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A time-dependent busy period queue length formula for the $M/E_k/1$ queue

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Abstract

In this paper, a closed-form time-dependent busy period queue length probability is obtained for the $M/E_k/1$ queue. This probability is frequently needed when we compare the length of the busy period and the maximum amount of service that can be rendered to the existing customers. The transient probability is given in terms of the generalized modified Bessel function of the second type of Griffiths et al. [1]. The queue length probability for the $M/M/1$ queue is also presented as a special case.

Keywords: Transient analysis, Markovian queue, Erlang distribution

Mathematics Subject Classification: 60K25

1. Introduction

Transient queue length probabilities play an important role in understanding queueing systems. However, transient solutions are not many due to mathematical complexities. Readers are referred to Leonenko et al. [3] and the references therein for a comprehensive survey on the transient analysis of Markovian queueing systems.

Early studies on the transient queue length distribution of the $M/E_k/1$ queue were made by Luchak [4] and Saaty [5]. Recently, Griffiths et al. [2] derived the closed-form transient queue length distribution in terms of the *generalized modified Bessel function of the second type* defined as

$$\tilde{I}_n^{k,i}(t) = \left(\frac{t}{2}\right)^{n+k-i} \sum_{r=0}^{\infty} \frac{\left(\frac{t}{2}\right)^{r(k+1)}}{(k(r+1)-i)!(n+r)!}, \quad (1)$$

where, $t \geq 0$, $n = 0, \pm 1, \pm 2, \dots$, $k = 1, 2, \dots$, and $i = 1, 2, \dots, k$.

In this paper, we derive the closed-form transient busy period queue length probability of the $M/E_k/1$ queue. Our transient formula is different from those of earlier works in that ours is the queue length

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probability within a busy period. This probability is frequently needed when we compare the length of the busy period and the maximum amount of service that can be rendered to the existing customers. We also present the transient mean waiting time using the queue length distribution.

2. Main results

2.1. Queue length distribution

We consider the $M/E_k/1$ queue in which customers arrive according to the Poisson process with rate λ and the service times follow iid Erlang distributions of order k with mean service time $E(S) = 1/\mu$. Let $N(t)$ be the number of customers in the system at time t (including the one in service) and $J(t)$ be the remaining service phases of the customer in service at time t . We assume that a customer begins to be served at time $t = 0$. Thus we have $J(0) = k$. Let us define the first passage time τ_j as

$$\tau_j = \inf\{t > 0, N(t) = 0 | N(0) = j, J(0) = k\}.$$

We note that τ_j is the length of the busy period that starts with j customers in the system.

Let us define the following probabilities:

$$Q_{n,i}^{(j)}(t) = Pr[N(t) = n, J(t) = i, \tau_j > t | N(0) = j, J(0) = k], \quad (n, j \geq 1), \quad (1 \leq i \leq k), \quad (t > 0),$$

$$Q_0^{(j)}(t) = Pr[\tau_j < t | N(0) = j, J(0) = k], \quad (j \geq 1), \quad (t > 0).$$

The main purpose of this paper is to obtain $Q_{n,i}^{(j)}(t)$ in a closed form.

Since $\{N(t), J(t)\}$ is a continuous time Markov chain with absorbing barrier at $N(\tau_j) = 0$, it is not difficult to setup the following system equations :

$$\begin{cases} \frac{d}{dt} Q_0^{(j)}(t) = k\mu Q_{1,1}^{(j)}(t), & n = 0, \\ \frac{d}{dt} Q_{1,i}^{(j)}(t) = -(\lambda + k\mu)Q_{1,i}^{(j)}(t) + k\mu Q_{1,i+1}^{(j)}(t), & n = 1, \quad 1 \leq i \leq k-1, \\ \frac{d}{dt} Q_{1,k}^{(j)}(t) = -(\lambda + k\mu)Q_{1,k}^{(j)}(t) + k\mu Q_{2,1}^{(j)}(t), & n = 1, \quad i = k, \\ \frac{d}{dt} Q_{n,i}^{(j)}(t) = -(\lambda + k\mu)Q_{n,i}^{(j)}(t) + \lambda Q_{n-1,i}^{(j)}(t) + k\mu Q_{n,i+1}^{(j)}(t), & n \geq 2, \quad 1 \leq i \leq k-1, \\ \frac{d}{dt} Q_{n,k}^{(j)}(t) = -(\lambda + k\mu)Q_{n,k}^{(j)}(t) + \lambda Q_{n-1,k}^{(j)}(t) + k\mu Q_{n+1,1}^{(j)}(t), & n \geq 2, \quad i = k. \end{cases} \quad (2)$$

To solve Eq. (2), we define the generating function used in Griffiths et al. [1][2] as

$$G^{(j)}(z, t) = \sum_{n=1}^{\infty} \sum_{i=1}^k z^{(n-1)k+i} Q_{n,i}^{(j)}(t), \quad (t > 0). \quad (3)$$

We note that $G^{(j)}(z, t)$ is analytic within and on the unit circle $|z| = 1$.

Multiplying Eq. (2) by $z^{(n-1)k+i}$ and summing over all n and i , we obtain

$$\frac{\partial}{\partial t} G^{(j)}(z, t) = G^{(j)}(z, t) \phi(z) - k\mu Q_{1,1}^{(j)}(t), \quad (t > 0), \quad (4)$$

where, $\phi(z) = \lambda z^k - (\lambda + k\mu) + \frac{k\mu}{z}$.

Since the busy period starts with j customers, the initial condition is given by

$$G^{(j)}(z, 0) = z^{j \cdot k}. \quad (5)$$

Solving Eq. (4) with the above initial condition, we easily get

$$\begin{aligned} G^{(j)}(z, t) &= e^{\phi(z)t} z^{jk} - k\mu e^{\phi(z)t} \int_0^t Q_{1,1}^{(j)}(u) e^{-\phi(z)u} du \\ &= \left[e^{t(\lambda z^k + \frac{k\mu}{z})} z^{jk} - k\mu \int_0^t Q_{1,1}^{(j)}(u) e^{u(\lambda + k\mu)} e^{(t-u)(\lambda z^k + \frac{k\mu}{z})} du \right] e^{-t(\lambda + k\mu)}, \quad (t > 0). \end{aligned} \quad (6)$$

From the result in Griffiths et al. [1][2], we have the following equality

$$e^{t(\lambda z^k + \frac{k\mu}{z})} = \sum_{n=-\infty}^{\infty} \sum_{i=1}^k (\beta z)^{k(n-1)+i} \cdot \tilde{\mathbf{I}}_n^{k,i}(\alpha t), \quad (7)$$

where, $\alpha = 2^{k+1} \sqrt{\lambda(k\mu)^k}$ and $\beta = \sqrt[k+1]{\frac{\lambda}{k\mu}}$.

Remark 1. Note that Eq. (7) corrects the error in Griffiths et al. [2]. We show the proof in Appendix A.

Now, we apply Eq. (7) to Eq. (6) and obtain

$$\begin{aligned}
G^{(j)}(z, t) &= \sum_{n=1}^{\infty} \sum_{i=1}^k z^{(n-1)k+i} Q_{n,i}^{(j)}(t) \\
&= \left[e^{t(\lambda z^k + \frac{k\mu}{z})} z^{jk} - k\mu \int_0^t Q_{1,1}^{(j)}(u) e^{u(\lambda+k\mu)} e^{(t-u)(\lambda z^k + \frac{k\mu}{z})} du \right] e^{-t(\lambda+k\mu)} \\
&= \left[\sum_{n=1}^{\infty} \sum_{i=1}^k (\beta z)^{k(n-1)+i} \cdot z^{jk} \cdot \tilde{\mathbf{I}}_n^{k,i}(\alpha t) \right. \\
&\quad \left. - k\mu \int_0^t Q_{1,1}^{(j)}(u) e^{u(\lambda+k\mu)} \sum_{n=1}^{\infty} \sum_{i=1}^k (\beta z)^{k(n-1)+i} \cdot \tilde{\mathbf{I}}_n^{k,i}[\alpha(t-u)] du \right] \cdot e^{-t(\lambda+k\mu)} \\
&= \left[\sum_{n=1}^{\infty} \sum_{i=1}^k \beta^{k(n-j-1)+i} \cdot z^{k(n-1)+i} \cdot \tilde{\mathbf{I}}_{n-j}^{k,i}(\alpha t) \right. \\
&\quad \left. - k\mu \int_0^t Q_{1,1}^{(j)}(u) e^{u(\lambda+k\mu)} \sum_{n=1}^{\infty} \sum_{i=1}^k (\beta z)^{k(n-1)+i} \cdot \tilde{\mathbf{I}}_n^{k,i}[\alpha(t-u)] du \right] e^{-t(\lambda+k\mu)},
\end{aligned} \tag{8}$$

from which we get

$$\begin{aligned}
Q_{n,i}^{(j)}(t) &= \left[\beta^{k(n-j-1)+i} \cdot \tilde{\mathbf{I}}_{n-j}^{k,i}(\alpha t) - k\mu \int_0^t Q_{1,1}^{(j)}(u) e^{u(\lambda+k\mu)} \beta^{k(n-1)+i} \cdot \tilde{\mathbf{I}}_n^{k,i}[\alpha(t-u)] du \right] e^{-t(\lambda+k\mu)}, \\
&\quad (n, j \geq 1), (1 \leq i \leq k), (t > 0).
\end{aligned} \tag{9}$$

Now, it remains to determine the probability $Q_{1,1}^{(j)}(t)$ to complete Eq. (9). Using the result in Luchak [4], we obtain

$$Q_{1,1}^{(j)}(t) = \frac{j \cdot k}{k\mu t} \left[\frac{(k\mu t)^{j \cdot k}}{(j \cdot k)!} + \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \cdot \frac{(k\mu t)^{k(n+j)}}{[k(n+j)]!} \right] \cdot e^{-(1+\frac{\lambda}{k\mu})k\mu t}. \tag{10}$$

Using the first equality in Eq. (2), we also get

$$\frac{d}{dt} Q_0^{(j)}(t) = k\mu Q_{1,1}^{(j)}(t) = \frac{j \cdot k}{t} \left[\frac{(k\mu t)^{j \cdot k}}{(j \cdot k)!} + \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \cdot \frac{(k\mu t)^{k(n+j)}}{[k(n+j)]!} \right] \cdot e^{-(1+\frac{\lambda}{k\mu})k\mu t}. \tag{11}$$

We again note that $Q_0^{(j)}(t) = Pr(\tau_j < t)$. Thus above $\frac{d}{dt} Q_0^{(j)}(t)$ is the probability density function of the length of the busy period of the $M/E_k/1$ queue that starts with j initial customers.

2.2. Mean waiting time

In this section, we derive the transient mean waiting time $W_q^{(j)}(t)$ at time t in a busy period. Let $m^{(j)}(t)$ be the expected number of the service phases at time t in a busy period under the condition that the busy period starts with j initial customers at time 0. Then, using Eq. (6), we have

$$m^{(j)}(t) = \left. \frac{d}{dz} G^{(j)}(z, t) \right|_{z=1} = k(\lambda - \mu)t + j \cdot k - k^2 \cdot \mu \cdot (\lambda - \mu) \cdot \int_0^t (t - u) Q_{1,1}^{(j)}(u) du, \quad (j \geq 1), \quad (t > 0). \quad (12)$$

We note that customers can be served only if they arrive in the busy system at time t . Therefore, the transient mean waiting time needs to be normalized. Since it takes exponentially distributed random time (with mean $1/k\mu$) to finish each service phase, we have

$$W_q^{(j)}(t) = \frac{m^{(j)}(t)}{k \cdot \mu \cdot \tilde{Q}_0^{(j)}(t)} = \frac{\frac{\lambda}{\mu}t + \frac{j}{\mu} - t - (\lambda - \mu) \cdot k \cdot \int_0^t (t - u) Q_{1,1}^{(j)}(u) du}{\tilde{Q}_0^{(j)}(t)}, \quad (j \geq 1), \quad (t > 0), \quad (13)$$

where, $\tilde{Q}_0^{(j)}(t) = [1 - Q_0^{(j)}(t)]$ and can be obtained by using Eq. (11).

Remark 2. We again note that $\tilde{Q}_0^{(j)}(t)$ is the probability that the system is busy at time t and takes the role of the normalization constant.

Remark 3. Therefore, $W_q^{(j)}(t)$ is the conditional mean waiting time which only counts the customers who are served by the server.

3. A special case: Transient busy period queue length probability of the $M/M/1$ queue

In this section, we derive the transient busy period queue length distribution of $M/M/1$ queue by using Eq. (9) and compare it with the previous results in Luchak [4] and Saaty [5].

For the $M/M/1$ queue with arrival rate λ and service rate μ , let us define the following probabilities:

$$\tilde{P}_n^{(j)}(t) = Pr[N(t) = n, \tau_j > t | N(0) = j], \quad (n, j \geq 1), \quad (t > 0),$$

$$\tilde{P}_0^{(j)}(t) = Pr[\tau_j \leq t | N(0) = j], \quad (j \geq 1), \quad (t > 0).$$

From Luchak [4] and Saaty [5], it is known that

$$\tilde{P}_n^{(j)}(t) = \left(\sqrt{\frac{\mu}{\lambda}} \right)^{j-n} e^{-(\lambda+\mu)t} \left[I_{n-j} \left(2\sqrt{\lambda\mu} t \right) - I_{n+j} \left(2\sqrt{\lambda\mu} t \right) \right], \quad (n, j \geq 1), \quad (t > 0), \quad (14)$$

$$\frac{d}{dt} \tilde{P}_0^{(j)}(t) = \frac{j \left(\sqrt{\mu/\lambda} \right)^j e^{-(\lambda+\mu)t} \cdot I_j \left(2\sqrt{\mu\lambda} t \right)}{t}, \quad (j \geq 1), \quad (t > 0), \quad (15)$$

where, $I_n(t)$ is the modified Bessel function of the first kind defined by

$$I_n(t) = \sum_{r=0}^{\infty} \frac{(t/2)^{2r+n}}{r!(r+n)!}.$$

Now, we derive Eq. (14) from Eq. (9). Using $k = 1$ and $i = 1$ in Eq. (9) and we get

$$\begin{aligned} \tilde{P}_n^{(j)}(t) &= \left[Q_{n,i}^{(j)}(t) \right]_{k=1, i=1} \\ &= e^{-(\lambda+\mu)t} \left[\left(\sqrt{\lambda/\mu} \right)^{(n-j)} \cdot \tilde{I}_{n-j}^{1,1} \left(2\sqrt{\lambda\mu} t \right) \right. \\ &\quad \left. - \mu \int_0^t \left[Q_{1,1}^{(j)}(u) \right]_{k=1} \cdot e^{(\lambda+\mu)u} \left(\sqrt{\lambda/\mu} \right)^n \cdot \tilde{I}_n^{1,1} \left(2\sqrt{\lambda\mu} (t-u) \right) du \right]. \end{aligned} \quad (16)$$

Since $\tilde{I}_n^{1,1}(t) = I_n(t)$ and $\tilde{P}_1^{(j)}(u) = \left[Q_{1,1}^{(j)}(u) \right]_{k=1}$, it follows that

$$\begin{aligned} \tilde{P}_n^{(j)}(t) &= e^{-(\lambda+\mu)t} \left(\sqrt{\lambda/\mu} \right)^{n-j} \cdot I_{n-j} \left(2\sqrt{\lambda\mu} t \right) \\ &\quad - \mu e^{-(\lambda+\mu)t} \int_0^t \tilde{P}_1^{(j)}(u) \cdot e^{(\lambda+\mu)u} \left(\sqrt{\lambda/\mu} \right)^n \cdot I_n \left(2\sqrt{\lambda\mu} (t-u) \right) du. \end{aligned} \quad (17)$$

For the second term of Eq. (17), we use the first equality of Eq. (2) and obtain

$$\begin{aligned} \tilde{P}_n^{(j)}(t) &= e^{-(\lambda+\mu)t} \left(\sqrt{\lambda/\mu} \right)^{n-j} \cdot I_{n-j} \left(2\sqrt{\lambda\mu} t \right) \\ &\quad - e^{-(\lambda+\mu)t} \int_0^t \frac{d}{dt} \tilde{P}_0^{(j)}(t) \cdot e^{(\lambda+\mu)u} \left(\sqrt{\lambda/\mu} \right)^n \cdot I_n \left(2\sqrt{\lambda\mu} (t-u) \right) du. \end{aligned} \quad (18)$$

Using Eq. (15) in Eq. (18), we obtain

$$\begin{aligned} \tilde{P}_n^{(j)}(t) &= e^{-(\lambda+\mu)t} \left(\sqrt{\lambda/\mu} \right)^{n-j} \cdot I_{n-j} \left(2\sqrt{\lambda\mu} t \right) \\ &\quad - e^{-(\lambda+\mu)t} \int_0^t \frac{j}{u} \left(\sqrt{\lambda/\mu} \right)^{n-j} \cdot I_j \left(2\sqrt{\lambda\mu} u \right) \cdot I_n \left(2\sqrt{\lambda\mu} (t-u) \right) du \\ &= e^{-(\lambda+\mu)t} \left(\sqrt{\frac{\mu}{\lambda}} \right)^{j-n} \cdot \left[I_{n-j} \left(2\sqrt{\lambda\mu} t \right) - \int_0^t \left\{ \frac{j}{u} \cdot I_j \left(2\sqrt{\lambda\mu} u \right) \right\} \cdot \left\{ I_n \left(2\sqrt{\lambda\mu} (t-u) \right) \right\} du \right]. \end{aligned} \quad (19)$$

Now, $\int_0^t \left\{ \frac{j}{u} \cdot I_j \left(2\sqrt{\lambda\mu} u \right) \right\} \cdot \left\{ I_n \left(2\sqrt{\lambda\mu} (t-u) \right) \right\} du$ in the above equation can be reduced to $I_{n+j} \left(2\sqrt{\lambda\mu} t \right)$ by the following theorem.

Theorem 1. *Let us define*

$$A(t) = \frac{j}{t} \cdot I_j \left(2\sqrt{\lambda\mu} t \right), \quad B(t) = I_n \left(2\sqrt{\lambda\mu} t \right).$$

Let $(A * B)(t)$ be the convolution of $A(t)$ and $B(t)$. Then, we have

$$(A * B)(t) = \int_0^t \left\{ \frac{j}{u} \cdot I_j \left(2\sqrt{\lambda\mu} u \right) \right\} \cdot \left\{ I_n \left(2\sqrt{\lambda\mu} (t - u) \right) \right\} du = I_{n+j} \left(2\sqrt{\lambda\mu} t \right). \quad (20)$$

Proof. Let us define $A^*(\theta)$ and $B^*(\theta)$ as the Laplace transforms of $A(t)$ and $B(t)$, respectively. Taking Laplace transform of $A(t)$, we obtain

$$A^*(\theta) = \mathcal{L} \left[\frac{j}{t} \cdot I_j \left(2\sqrt{\lambda\mu} t \right) \right] = \left(2\sqrt{\lambda\mu} \right)^j \left(\theta + \sqrt{\theta^2 - \left(2\sqrt{\lambda\mu} \right)^2} \right)^{-j}. \quad (21)$$

Similarly, we obtain

$$B^*(\theta) = \mathcal{L} \left[I_n \left(2\sqrt{\lambda\mu} t \right) \right] = \left(2\sqrt{\lambda\mu} \right)^n \frac{\left(\theta + \sqrt{\theta^2 - \left(2\sqrt{\lambda\mu} \right)^2} \right)^{-n}}{\sqrt{\theta^2 - \left(2\sqrt{\lambda\mu} \right)^2}}. \quad (22)$$

Multiplying $A^*(\theta)$ by $B^*(\theta)$, we obtain

$$A^*(\theta)B^*(\theta) = \mathcal{L} [(A * B)(t)] = \left(2\sqrt{\lambda\mu} \right)^{(n+j)} \frac{\left(\theta + \sqrt{\theta^2 - \left(2\sqrt{\lambda\mu} \right)^2} \right)^{-(n+j)}}{\sqrt{\theta^2 - \left(2\sqrt{\lambda\mu} \right)^2}}. \quad (23)$$

Noting that Eq. (23) is none other than the Laplace transform of $I_{n+j} \left(2\sqrt{\lambda\mu} t \right)$ finishes the proof. \square

Now, using Eq. (20) in Eq. (19), we obtain Eq. (14), which confirms the results in Luchak [4] and Saaty [5].

4. Numerical examples

In this section, some numerical results are presented. We use Eqs. (9) and (10) to compute the probabilities $Q_0^{(2)}(t)$, $Q_{1,2}^{(2)}(t)$, and $Q_{2,2}^{(2)}(t)$. Figure 1 shows the computation results of the probabilities.

(Figure 1 will be placed here)

Figure 1(a) shows the probabilities under the condition that the offered load ($\rho = \lambda E(S)$) is less than 1. It shows that $Q_0^{(2)}(t)$ converges to 1 as the time increases because the length of the busy period is finite when $\rho < 1$. Figure 1(b) shows the probabilities under the condition that $\rho \geq 1$.

Now, we show the numerical examples for the mean waiting time at time t . Using Eqs. (11) and (13), we show the computational results in Figure 2.

(Figure 2 will be placed here)

Figure 2(a) shows the mean waiting times under two different ρ settings. With the same parameter settings, Figure 2(b) shows the computational results for $\tilde{Q}_0^{(2)}(t)$.

Error correction in Griffiths et al. [2]

To obtain the transient queue length distribution of the $M/E_k/1$ queue, Griffiths et al. [2] used the following identity:

$$e^{\frac{1}{2}t(\lambda z^k + \frac{k\mu}{z})} = \sum_{n=-\infty}^{\infty} \sum_{i=1}^k (\beta z)^{k(n-1)+i} \cdot \tilde{\mathbf{I}}_n^{k,i}(\alpha t), \quad (\text{A.1})$$

where $\alpha = 2^{k+1}\sqrt{\lambda(k\mu)^k}$ and $\beta = \sqrt[k+1]{\frac{\lambda}{k\mu}}$. However, there is an error in the above equation and we correct it in this section.

Theorem 2. (Correction of (A.1)) *For the generating function of the generalized Bessel function of the second type, we have the following identity:*

$$e^{t(\lambda z^k + \frac{k\mu}{z})} = e^{\frac{1}{2}t(2\lambda z^k + \frac{2k\mu}{z})} = \sum_{n=-\infty}^{\infty} \sum_{i=1}^k (\beta z)^{k(n-1)+i} \cdot \tilde{\mathbf{I}}_n^{k,i}(\alpha t), \quad (\text{A.2})$$

Proof. By using Theorem 1 in Griffiths et al. [1], we obtain

$$\begin{aligned} e^{t(\lambda z^k + \frac{k\mu}{z})} &= e^{\frac{1}{2}t(2\lambda z^k + \frac{2k\mu}{z})} \\ &= \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}t \ 2\lambda z^k)^m}{m!} \frac{(\frac{1}{2}t \ \frac{2k\mu}{z})^r}{r!} = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{l=1}^k \frac{(\frac{1}{2}t \ 2\lambda z^k)^m}{m!} \frac{(\frac{1}{2}t \ \frac{2k\mu}{z})^{r+k+l-1}}{(kr+l-1)!} \end{aligned} \quad (\text{A.3})$$

We change the variables $m = n - r$, $-\infty \leq n \leq \infty$ and $i = k - l + 1$, $i = 1, 2, \dots, k$ in the above equation and get

$$e^{t(\lambda z^k + \frac{k\mu}{z})} = \sum_{n=-\infty}^{\infty} \sum_{i=1}^k \sum_{r=0}^{\infty} \frac{(\frac{1}{2}t)^{n+k-i+r(k+1)} \cdot z^{k(n-1)+i} \cdot (2\lambda)^{n+r} \cdot (2k\mu)^{k(r+1)-i}}{(r+n)![k(r+1)-i]!}. \quad (\text{A.4})$$

Now, we note that

$$\begin{aligned} (2\lambda)^{n+r} \cdot (2k\mu)^{k(r+1)-i} &= \left(2^{k+1} \sqrt{\lambda(k\mu)^k} \right)^{n+k-i+r(k+1)} \cdot \left(\sqrt[k+1]{\frac{\lambda}{k\mu}} \right)^{k(n-1)+i} \\ &= \alpha^{n+k-i+r(k+1)} \cdot \beta^{k(n-1)+i}. \end{aligned} \quad (\text{A.5})$$

Using Eq. (A.5) in (A.4), we obtain

$$e^{t(\lambda z^k + \frac{k\mu}{z})} = \sum_{n=-\infty}^{\infty} \sum_{i=1}^k (\beta z)^{k(n-1)+i} \cdot \left[\left(\frac{1}{2} \alpha t \right)^{n+k-i} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2} \alpha t \right)^{r(k+1)}}{(r+n)!(k(r+1)-i)!} \right]. \quad (\text{A.6})$$

Then, using the definition of the generalized Bessel function of the second type in Eq. (A.6), we obtain Eq. (A.2). \square

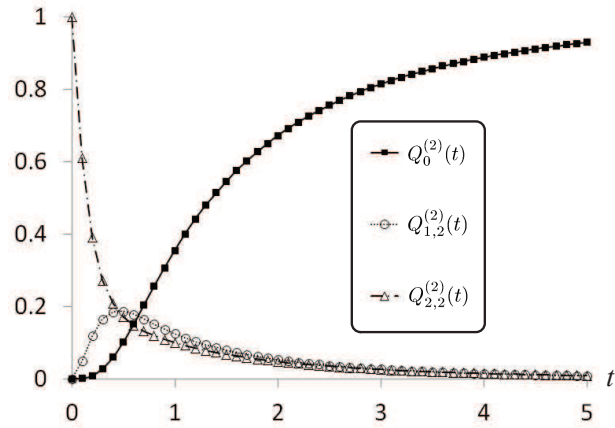
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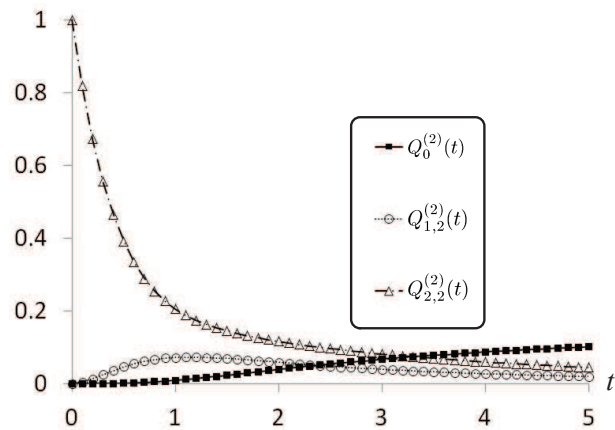
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References

- [1] Griffiths, G.D., Leonenko, G.M., Williams, J.E., 2006a. New generalization of the modified Bessel function and its generating function, *Fractional Calculus & Applied Analysis* **8(3)** 267–276.
- [2] Griffiths, G.D., Leonenko, G.M., Williams, J.E., 2006b. The transient solution to $M/E_k/1$ queue, *Operations Research Letters* **34** 349–354.
- [3] Leonenko, G.M., 2009. A new formula for the transient solution of the Erlang queueing Model, *Statistics and Probability Letters* **79** 400–406.
- [4] Luchak, G., 1958, The continuous time Solution of the equations of the single channel queue with a general class of service-time distributions by the method of generating functions, *Journal of the Royal Statistical Society. Series B (Methodological)* **20(1)** 176-181.
- [5] Saaty, L.T., 1961, *Elements of Queueing Theory*, Mcgraw Hill, New York.



(a) $\lambda = 1$, $E(S) = 0.5$, $k = 2$, ($\rho = \lambda E(S) = 0.5$)



(b) $\lambda = 1$, $E(S) = 2$, $k = 2$, ($\rho = 2$)

Figure 1: Probabilities $Q_0^{(2)}(t)$, $Q_{1,2}^{(2)}(t)$ and $Q_{2,2}^{(2)}(t)$.

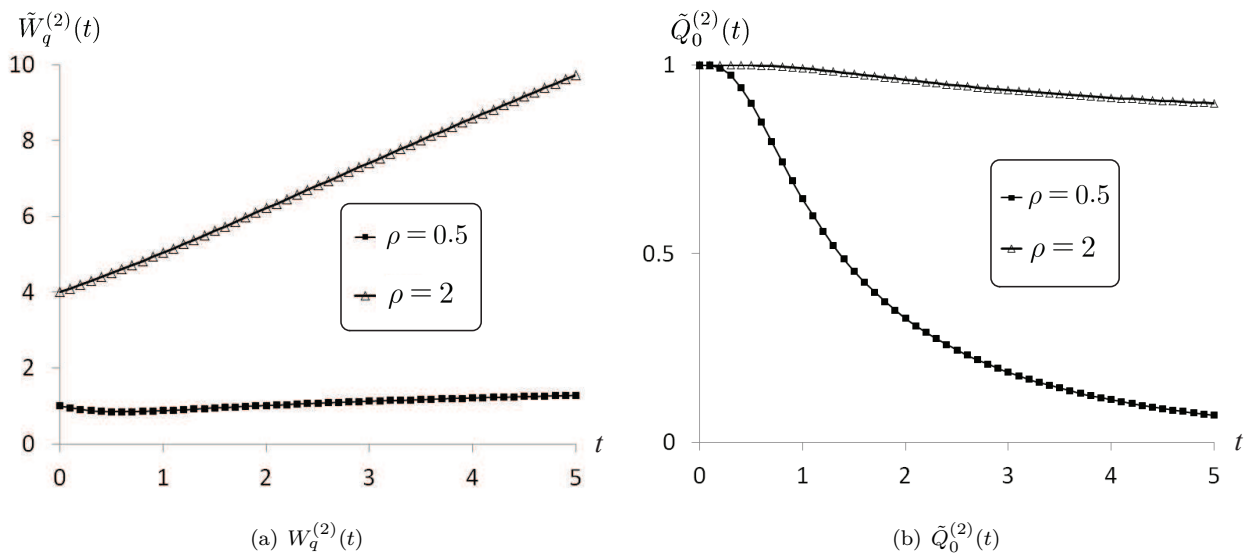


Figure 2: Computational results for $W_q^{(j)}(t)$ and $Q_0^{(j)}(t)$.