = \frac{2\pi\delta_{\kappa,0}}{\gamma_n^{-1/2,-1/2}}, \quad \kappa := \text{mod}(l+1, n),$$

where  $\delta_{\kappa,0}$  is the Kronecker delta. Hence, it follows from (3.14) that

$$|E_n^{CG}[u]| \leq \frac{2\pi M}{\rho^{2n}} \left[ 1 + \frac{1}{\rho^2} \sum_{j=1}^{\infty} \frac{1}{\rho^{2(jn-1)}} \right] = \frac{2\pi M}{\rho^{2n} - 1}, \quad n \geq 1,$$

which is the same as (3.3) derived in [12].

*Remark 3.4.* We find from (3.12) that for  $\alpha = 1/2$ ,

$$\mu_{n,2l}^{1/2,1/2} = \begin{cases} (-1)^\kappa \frac{\pi}{2} \frac{1}{\gamma_n^{1/2,1/2}} & \text{if } \kappa := \text{mod}(l, n+1) = 0, 1, \\ 0 & \text{otherwise,} \end{cases}$$

which implies

$$\begin{aligned}
 \sum_{l=0}^{\infty} |\mu_{n,2l+2}^{1/2,1/2} - \mu_{n,2l}^{1/2,1/2}| \frac{1}{\rho^{2l}} &= |\mu_{n,2}^{1/2,1/2} - \mu_{n,0}^{1/2,1/2}| + \sum_{j=1}^{\infty} |\mu_{n,2j(n+1)}^{1/2,1/2}| \frac{1}{\rho^{2j(n+1)-2}} \\
 &\quad + \sum_{j=1}^{\infty} \left( |\mu_{n,2j(n+1)+2}^{1/2,1/2} - \mu_{n,2j(n+1)}^{1/2,1/2}| \frac{1}{\rho^{2j(n+1)}} + |\mu_{n,2j(n+1)+2}^{1/2,1/2}| \frac{1}{\rho^{2j(n+1)+2}} \right) \\
 &= \frac{\pi}{2} \frac{1}{\gamma_n^{1/2,1/2}} \left( 2 + (\rho + \rho^{-1})^2 \sum_{j=1}^{\infty} \frac{1}{\rho^{2j(n+1)}} \right) = \frac{\pi}{2} \frac{1}{\gamma_n^{1/2,1/2}} \left( 2 + \frac{(\rho + \rho^{-1})^2}{\rho^{2n+2} - 1} \right).
 \end{aligned}$$

Hence, it follows from (3.14) that for the Chebyshev–Gauss quadrature of the second kind,

$$|E_n^{GG}[u]| \leq \frac{\pi M}{2\rho^{2n}} \left( 1 + \frac{1}{\rho^2} \left( 2 + \frac{(\rho + \rho^{-1})^2}{\rho^{2n+2} - 1} \right) \right) = \frac{\pi M(\rho^2 + 2 + \rho^{-2n-4})}{2(\rho^{2n+2} - 1)}.$$

Note that Hunter [27, (4.8)] obtained the following estimate by a delicate technique:

$$(3.18) \quad |E_n^{GG}[u]| \leq \frac{\pi M(\rho^2 + 2 + \rho^{-2})}{2(\rho^{2n+2} - 1)}.$$

We see that (3.17) is sharper.

For general  $\alpha > -1$ , the derivation of an explicit bound for

$$(3.19) \quad \Theta_n^\alpha := \max_{l \geq 0} \theta_{n,l}^\alpha, \quad \theta_{n,l}^\alpha := \gamma_n^{\alpha,\alpha} |\mu_{n,2l+2}^{\alpha,\alpha} - \mu_{n,2l}^{\alpha,\alpha}|, \quad n \geq 1,$$

seems nontrivial. We have only empirical results bases on computation. Some indications are listed as follows:

- (i) Observe from (3.16) that for fixed  $n$ ,  $\{\theta_{n,l}^{1/2}\}_{l \geq 0}$  are  $(n + 1)$ -periodic (see Figure 3.1(a)), and the maximum is attained at  $l = j(n + 1), j = 0, 1, \dots$ . We compute ample samples of  $n, l$ , and  $\alpha$ , and find very similar “periodic” behaviors (see Figure 3.1(b)–(c) for  $\alpha = 0, 1$ ).
- (ii) Another interesting empirical observation is that for fixed  $\alpha$ , the maximum value  $\Theta_n^\alpha$  converges to a constant value, and it decreases as  $\alpha$  increases (see Figure 3.1(d)). Note that for the Legendre case,  $\Theta_n^0 \approx 4$ .

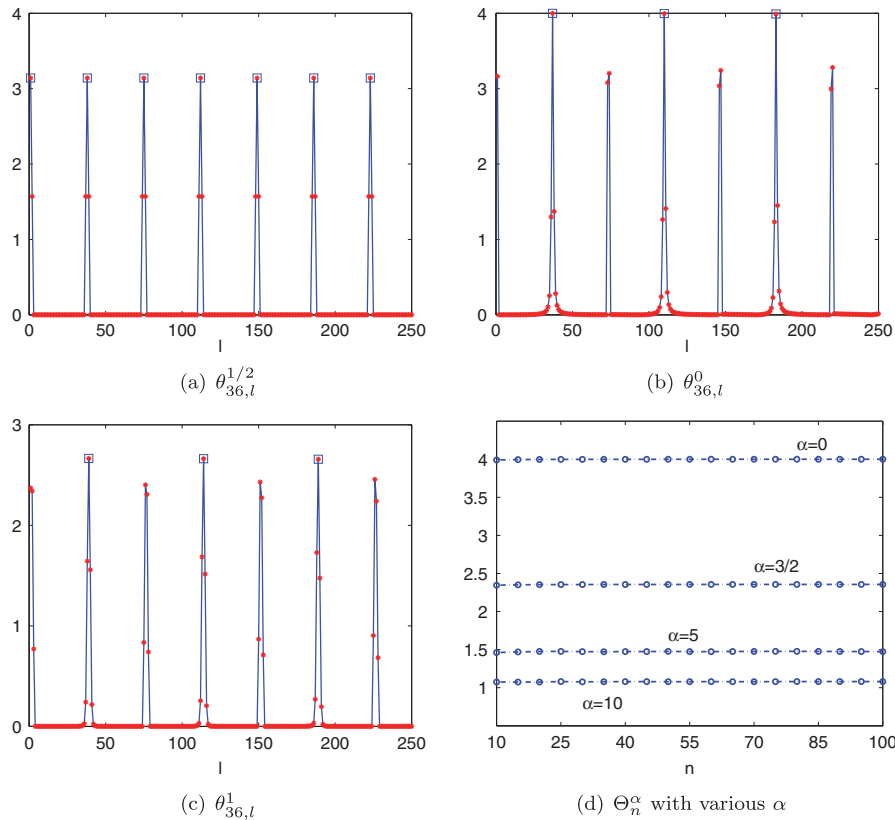


FIG. 3.1. (a)–(c) Profiles of  $\theta_{n,l}^\alpha$  with  $n = 36, \alpha = 1/2, 0, 1$ , and  $0 \leq l \leq 250$ . We indicate by  $\square$  the location where the maximum value  $\Theta_n^\alpha$  is attached. (d) The maximum value  $\Theta_n^\alpha$  with  $\alpha = 0, 3/2, 5, 10$ , and  $10 \leq n \leq 100$ , where we compute  $\{\theta_{n,l}^\alpha\}$  for  $l$  up to 1000.

Now, we turn to the second approach. The main result is summarized below.

**THEOREM 3.3.** *Let  $u \in \mathcal{A}_\rho$  with  $\rho > 1$ . Then for  $\alpha > -1$ ,  $\alpha \neq -1/2$ , and  $n \gg 1$ , we have*

$$(3.20) \quad |E_n^{GG}[u]| \leq \frac{D_n M \sqrt{\pi}}{\rho^{2n}} \left( \frac{\sqrt{\pi}}{4^\alpha} + \frac{\Gamma(\alpha+1)}{\sqrt{\Gamma(2\alpha+2)}} \frac{2\sqrt{2}}{\rho^2-1} \right) \begin{cases} (1+\rho^{-2})^{\alpha+1/2}, & \alpha > -1/2, \\ (1-\rho^{-2})^{\alpha+1/2}, & \alpha < -1/2, \end{cases}$$

and in particular, for the Legendre case,

$$(3.21) \quad |E_n^{LG}[u]| \leq \frac{D_n M \pi \sqrt{1+\rho^{-2}}}{\rho^{2n}} \left( 1 + \frac{1}{2(\rho^2-1)} \right),$$

where the constant  $D_n \approx 1$ .

*Proof.* We carry out the proof by using the important relation, due to (3.1) and (B.1),

$$(3.22) \quad |E_n^{GG}[u]| \leq \frac{\gamma_n^{\alpha,\alpha}}{\min_{z \in \mathcal{E}_\rho} |P_n^{(\alpha,\alpha)}(z)|} \left| \frac{1}{\pi i} \oint_{\mathcal{E}_\rho} Q_n^{(\alpha,\alpha)}(z) u(z) dz \right| = \frac{\gamma_n^{\alpha,\alpha} |\hat{u}_n^{\alpha,\alpha}|}{\min_{z \in \mathcal{E}_\rho} |P_n^{(\alpha,\alpha)}(z)|}$$

for  $n \geq 1$ . Since the numerator has been estimated in Theorem 2.5 (also see (2.35)), it suffices to deal with the denominator.

By [43, (4.6)], we have

$$(3.23) \quad |P_n^{(\alpha,\alpha)}(z)| \geq \tilde{D}_n |A_n^\alpha| \frac{n^{\alpha-1/2} \rho^n}{|\Gamma(\alpha+1/2)|} \begin{cases} (1+\rho^{-2})^{-\alpha-1/2} & \text{if } \alpha > -1/2, \\ (1-\rho^{-2})^{-\alpha-1/2} & \text{if } \alpha < -1/2, \end{cases} \\ \geq \frac{\tilde{D}_n}{\Upsilon_n^{2\alpha+1,\alpha+1}} \frac{4^\alpha \rho^n}{\sqrt{\pi n}} \begin{cases} (1+\rho^{-2})^{-\alpha-1/2} & \text{if } \alpha > -1/2, \\ (1-\rho^{-2})^{-\alpha-1/2} & \text{if } \alpha < -1/2, \end{cases}$$

where  $\tilde{D}_n \approx 1$  for  $n \gg 1$  and  $\Upsilon_n^{2\alpha+1,\alpha+1}$  is defined in (2.11). Note that in the last step, we dealt with  $|A_n^\alpha|$  in (3.8) as

$$|A_n^\alpha| = \frac{|\Gamma(2\alpha+1)|}{\Gamma(\alpha+1)} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+2\alpha+1)} = \frac{4^\alpha |\Gamma(\alpha+1/2)|}{\sqrt{\pi}} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+2\alpha+1)} \\ \geq \frac{4^\alpha}{\Upsilon_n^{2\alpha+1,\alpha+1}} \frac{|\Gamma(\alpha+1/2)|}{\sqrt{\pi n^\alpha}},$$

where we used Lemma 2.1 and the property of the Gamma function (see [1]):

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z} \sqrt{\pi} \Gamma(2z).$$

Therefore, a combination of (2.12), (2.35), and (3.22)–(3.23) leads to the desired result, where we denote

$$D_n = \frac{\Upsilon_n^{2\alpha+1,\alpha+1}}{\tilde{D}_n} C_n \bar{C}_n$$

and recall that  $C_n$  and  $\bar{C}_n$  are defined in (2.13) (with  $\alpha = \beta$ ) and (2.36), respectively.

The use of the refined estimate (2.49) yields (3.21).  $\square$

*Remark 3.5.* Note that the constant  $\tilde{D}_n$  in (3.23) also depends on  $\alpha$ ,  $\rho$ , and a small parameter  $\varepsilon > 0$ . Asymptotically, it behaves like  $1 + O(n^{\varepsilon-1})$  (see [43, Theorem 4.1]). However, it is involved and tedious to track the explicit dependence on the parameters, so we just state the asymptotic results in the above theorem.

**4. Numerical results and comparisons.** In this section, we present various numerical results to show the tightness of the bounds derived in this paper and to compare them with other existing ones mentioned in the previous part.

In the first example, we purposely choose the Chebyshev and Legendre expansions with known expansion coefficients,

$$(4.1) \quad u_1(x) = \frac{3}{5-4x} = T_0(x) + \sum_{n=1}^{\infty} \frac{T_n(x)}{2^{n-1}}, \quad u_2(x) = \frac{2}{\sqrt{5-4x}} = \sum_{n=0}^{\infty} \frac{P_n(x)}{2^n},$$

which follow from generating functions of Chebyshev and Legendre polynomials (cf. [36]).

Note that the function  $u_1$  has a simple pole at  $z = 5/4$ , so the semi-major axis (cf. (2.3)) should satisfy

$$1 < a = (\rho + \rho^{-1})/2 < 5/4 \Rightarrow 1 < \rho < 2.$$

One also verifies that

$$M = \max_{z \in \mathcal{E}_\rho} |u_1(z)| = \frac{3\rho}{(2\rho - 1)(2 - \rho)}.$$

Then the estimate (2.40) reduces to

$$\hat{u}_n^C = \frac{1}{2^{n-1}} \leq \frac{6}{(2\rho - 1)(2 - \rho)\rho^{n-1}} := B_n^C(\rho), \quad 1 < \rho < 2, \quad n \geq 1.$$

Similarly, for the Legendre expansion of  $u_2$ , the result (2.49) becomes

$$\hat{u}_n^{0,0} = \frac{1}{2^n} \leq \frac{\sqrt{\pi n}}{\rho^n} \left( 1 + \frac{n+2}{2n+3} \frac{1}{\rho^2 - 1} \right) \exp\left( \frac{8n-1}{12n(2n-1)} \right) \sqrt{\frac{4\rho}{(2\rho-1)(2-\rho)}} := B_n^L(\rho)$$

for  $1 < \rho < 2$  and  $n \geq 1$ .

We take  $\rho = 1.98$ , and plot the exact coefficients  $\hat{u}_n^C$  and  $\hat{u}_n^{0,0}$  and the bounds  $B_n^C$  and  $B_n^L$  in Figure 4.1(a) and (b), respectively. Actually, the bound for the Chebyshev case (see (1.1)) can be considered as one benchmark for illustrating tightness of the upper bound. Indeed, the result for the Legendre case stated in Theorem 2.7 seems as sharp as that for the Chebyshev case.

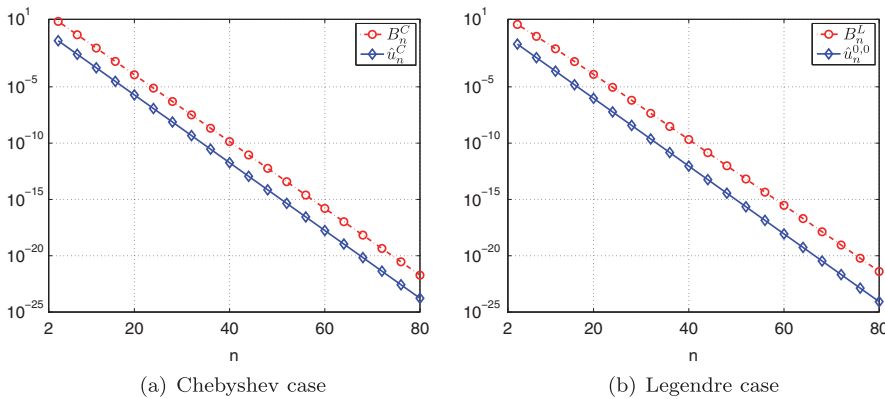


FIG. 4.1. Expansion coefficients of  $u_1, u_2$  in (4.1) against their error bounds.

Next, we compare the bounds for the Legendre expansion coefficients in Theorem 2.7 and (2.55) (obtained by [42]). For clarity, we drop the common part  $M\sqrt{n}/\rho^n$  and denote the remaining factors in the upper bounds (2.49) and (2.55) by

$$b_n(\rho) = \sqrt{\pi} \left( 1 + \frac{n+2}{2n+3} \frac{1}{\rho^2-1} \right) \exp \left( \frac{8n-1}{12n(2n-1)} \right), \quad \tilde{b}(\rho) = 2 \left( 1 + \frac{1}{\rho^2-1} \right).$$

In Figure 4.2(a), we plot the difference  $e_n(\rho) := \tilde{b}(\rho) - b_n(\rho)$  for various  $\rho$  and  $1 \leq n \leq 80$ . We see that  $e_n(\rho) > 0$  and the difference is of magnitude around 6, when  $\rho$  is close to 1. Moreover, for fixed  $\rho$ , the difference is roughly a constant for slightly large  $n$ . In Figure 4.2(b), we plot some sample  $e_n(\rho)$  for  $\rho$  close to 1, and we see that our bound is much sharper.

We next make a similar comparison of bounds for Jacobi and Gegenbauer expansions. For example, for  $\alpha = 1$  and  $\beta = 0$ , we extract the factors in (1.2) and (2.57) by dropping  $M\sqrt{n}/\rho^n$ . We plot in Figure 4.3(a) the difference of two remaining parts (i.e., that of (1.2) subtracts that of (2.57)). Once again, our bound is much tighter. Likewise, we depict in Figure 4.3(b) the extracted bounds from (1.2) and (2.28) with

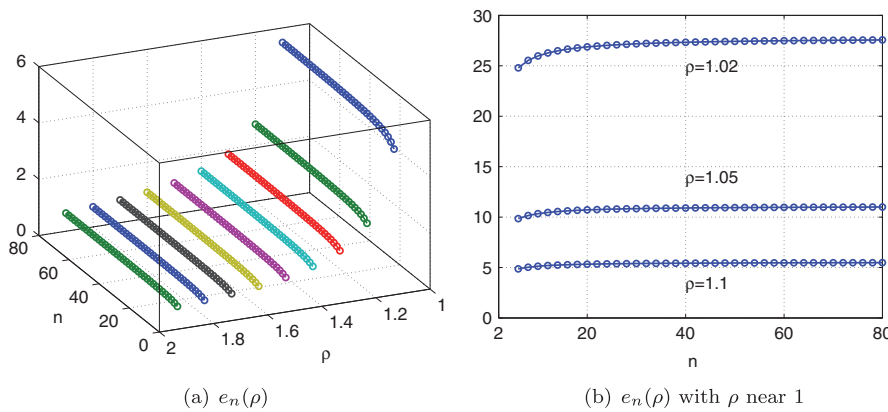


FIG. 4.2. (a) Comparison of error bounds for Legendre expansions in (2.49) and (2.55). (b) Samples of  $e_n(\rho)$  for  $\rho$  close to 1.

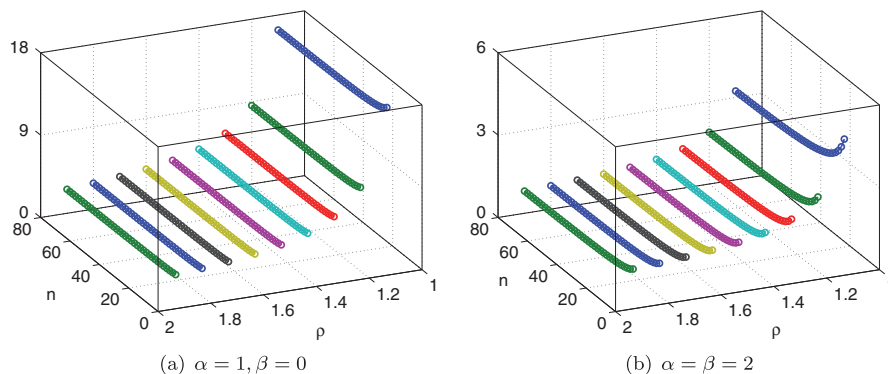


FIG. 4.3. (a) Comparison of error bounds for Jacobi expansion with  $\alpha = 1, \beta = 0$  in (1.2) and (2.57). (b) Comparison of error bounds for Gegenbauer expansion with  $\alpha = \beta = 2$  in (1.2) and (2.28).

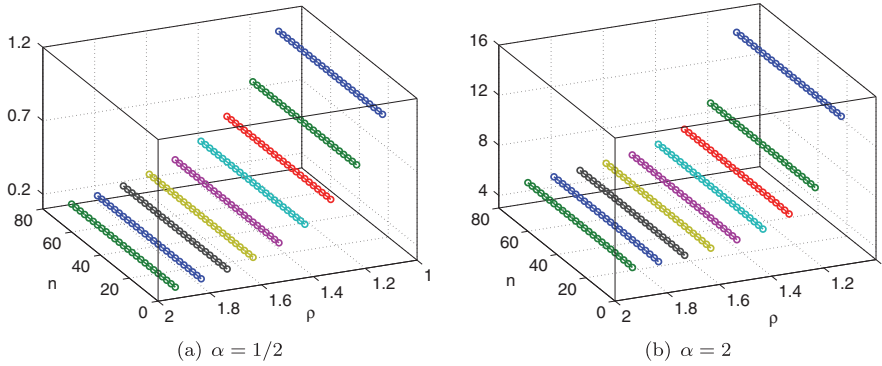


FIG. 4.4. (a) Comparison of error bounds for the Gegenbauer–Gauss quadrature with  $\alpha = 1/2$  in (3.18) and (3.17). (b) Comparison of error bounds for the Gegenbauer–Gauss quadrature with  $\alpha = 2$  in (3.4) and (3.11).

$\alpha = \beta = 2$ . The situation is akin to the Legendre case, where the bounds obtained in this paper are sharper.

Finally, we turn to the comparison of error bounds for the Gegenbauer–Gauss quadrature remainder. For  $\alpha = 1/2$ , we extract the factors in (3.17) and (3.18) by dropping  $M/\rho^{2n}$  as before. We plot in Figure 4.4(a) the difference of two remaining parts in (3.18) and in (3.17). Once again, our bound is much tighter. Likewise, we depict in Figure 4.4(b) the extracted bounds from (3.4) and (3.11) with  $\alpha = 2$ , and observe similar behaviors.

**Concluding remarks.** In this paper, we derived various new and sharp error bounds for Jacobi polynomial expansions and the Gegenbauer–Gauss quadrature of analytic functions with analyticity characterized by the Bernstein ellipse. We adopted an argument that could recover the best known bounds and attempted to make the dependence of the estimates on the parameters explicitly. Both analytic estimates and numerical comparisons with available ones demonstrated the sharpness of the error bounds.

**Appendix A. Jacobi polynomials.** We collect some properties of Jacobi polynomials used in the paper. For  $\alpha, \beta > -1$ , the Jacobi polynomials (see, e.g., [36]), denoted by  $P_n^{(\alpha, \beta)}(x)$ ,  $x \in I := (-1, 1)$ , are defined by the Rodrigues formula

$$(A.1) \quad (1-x)^\alpha(1+x)^\beta P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \left[ (1-x)^{\alpha+n}(1+x)^{\beta+n} \right], \quad n \geq 0.$$

The Jacobi polynomials satisfy

$$(A.2a) \quad (1-x)P_n^{(\alpha+1, \beta)}(x) = \frac{2}{2n + \alpha + \beta + 2} \left( (n + \alpha + 1)P_n^{(\alpha, \beta)}(x) - (n + 1)P_{n+1}^{(\alpha, \beta)}(x) \right),$$

$$(A.2b) \quad (1+x)P_n^{(\alpha, \beta+1)}(x) = \frac{2}{2n + \alpha + \beta + 2} \left( (n + \beta + 1)P_n^{(\alpha, \beta)}(x) + (n + 1)P_{n+1}^{(\alpha, \beta)}(x) \right).$$

As a direct consequence of (A.2), we have that for any  $k, l \in \mathbb{N} = \{0, 1, \dots\}$ ,

$$(A.3) \quad (1-x)^k(1+x)^l P_n^{(\alpha+k, \beta+l)}(x) = \sum_{i=n}^{n+k+l} d_i^{\alpha+k, \beta+l} P_i^{(\alpha, \beta)}(x),$$



where  $\{d_i^{\alpha+k, \beta+l}\}_{i=n}^{n+k+l}$  is a unique set of constants (with  $d_n^{\alpha, \beta} = 1$ ), computed from (A.2) recursively. Here, we sketch the proof of (A.3). To this end, let  $\{c_j\}$  be a set of generic constants. Using (A.2a) and (A.2b) repeatedly leads to

$$\begin{aligned} & (1-x)^k(1+x)^l P_n^{(\alpha+k, \beta+l)}(x) \\ &= (1-x)^{k-1}(1+x)^l (c_1 P_n^{(\alpha+k-1, \beta+l)}(x) + c_2 P_{n+1}^{(\alpha+k-1, \beta+l)}(x)) \\ &= \dots = (1+x)^l \sum_{m=n}^{n+k} c_m P_m^{(\alpha, \beta+l)}(x) = \dots = \sum_{m=n}^{n+k+l} c_m P_m^{(\alpha, \beta)}(x). \end{aligned}$$

This yields (A.3). We point out that for  $\alpha = \beta = 0$ ,  $\{(1-x)^k(1+x)^l P_n^{(k, l)}(x)\}$  (up to a certain constant factor) are defined as generalized Jacobi polynomials in [24].

The following formula, derived from [2, Lemma 7.1.1] (also see [35, Theorem 3.21]), was used for the derivation of (2.22):

$$\begin{aligned} \text{(A.4)} \quad \hat{c}_j^n &:= \hat{c}_j^n(\alpha, \beta, a, b) = \frac{1}{\gamma_n^{\alpha, \beta}} \int_{-1}^1 P_{n+j}^{(a, b)}(x) P_n^{(\alpha, \beta)}(x) \omega^{\alpha, \beta}(x) dx \\ &= \frac{\Gamma(n+j+a+1)}{\Gamma(n+j+a+b+1)} \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)} \\ &\quad \times \sum_{m=0}^j \frac{(-1)^m \Gamma(2n+j+m+a+b+1)\Gamma(n+m+\alpha+1)}{m!(j-m)!\Gamma(n+m+a+1)\Gamma(2n+m+\alpha+\beta+2)} \end{aligned}$$

for  $a, b, \alpha, \beta > -1$  and  $n, j \geq 0$ .

Let  $T_n(x) = \cos(n \arccos(x))$  be the Chebyshev polynomial of the first kind of degree  $n$ . Then the second-kind Chebyshev polynomial, denoted by  $U_n(x)$ , can be expressed by

$$\text{(A.5)} \quad U_n(x) = \frac{\sin((n+1) \arccos(x))}{\sqrt{1-x^2}} = \frac{T'_{n+1}(x)}{n+1} = \sqrt{\frac{\pi}{2}} \frac{P_n^{(1/2, 1/2)}(x)}{\sqrt{\gamma_n^{1/2, 1/2}}}.$$

The Chebyshev polynomials enjoy the following important properties:

$$\text{(A.6a)} \quad P_n^{(-1/2, -1/2)}(x) = P_n^{(-1/2, -1/2)}(1)T_n(x) = \frac{\Gamma(n+1/2)}{\sqrt{\pi}n!}T_n(x),$$

$$\text{(A.6b)} \quad T'_n(x) = 2n \sum_{\substack{k=0 \\ k+n \text{ odd}}}^{n-1} \frac{1}{c_k} T_k(x),$$

where  $c_0 = 2$  and  $c_k = 1$  for  $k \geq 1$ .

**Appendix B. Proof of Lemma 2.2.** We first show that

$$\text{(B.1)} \quad \hat{u}_n^{\alpha, \beta} = \frac{1}{\pi i} \oint_{\mathcal{E}_\rho} Q_n^{(\alpha, \beta)}(z) u(z) dz,$$

where

$$\text{(B.2)} \quad Q_n^{(\alpha, \beta)}(z) := \frac{1}{2\gamma_n^{\alpha, \beta}} \int_{-1}^1 \frac{P_n^{(\alpha, \beta)}(x) \omega^{\alpha, \beta}(x)}{z-x} dx,$$

and  $\gamma_n^{\alpha, \beta}$  is given by (2.8). Recall the Cauchy integral formula:

$$(B.3) \quad \frac{d^n}{dx^n} u(x) = \frac{n!}{2\pi i} \oint_{\mathcal{E}_\rho} \frac{u(z)}{(z-x)^{n+1}} dz.$$

Using (1.2), the Rodrigues formula (A.1), (B.3), and integration by parts leads to

$$(B.4) \quad \begin{aligned} \hat{u}_n^{\alpha,\beta} &= \frac{1}{\gamma_n^{\alpha,\beta}} \int_{-1}^1 u(x) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx \\ &= \frac{1}{\gamma_n^{\alpha,\beta}} \frac{(-1)^n}{2^n n!} \int_{-1}^1 \omega^{\alpha+n,\beta+n}(x) \frac{d^n}{dx^n} u(x) dx \\ &= \frac{1}{\gamma_n^{\alpha,\beta}} \frac{1}{2^n n!} \int_{-1}^1 \left( \frac{n!}{2\pi i} \oint_{\mathcal{E}_\rho} \frac{u(z)}{(z-x)^{n+1}} dz \right) \omega^{\alpha+n,\beta+n}(x) dx \\ &= \frac{1}{2^n \gamma_n^{\alpha,\beta}} \frac{1}{2\pi i} \oint_{\mathcal{E}_\rho} \left( \int_{-1}^1 \frac{\omega^{\alpha+n,\beta+n}(x)}{(z-x)^{n+1}} dx \right) u(z) dz. \end{aligned}$$

We find from integration by parts that

$$(B.5) \quad \int_{-1}^1 \frac{\omega^{\alpha+n,\beta+n}(x)}{(z-x)^{n+1}} dx = \frac{(-1)^n}{n!} \int_{-1}^1 \frac{1}{z-x} \frac{d^n}{dx^n} \omega^{\alpha+n,\beta+n}(x) dx.$$

Inserting (B.5) into (B.4), we derive from the Rodrigues formula (A.1) that

$$\hat{u}_n^{\alpha,\beta} = \frac{1}{2\pi i} \frac{1}{\gamma_n^{\alpha,\beta}} \oint_{\mathcal{E}_\rho} \left( \int_{-1}^1 \frac{\omega^{\alpha,\beta}(x) P_n^{(\alpha,\beta)}(x)}{z-x} dx \right) u(z) dz = \frac{1}{\pi i} \oint_{\mathcal{E}_\rho} Q_n^{(\alpha,\beta)}(z) u(z) dz,$$

where  $Q_n^{(\alpha,\beta)}(z)$  is given in (B.2).

Since  $z = (w + w^{-1})/2$ , we have from the generating function of the Chebyshev polynomial of the second kind (cf. [1]) that

$$(B.6) \quad \frac{1}{z-x} = \frac{2}{w} \frac{1}{w^{-2} - 2xw^{-1} + 1} = \frac{2}{w} \sum_{k=0}^{\infty} \frac{U_k(x)}{w^k}.$$

Inserting it into (B.2), we find from the orthogonality of the Jacobi polynomials (cf. (2.7)) that

$$(B.7) \quad \begin{aligned} Q_n^{(\alpha,\beta)}(z) &= \frac{1}{\gamma_n^{\alpha,\beta}} \sum_{k=0}^{\infty} \frac{1}{w^{k+1}} \int_{-1}^1 U_k(x) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx \\ &= \frac{1}{\gamma_n^{\alpha,\beta}} \sum_{k=n}^{\infty} \frac{1}{w^{k+1}} \int_{-1}^1 U_k(x) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx \\ &= \frac{1}{\gamma_n^{\alpha,\beta}} \sum_{j=0}^{\infty} \frac{1}{w^{n+j+1}} \int_{-1}^1 U_{n+j}(x) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx \\ &= \sum_{j=0}^{\infty} \frac{\sigma_{n,j}^{\alpha,\beta}}{w^{n+j+1}}, \end{aligned}$$

where we defined

$$\sigma_{n,j}^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} \int_{-1}^1 U_{n+j}(x) P_n^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx.$$

Substituting the last identity of (B.7) into (B.1) leads to the desired formula (2.16).  $\square$

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