<table>
<thead>
<tr>
<th>Title</th>
<th>The varieties for some Specht modules</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Lim, Kay Jin</td>
</tr>
<tr>
<td>Citation</td>
<td>Lim, K. J. (2009). The varieties for some Specht modules. Journal of Algebra, 321(8), 2287-2301.</td>
</tr>
<tr>
<td>Date</td>
<td>2009</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10220/18897">http://hdl.handle.net/10220/18897</a></td>
</tr>
<tr>
<td>Rights</td>
<td>© 2009 Elsevier Inc. This is the author created version of a work that has been peer reviewed and accepted for publication by Journal of Algebra, Elsevier Inc. It incorporates referee's comments but changes resulting from the publishing process, such as copyediting, structural formatting, may not be reflected in this document. The published version is available at: [DOI: <a href="http://dx.doi.org/10.1016/j.jalgebra.2009.01.016">http://dx.doi.org/10.1016/j.jalgebra.2009.01.016</a>].</td>
</tr>
</tbody>
</table>
THE VARIETIES FOR SOME SPECHT MODULES

KAY JIN LIM

ABSTRACT. J. Carlson introduced the cohomological and rank variety for a module over a finite group algebra. We give a general form for the largest component of the variety for the Specht module for the partition \((p^p)\) of \(p^2\) restricted to a maximal elementary abelian \(p\)-subgroup of rank \(p\). We determine the varieties of a large class of Specht modules corresponding to \(p\)-regular partitions. To any partition \(\mu\) of \(np\) of not more than \(p\) parts with empty \(p\)-core we associate a unique partition \(\Phi(\mu)\) of \(np\), where the rank variety of the restricted Specht module \(S^{\mu} \downarrow_{E_n}\) to a maximal elementary abelian \(p\)-subgroup \(E_n\) of rank \(n\) is \(V_{\Phi(\mu)}^{E_n}(k)\) if and only if \(V_{E_n}^{E_n}(S^{\Phi(\mu)}) = V_{E_n}^{E_n}(k)\). In some cases where \(\Phi(\mu)\) is a 2-part partition, we show that the rank variety \(V_{E_n}^{E_n}(S^{\mu})\) is \(V_{E_n}^{E_n}(k)\). In particular, the complexity of the Specht module \(S^{\mu}\) is \(n\).

1. Introduction

Over last forty years, group cohomology has been studied extensively, especially its interaction with modular representation theory. Carlson [3] introduced the varieties for modules over group algebras. Various results have been published which relate the properties of the algebraic variety of a module and the structure of the module itself.

Hemmer and Nakano studied the support varieties for permutation modules and Young modules [7] over the symmetric groups. The varieties for most Specht modules remain unknown. A partition is made up of \((p \times p)\)-blocks if each part is a multiple of \(p\) and each part is repeated a multiple of \(p\) times. The VIGRE research group in Georgia (2004) conjectured that the variety for the Specht module corresponding to a partition \(\mu\) is the variety of the defect group of the block in which the Specht module lies unless and only unless the partition is made up of \((p \times p)\)-blocks.

It is not difficult to verify the conjecture when \(\mu\) has \(p\)-weight strictly less than \(p\). In the first part of this paper, we study the variety for the Specht module corresponding to the partition \((p^p)\). In the latter part, we verify the conjecture for a large class of partitions of \(p^2\) and \(np\) for some positive integer \(n\), where they are not made up of \((p \times p)\)-blocks.

We organize this paper as follows. A brief introduction to the varieties for modules and representation theory of symmetric groups is given in Section 2, the standard texts are [2] II, [9] and [11]. We also set up notations in this section. Our main results are stated in Section 3. In Section 4, in the case \(p\) is odd, we show that the dimension of the rank variety for the Specht module \(S^{(p^p)}\) is \(p - 1\). This therefore gives the complexity of the Specht module. In [4], Carlson gives an upper bound for the degree of the projectivized rank variety for a module over an elementary abelian \(p\)-group. Later in this section, we show that the radical ideal corresponding
to the largest component of \( S^{(p^n)} \downarrow_{E_n} \) is generated by a single non-zero polynomial, where a general formula is given. This general polynomial shows that the degree of the projectivized rank variety for the module \( S^{(p^n)} \downarrow_{E} \) is non-zero and divisible by \((p - 1)^2\).

In Section 5 we determine the varieties of a large class of Specht modules corresponding to \( p \)-regular partitions. For each partition \( \mu \) of \( np \) with no more than \( p \) parts which has empty \( p \)-core, we associate a unique partition \( \Phi(\mu) \) of \( np \) such that each part of the partition is a multiple of \( p \) and with the property that the rank variety \( V^\sharp_{E_n}(S^\mu) \) is \( V^\sharp_{E_n}(k) \) if and only if \( V^\sharp_{E_n}(S^{\Phi(\mu)}) = V^\sharp_{E_n}(k) \). In some cases where \( \Phi(\mu) \) is a 2-part partition, we show that the rank variety for the Specht module \( S^\mu \) restricted to a maximal elementary abelian \( p \)-subgroup \( E \) of the symmetric group \( S_{np} \) of rank \( n \) is precisely the rank variety for the trivial module. In this case, the complexity of the Specht module is \( n \). At the end of this section, we give a complete list of the varieties for the modules \( S^\mu \downarrow_{E_3} \) where \( \mu \) are partitions of 9, over \( p = 3 \).

2. Background Materials and Notations

Throughout this paper \( p \) is a prime and \( k \) is an algebraically closed field of characteristic \( p \). Let \( G \) be a finite group whose order is divisible by \( p \) and \( M \) be a finitely generated \( kG \)-module. Carlson [3] introduced the cohomological variety \( V_G(M) \) for the module \( M \). Avrunin and Scott show that the variety \( V_G(M) \) has a stratification \([2]\)

\[
V_G(M) = \bigcup_{E \in \mathcal{E}(G)} \mathrm{res}^*_G V_E(M)
\]

where \( \mathcal{E}(G) \) is a set of representatives for the conjugacy classes of elementary abelian \( p \)-subgroups of \( G \) and \( \mathrm{res}^*_G : V_E(k) \to V_G(k) \) is the map induced by the restriction map \( \mathrm{res}_G : \mathrm{Ext}^*_G(k, k) \to \mathrm{Ext}^*_E(k, k) \). The complexity of the module \( M \) is precisely the dimension \( \dim V_G(M) \) of the variety \( V_G(M) \). Since the map \( \mathrm{res}^*_G \) is a finite map, we have

\[
\dim V_G(M) = \max_{E \in \mathcal{E}(G)} \{ \dim \mathrm{res}^*_G V_E(M) \} = \max_{E \in \mathcal{E}(G)} \{ \dim V_E(M) \}
\]

Let \( E \) be an elementary abelian group of order \( p^n \) with generators \( g_1, \ldots, g_n \) and \( M \) be a finitely generated \( kE \)-module. For a generic affine point \( 0 \neq \alpha = (\alpha_1, \ldots, \alpha_n) \in k^n \), the Jordan type of the restriction \( M \downarrow_{k(\alpha)} \) is called the generic Jordan type of the module \( M \), where \( u_\alpha = 1 + \sum_{i=1}^n \alpha_i(g_i - 1) \) (see [3] and [6]). Generic Jordan type is compatible with direct sum and tensor product of two \( kE \)-modules, where the tensor product is given by the diagonal \( E \)-action (Proposition 4.7 [6]). The stable generic Jordan type of \( M \) is the generic Jordan type of \( M \) modulo projective direct summands. The rank variety \( V^\sharp_E(M) \) of the \( kE \)-module \( M \) is the set

\[
\{0\} \cup \{0 \neq \alpha \in k^n \mid M \downarrow_{k(\alpha)} \text{ is not } k\langle \alpha \rangle \text{-free} \}
\]

It is isomorphic to the cohomological variety of the module \( M \) (via the Frobenius map when \( p \) is odd) [1]. Note that \( M \) is not generically free if and only if its rank variety \( V^\sharp_E(M) \) is \( V^\sharp_E(k) \).

**Proposition 2.1.** Let \( M, N \) be \( kG \)-modules.
(i) $V_G(M) = \{0\}$ if and only if $M$ is projective $kG$-module.
(ii) $V_G(M \oplus N) = V_G(M) \cup V_G(N)$ and $V_G(M \otimes_k N) = V_G(M) \cap V_G(N)$ with the diagonal $G$-action.
(iii) If $p \nmid \dim_k M$, then $V_G(M) = V_G(k)$.
(iv) Suppose that $M$ is an indecomposable $kG$-module in a block with defect group $D$. Then $V_G(M) = \res_{G,D}^* V_D(M)$. Let $|G| = p^a s$ with $\gcd(s, p) = 1$. Then $p^a/|D|$ divides $\dim_k M$. Furthermore, if
\[
\gcd\left(\frac{|D| \dim_k M}{p^a}, p\right) = 1
\]
then $V_G(M) = \res_{G,D}^* V_D(k)$.
(v) If $H$ be a subgroup of $G$, then $V_H(M) = (\res_{G,H}^*)^{-1} V_G(M)$.
(vi) The variety $V_G(M)$ is a homogeneous subvariety of $V_G(k)$. If $M$ is indecomposable, then the projectivized variety $\overline{V_G(M)}$ is connected when endowed with the Zariski topology.
(vii) Suppose further that $G$ is an elementary abelian $p$-group of rank $n$ and $r = \dim V_G(M)$. Then $p^{r-r}$ divides $\dim_k M$.

The standard texts for the representation of symmetric groups are [9] and [11].

Let $S_m$ denote the symmetric group on $m$ letters. For each $s \leq n$, let $E_s$ be the elementary abelian subgroup of $S_{np}$ generated by the $p$-cycles $((i-1)p+1, \ldots, ip)$ for $1 \leq i \leq s$. A partition $\mu$ of $m$ is a sequence of positive integers $(\mu_1, \mu_2, \ldots, \mu_s)$ such that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_s$ and $\sum_{i=1}^s \mu_i = m$; in this case we write $|\mu| = m$. The Young subgroup $S_\mu$ of $S_m$ is
\[
S_{\{1,\ldots,\mu_1\}} \times S_{\{\mu_1+1,\ldots,\mu_1+\mu_2\}} \times \ldots \times S_{\{\mu_1+\ldots+\mu_{s-1}+1,\ldots,\mu_1+\ldots+\mu_s\}}
\]
The permutation module $M^\mu \cong kS_\mu \uparrow S_m$ is the $kS_m$-module $k$-spanned by all $\mu$-tabloids $\{t\}$ where $S_m$ acts by permuting the numbers assigned to the nodes of $t$. The Specht module $S^\mu$ is the submodule of $M^\mu$ $k$-spanned by the $\mu$-polytabloids $e_t = \sum_{\sigma \in C_t} \sgn(\sigma) \sigma \{t\}$ where $C_t$ is the column stabilizer of the $\mu$-tableau $t$. It has dimension given by the hook formula
\[
\dim_k S^\mu = \frac{m!}{\prod_{i,j \in [\mu]} h_{ij}}
\]
where $h_{ij}$ is the hook length of the $(i, j)$-node of the Young diagram $[\mu]$ of $\mu$. The number of skew $p$-hooks removed to obtain the $p$-core $\bar{\mu}$ of $\mu$ is precisely the number of hook lengths of $\mu$ divisible by $p$, we write $m_\mu$ for this common number. A defect group $D_\mu$ of the block containing the Specht module $S^\mu$ is a Sylow $p$-subgroup of the symmetric group $S_{m_\mu}$.

The Young module $Y^\mu$ corresponding to the partition $\mu$ is an indecomposable direct summand of $M^\mu$ with the property that each indecomposable summand of $M^\mu$ is isomorphic to some $Y^\lambda$ with $\lambda \geq \mu$ by the lexicographic ordering (Theorem 3.1 of [11]) and $Y^\mu$ occurs with multiplicity one. Each $Y^\mu$ has a Specht filtration and the filtration multiplicity is well-defined [5]. The permutation module $M^{(r-k,k)}$
Proposition 2.2.

(i) [The Branching Theorem (§9 of [9])] Let $\mu$ be a partition of $m$ and $\Omega(\mu)$ be the set consisting of all partitions obtained by removing a node from $\mu$, then the $k\mathfrak{S}_{m-1}$-module $S^\mu\downarrow_{\mathfrak{S}_{m-1}}$ has a filtration with Specht factors $S^\lambda$ one for each $\lambda \in \Omega(\mu)$.

(ii) [Nakayama’s Conjecture] Let $\mu, \lambda$ be partitions of $m$. Then the Specht modules $S^\mu, S^\lambda$ lie in the same block if and only if $\tilde{\mu} = \tilde{\lambda}$. So any $k\mathfrak{S}_m$-module which has a filtration with Specht factors decomposes into direct summands according to $p$-cores.

(iii) Let $C_p$ be the cyclic group of order $p$ and $J_i$ be the Jordan block of size $0 \leq i \leq p$. Then there is an extension of indecomposable $kC_p$-modules

$$0 \to J_i \to J_a \oplus J_b \to J_j \to 0$$

with $a \geq b$ if and only if $\max\{i,j\} \leq a \leq p$, $0 \leq b \leq \min\{i,j\}$ and $a + b = i + j$.

(iv) If $\mu$ is a partition of $np$ with non-empty $p$-core, then $S^\mu\downarrow_{E_n}$ is generically free.

(v) Let $\mu'$ be the conjugate of a partition $\mu$ of $m$. Then $V_{\mathfrak{S}_m}(S^{\mu'}) = V_{\mathfrak{S}_m}(S^\mu)$. In particular, $V_{E_p}(S^{\mu'}) = V_{E_p}(S^\mu)$ for any elementary abelian $p$-subgroup $E$ of $\mathfrak{S}_m$.

Proof. (iv) By Proposition 2.1 (iv), $V_{\mathfrak{S}_{np}}(S^\mu) = \text{res}^\mathfrak{S}_{mn, D_\mu} V_{D_\mu}(S^\mu)$ where $D_\mu$ is a Sylow $p$-subgroup of $\mathfrak{S}_{mn,p}$. Since $m_\mu < n$, any maximal elementary abelian $p$-subgroup of $D_\mu \leq \mathfrak{S}_{mn,p}$ has rank strictly less than $n$. So

$$\dim V_{E_n}(S^{\mu}) \leq \dim V_{\mathfrak{S}_{np}}(S^\mu) = \dim V_{D_\mu}(S^\mu) \leq \max_{E \leq D_\mu} \{\dim V_{E}(S^{\mu})\} < n$$

where $E$ runs through all elementary abelian $p$-subgroups of $D_\mu$ by the Quillen Stratification Theorem.

(v) Note that $S^{\mu'} \otimes_k S^{(1^n)} \cong (S^\mu)^*$ (see 8.15 [9]). By 5.7.3 and 5.8.5 of [2] II, we have $V_{\mathfrak{S}_m}(S^{\mu'}) = V_{\mathfrak{S}_m}(S^{\mu'}) \cap V_{\mathfrak{S}_m}(k) = V_{\mathfrak{S}_m}((S^\mu)^*) = V_{\mathfrak{S}_m}(S^\mu)$. \hfill $\square$

3. Main Results

The VIGRE research group in Georgia (2004) conjectured that the variety for the Specht module corresponding to a partition $\mu$ is the variety of the defect group where the Specht module lies in if and only if the partition is not made up of $(p \times p)$-blocks. Let $W_i(S^{(\mu^p)})$ be the union of all irreducible components of $V_{E_p}^\sharp(S^{(\mu^p)}) \subseteq V_{E_p}^\sharp(k)$ of dimension $i$. In Section 4 we prove the following theorem.

Theorem 3.1. Suppose that $p$ is an odd prime and $\mu = (p^i)$ is the $(p \times p)$-partition.
(i) \( \dim V_{\mathfrak{S}_p}(S^\mu) = p - 1 \).
(ii) We have \( I(W_{p-1}(S^{(p^p)})) = \sqrt{f} = \langle f \rangle \) where

\[
f(x_1, \ldots, x_p) = (x_1 \cdots x_p)^{p-1} \tilde{f} + \sum_{i=1}^{p} x_1^{n(p-1)} \cdots x_i^{n(p-1)} \cdots x_p^{n(p-1)}
\]

for some homogeneous polynomial \( \tilde{f} \in k[x_1, \ldots, x_p]^{(p^p)^p \times \mathfrak{S}_p} \) and positive integer \( n \) where \( x_1^{n(p-1)} \cdots x_i^{n(p-1)} \cdots x_p^{n(p-1)} \) is the product of all \( x_j^{n(p-1)} \)'s, \( 1 \leq j \leq p \), except the term \( x_i^{n(p-1)} \).

In Section 3, we determine the varieties of a large class of Specht modules that are not made up of \((p \times p)\)-blocks. We associate to any partition \( \mu = (\mu_1, \ldots, \mu_s) \) of \( np \) not more than \( p \) parts with empty \( p \)-core a unique partition \( \Phi(\mu) \) of \( np \) such that \( \Phi(\mu) = (n_1p, n_2p, \ldots, n_rp) \) and \( r \leq s \).

**Hypothesis 3.2.** Suppose that \( \mu \) is a partition of \( np \) not more than \( p \) parts with empty \( p \)-core and one of the following conditions hold.

(H1) The prime \( p \) is odd, \( n = p \) and \( \Phi(\mu) \) is a 2-part partition \((p^2 - mp, mp)\) of \( p^2 \) for some \( 1 \leq m < p/2 \).
(H2) The prime \( p \) is odd and \( \Phi(\mu) \) is a 2-part partition \((np - \varepsilon p, \varepsilon p)\) such that \( n \not\equiv 2 \) (mod \( p \)) and \( \varepsilon \in \{1, 2\} \).
(H3) \( \Phi(\mu) = (np) \).
(H4) \( p = 2 \) and \( \Phi(\mu) \) is the partition \((2n - 2, 2) \neq (2, 2) \) or \((2n - 4, 4) \neq (4, 4) \).

**Theorem 3.3.** If \( \mu \) is a partition satisfying Hypothesis 3.2, then \( V^\mu_{E_n}(k) = V^\mu_{E_n}(k) \). In this case, the complexity of \( S^\mu \) is \( n \).

4. The Variety for the Specht Module \( S^{(p^p)} \)

**Theorem 4.1.** Suppose that the Specht module \( S^\mu \) corresponding to a partition \( \mu \) of \( n \) lies inside a block with abelian defect \( D_\mu \). We have \( V_{\mathfrak{S}_n}(S^\mu) = \text{res}_{\mathfrak{S}_n, D_\mu} V_{D_\mu}(k) \) and the complexity of the Specht module \( S^\mu \) is exactly the \( p \)-weight of \( \mu \).

**Proof.** Let \( n! = p^m b \) with \( \gcd(a, b) = 1 \) and \( m_\mu \) be the \( p \)-weight of \( \mu \). Since the number of hook lengths of \([\mu]\) divisible by \( p \) is at least as many as \( m_\mu \), we have

\[
\mathbb{Z} \ni \frac{|D_\mu|}{p^m} (\dim_k S^\mu)_p = \frac{p^{m_\mu}}{p^m} \prod_{(i,j) \in [\mu]} (h_{ij})_p = 1
\]

By Proposition 2.1 (iv), \( V_{\mathfrak{S}_n}(S^\mu) = \text{res}_{\mathfrak{S}_n, D_\mu} V_{D_\mu}(k) \). The complexity of the \( S^\mu \) is exactly the \( p \)-rank of the defect group \( D_\mu \), i.e., \( m_\mu \).

For \( p = 2 \), let \( F \) be the maximal abelian subgroup of \( \mathfrak{S}_4 \) generated by \((12)(34)\) and \((13)(24)\). The module \( S^{(2^2)} \downarrow_F \) is isomorphic to \( k \oplus k \), so \( \dim V_{\mathfrak{S}_4}(S^{(2^2)}) = 2 \). For the rest of Section 4, \( p \) is an odd prime.
4.1. Some vanishing ideals. Given the symmetric group $G_m$ of degree $m$, the group $(F_p^*)^m \times G_m$ is defined by the group actions $(\beta, \sigma)(\beta', \sigma') = (\beta \cdot \sigma(\beta'), \sigma\sigma')$ where \(\sigma(\beta') = (\beta'_\sigma^{-1(1)}, \ldots, \beta'_\sigma^{-1(m)})\) for all $\sigma, \sigma' \in G_m$ and $\beta = (\beta_1, \ldots, \beta_m), \beta' = (\beta'_1, \ldots, \beta'_m) \in (F_p^*)^m$. This group acts on the polynomial ring $k[x_1, \ldots, x_m]$ via
\[(\beta, \sigma)x_i := \beta_{\sigma(i)}x_{\sigma(i)}\]
for all $1 \leq i \leq m$ and $(\beta, \sigma) \in (F_p^*)^m \times G_m$. We think of the action in two stages, first by $(F_p^*)^m$ and then followed by $G_m$.

**Lemma 4.2.** Let $p$ be an odd prime, $m \geq 3$, $f \in k[x_1, \ldots, x_m](F_p^*)^m \times A_m$ and $x_i$ be a factor of $f$ for some $1 \leq i \leq m$. Then $x_1^{p-1} \ldots x_m^{p-1}$ is a factor of $f$.

**Proof.** Since $f \in (k[x_1, \ldots, x_m](F_p^*)^m)^A_m = k[x_1^{p-1}, \ldots, x_m^{p-1}]^A_m$, it follows that if $x_i$ divides $f$ then $x_i^{p-1}$ divides $f$. As $f$ is invariant under the action of $A_m$, all $x_i^{p-1}$'s divide $f$. All factors $x_i^{p-1}$ are pairwise coprime, so the product divides $f$. \(\square\)

**Lemma 4.3.** Suppose that $m \geq 3$, $G = (F_p^*)^m \times G_m$ and $f$ is a non-zero polynomial in $k[x_1, \ldots, x_m]$ such that the ideal $\langle f \rangle$ generated by $f$ satisfies the property $G(\langle f \rangle) = \langle f \rangle$, i.e., for all $g \in G$, we have $gf \in \langle f \rangle$. Then $f$ is either fixed by the subgroup $(F_p^*)^m \times A_m$ of $G$ or divisible by $x_1 \ldots x_m$. In the first of these cases, $f$ is either fixed or negated by the action of $G$.

**Proof.** Write $R = k[x_1, \ldots, x_m]$ and $f = \sum a_{n_1 \ldots n_m}x_1^{n_1} \ldots x_m^{n_m}$ with $a_{n_1 \ldots n_m} \in k$. Suppose that for each $(\beta, \sigma) \in G$, we have $(\beta, \sigma)f = hf$ for some polynomial $h \in R$. Since the action of $(\beta, \sigma)$ on each monomial of $f$ is multiplication by some $\beta_i$’s and permuting $x_j$’s, the highest degree of a monomial appearing in $f$ is at least as large as the highest degree of a monomial appearing in $hf$. So, $h = c(\beta, \sigma)$ lies in $k$. Let $1 = (1, \ldots, 1)$ in $(F_p^*)^m$. Let $\beta(i)$ be the element in $(F_p^*)^m$ such that its $i$th coordinate is $\beta_i$ and 1 elsewhere. It is clear that $c(1, \sigma)^{-1} = c(1, \sigma^{-1})$ and
\[c(\beta, \sigma) = c(\beta(1), 1)c(\beta(2), 1) \ldots c(\beta(m), 1)c(1, \sigma)\]
for all $\beta \in (F_p^*)^m$ and $\sigma \in G_m$.

It suffices to examine the action of transpositions on $f$ to determine $c(1, \sigma)$. For any transposition $\sigma$, $f = \sigma^2 f = c(1, \sigma)^2 f$, so $c(1, \sigma) \in \{\pm 1\}$. This shows that $c(1, \xi) = 1$ for all $\xi \in A_m$. Suppose that there exists some monomial $a_{n_1 \ldots n_m}x_1^{n_1} \ldots x_m^{n_m}$ involved in $f$ with $n_i = 0$ for some $1 \leq i \leq m$. By the action of $\beta(i)$ on $f$ we have $a_{n_1 \ldots n_m} = c(\beta(i), 1)a_{n_1 \ldots n_m}$, i.e., $c(\beta(i), 1) = 1$. For any $1 \leq j \leq m$, let $\sigma \in A_m$ such that $\tau(i) = j$, then $\tau f = f$ gives $c(\beta(j), 1) = 1$. So $(F_p^*)^m \times A_m$ acts trivially on $f$ in this case. Otherwise, every monomial involved in $f$ is divisible by $x_1$ and so $f$ is divisible by $x_1 \ldots x_m$. \(\square\)

For any fixed $1 \leq i \leq m$, define $x_1 \ldots \widehat{x_i} \ldots x_m$ as the product of all $x_j$’s where $1 \leq j \neq i \leq m$. We write $I_i$ for the radical ideal of the polynomial ring $k[x_1, \ldots, x_m]$ generated by $x_1 \ldots \widehat{x_i} \ldots x_m$ and $x_i$. 
Lemma 4.4. For $m \geq 2$, the vanishing ideal of the union of the varieties $V(I_i)$ where $1 \leq i \leq m$ is $\{x_1 \ldots \hat{x_i} \ldots x_m \mid 1 \leq i \leq m\}$.

Proof. The algebraic variety given by the union of all $V(I_i)$'s consists of all planes at least two coordinates having value 0, which is precisely the algebraic variety defined by the ideal $J$ generated by $x_1 \ldots \hat{x_i} \ldots x_m$ for all $1 \leq i \leq m$. Furthermore, $J$ is a radical ideal, so both sides coincide using Hilbert’s Nullstellensatz. \qed

Proposition 4.5. Let $m \geq 2$ and $I$ be a radical ideal of $k[x_1, \ldots, x_m]$ generated by a single polynomial $f$ such that $V(f) \cap V(x_i) = V(x_1 \ldots \hat{x_i} \ldots x_m, x_i)$ for any $1 \leq i \leq m$. Then

$$f(x_1, \ldots, x_m) = x_1 \ldots x_m \tilde{f} + \sum_{i=1}^{m} a_i x_1^{n_{1i}} \ldots x_m^{n_{mi}}$$

such that $\tilde{f} \in k[x_1, \ldots, x_m]$ and

(i) $n_{ii} = 0$ for all $1 \leq i \leq m$,
(ii) $0 \neq a_i \in k$ for all $1 \leq i \leq m$,
(iii) $n_{ij} > 0$ for all $i \neq j$.

Proof. The given hypothesis implies $\bigcup V(x_1 \ldots \hat{x_i} \ldots x_m, x_i) \subseteq V(f)$. By Lemma 4.4, $f$ lies inside the ideal $\{x_1 \ldots \hat{x_i} \ldots x_m \mid 1 \leq i \leq m\}$, i.e.,

$$f(x_1, \ldots, x_m) = \sum_{i=1}^{m} f_i x_1 \ldots \hat{x_i} \ldots x_m$$

for some $f_1, \ldots, f_m \in k[x_1, \ldots, x_m]$. It is not difficult to see that we can rewrite $f$ as follows

$$f(x_1, \ldots, x_m) = x_1 \ldots x_m \tilde{f} + \sum_{i=1}^{m} a_i(x_1, \ldots, x_m) x_1^{n_{1i}} \ldots x_m^{n_{mi}}$$

such that

(i) $n_{ij} > 0$ if $j \neq i$ and $n_{ii} = 0$ for all $1 \leq i \leq m$,
(ii) $\tilde{f}, a_i \in k[x_1, \ldots, x_m]$ for each $1 \leq i \leq m$,
(iii) for all $1 \leq i, j \leq m$, $x_j$ does not divide $a_i$ and $a_i$ is independent of the variable $x_i$.

For each $1 \leq i \leq m$, the equation $f(x_1, \ldots, x_m)_{x_i=0} = a_i x_1^{n_{1i}} \ldots x_m^{n_{mi}}$ implies

$$V(a_i, x_i) = V(a_i) \cap V(x_i) \subseteq V(f) \cap V(x_i) = V(x_1 \ldots \hat{x_i} \ldots x_m, x_i)$$

By Hilbert’s Nullstellensatz, we have $(x_1 \ldots \hat{x_i} \ldots x_m, x_i) \subseteq (a_i, x_i)$. Suppose that $x_1 \ldots \hat{x_i} \ldots x_m = a_i w_i + x_i v_i$ for some $w_i, v_i \in k[x_1, \ldots, x_m]$. We write $w_i = u_i + x_i u_i'$ such that $u_i$ is independent of the variable $x_i$. The equation

$$x_1 \ldots \hat{x_i} \ldots x_m - a_i u_i = a_i x_i u_i' + x_i v_i$$

implies that $a_i x_i u_i' + x_i v_i = 0$ as the left-hand side is independent of $x_i$. So we have $x_1 \ldots \hat{x_i} \ldots x_m = a_i u_i$.

Since $x_j$ does not divide $a_i$ for all $j \neq i$, we have $u_i/x_1 \ldots \hat{x_i} \ldots x_m$ is a polynomial. Comparing degrees, we have $a_i \in k$. Note that $a_i \neq 0$; otherwise,

$$(x_1 \ldots \hat{x_i} \ldots x_m, x_i) \subseteq (x_i)$$
Now we have all the desired properties. □

4.2. **Proof of Theorem 3.1** Let $E = E_p$. Now we are ready to prove Theorem 3.1 (i).

**Proof of Theorem 3.1(i).** Recall that $\mu = (p^\nu)$. By the Branching Theorem, we have $S^\mu \doteq \mathfrak{S}_{p^2 - 1} \cong S^\tau$ where $\tau = (p^{\nu - 1}, p - 1)$. Note that $\tau$ has $p$-core $\tilde{\tau} = (p, 1^{p - 1})$ and the corresponding block has defect group $D = E_{p^2 - 2} \leq \mathfrak{S}_{p^2 - 1}$. Using the hook formula

$$|D|/(\dim_k S^\tau)_{p} = \frac{p^{p - 2}p}{p^{p - 1}} = 1$$

so $V_{\mathfrak{S}_{p^2 - 1}}(S^\mu) = V_{\mathfrak{S}_{p^2 - 1}}(S^\tau) = \text{res}_{\mathfrak{S}_{p^2 - 1}}^\epsilon V_{E_{p^2}}(k)$ by Proposition 2.1 (iv). This also shows that $\dim V_{\mathfrak{S}_{p^2 - 1}}(S^\mu) = p - 1$.

On the other hand, let $r = \dim V_{E}(S^\mu) \leq \dim V_{\mathfrak{S}_{p^2 - 1}}(S^\mu)$, using Proposition 2.1 (vii), $p^{r - 2}$ divides $(\dim_k S^\mu)_{p} = p$. Let $E_{p^2 - 1}(S^\mu)$ show that $G$ acts on the rank variety $V_{E_{p^2}}(M)$ of a $kE$-module $M$ and hence on the vanishing ideal $I(V_{E_{p^2}}^{2}(M))$ of $V_{E_{p^2}}^{2}(M)$ where the action is given by $(\beta, \sigma)x_i = \beta_{\alpha i}x_{\sigma(i)}$.

**Lemma 4.6.** Let $\tau = (p^{\nu - 1}, p - 1)$ and $E_{p^2 - 1} \leq \mathfrak{S}_{p^2 - 1}$. Then the rank variety $V_{E_{p^2 - 1}}^{2}(S^\tau) \subseteq k^{p - 1}$ is the union of all hyperplanes $V(x_i) \subseteq k^{p - 1}$ where $1 \leq i \leq p - 1$.

**Proof.** We know that $V_{\mathfrak{S}_{p^2 - 1}}(S^\tau) = \text{res}_{\mathfrak{S}_{p^2 - 1}}^\epsilon V_{E_{p^2}}(k)$. Apply Proposition 2.1 (v), we have $V_{E_{p^2 - 1}}(S^\tau) = \left(\text{res}_{\mathfrak{S}_{p^2 - 1}}^\epsilon V_{E_{p^2}}(k)\right)^{-1} \text{res}_{\mathfrak{S}_{p^2 - 1}}^\epsilon V_{E_{p^2}}(k)$. So the rank variety $V_{E_{p^2 - 1}}^{2}(S^\tau)$ lies in the union of hyperplanes $\bigcup_{i=1}^{p - 1} V(x_i)$. Since the rank variety contains $V(x_{p - 1})$ so it contains $V(x_i)$'s where $1 \leq i \leq p - 1$ given by the action of $\mathfrak{S}_{p^2 - 1}$ on the $p - 1$ coordinates of $V_{E_{p^2 - 1}}^{2}(S^\tau)$. □

Since $k[x_1, \ldots, x_m]$ is a unique factorization domain, prime ideals of height one are principal. Let $W_i(S(p^\nu))$ be the union of all irreducible components of $V_{E_{p^2}}^{2}(S(p^\nu)) \subseteq k^p$ of dimension $i$. Since $W_{p - 1}(S(p^\nu))$ has codimension 1, by the previous remark, it is defined by a single polynomial in variables $x_1, \ldots, x_p$. By Lemma 4.6

$$W_{p - 1}(S(p^\nu)) \cap V(x_i) \subseteq V_{E_{p^2}}^{2}(S(p^\nu)) \cap V(x_i) = \bigcup_{j \neq i} V(x_j, x_i)$$

and $\dim (W_{p - 1}(S(p^\nu)) \cap V(x_i)) = p - 2$, so $W_{p - 1}(S(p^\nu)) \cap V(x_i)$ contains one of the irreducible varieties $V(x_j, x_i)$. Since $W_{p - 1}(S(p^\nu))$ is invariant under the action of $G$, its intersection with $V(x_i)$ contains all of $V(x_j, x_i)$ where $1 \leq j \neq i \leq p$. We have proved the following lemma.

**Lemma 4.7.** For any $1 \leq i \leq p$, we have $W_{p - 1}(S(p^\nu)) \cap V(x_i) = \bigcup_{j \neq i} V(x_j, x_i)$. 
Proof of Theorem 3.1(ii). By Lemma 4.7 and Proposition 4.5, \( f \) has the form

\[
f(x_1, \ldots, x_p) = x_1 \ldots x_p \tilde{f} + \sum_{i=1}^{p} a_i x_1^{n_{1i}} \ldots x_p^{n_{pi}}
\]
such that \( \tilde{f} \in k[x_1, \ldots, x_p] \), \( n_{ij} > 0 \) for all \( j \neq i \), \( n_{ii} = 0 \) for all \( 1 \leq i \leq p \) and \( a_i \in k^* \) for all \( 1 \leq i \leq p \). The group \( (\mathbb{F}_p^\times)^p \times S_p \) acts on the radical ideal generated by \( f \). If \( f \) is divisible by \( x_1 \ldots x_p \), then \( V(f) \cap V(x_1) \) contains \( V(x_1) \), which contradicts Lemma 4.7. By Lemma 4.3, \( f \) is fixed by \( (\mathbb{F}_p^\times)^p \times S_p \) up to a sign and fixed by the subgroup \( S = (\mathbb{F}_p^\times)^p \rtimes A_p \).

We write \( w_i \) for \( x_1^{n_{1i}} \ldots x_p^{n_{pi}} \). Let \( \tau = (12 \ldots p) \in A_p \) be the \( p \)-cycle. For any \( 1 \leq j \leq p \), we have \( \tau^{j-1} f = f \). So

\[
(\tau^{j-1} \tilde{f} - \tilde{f}) x_1 \ldots x_p = (a_j w_j - a_1 \tau^{j-1} w_1) + \text{terms divisible by } x_j
\]

where both \( w_j, \tau^{j-1} w_1 \) are independent of \( x_j \). We must have \( a_j = a_1 \) and \( w_j = \tau^{j-1} w_1 \). Let \( a = a_1 = \ldots = a_p \neq 0 \). For any \( \beta = (\beta_1, \ldots, \beta_p) \in (\mathbb{F}_p^\times)^p \), let \( \beta(j) \) be the element in \( (\mathbb{F}_p^\times)^p \) such that its \( j \)-th coordinate is \( \beta_j \) and 1 elsewhere. Note that \( \beta(j) f = f \). So

\[
(\beta_j (\beta(j) \tilde{f}) - \tilde{f}) x_1 \ldots x_p = \sum_{i \neq j} (a - a\beta_j^{n_{ji}}) w_i
\]

For \( i \neq j \), \( x_i \) divides the left-hand term and all \( w_i \)’s such that \( r \neq i, j \) on the right-hand side. So \( a - a\beta_j^{n_{ji}} = 0 \), i.e., \( \beta_j^{n_{ji}} = 1 \). This shows that for \( i \neq j \) each \( n_{ji} \) is divisible by \( p - 1 \).

Note that \( V(1/af) = V(f) \), so we may assume that \( f \) has the form

\[
f(x_1, \ldots, x_p) = x_1 \ldots x_p \tilde{f} + \sum_{i=0}^{p-1} \tau^i (x_2^{n_{2i}(p-1)} \ldots x_p^{n_{pi}(p-1)}) \tag{\star}
\]

where \( w_j = \tau^{j-1} x_2^{n_{2j}(p-1)} \ldots x_p^{n_{pj}(p-1)} \). The sum and the term \( x_1 \ldots x_p \tilde{f} \) in \( \star \) are clearly invariant under the group \( S \). By Lemma 4.2, \( x_1 \ldots x_p \tilde{f} = x_1^{p-1} \ldots x_p^{p-1} \tilde{f} \) for some homogeneous polynomial \( \tilde{f} \) fixed by \( S \). For any \( j \neq 2 \), let \( (2j)f = \epsilon f \) where \( \epsilon \in \{\pm 1\} \).

So

\[
((2j) \tilde{f} - \epsilon \tilde{f}) x_1 \ldots x_p = (\epsilon a w_j - a(2j) w_2) + \text{terms divisible by } x_j
\]
gives us \( w_j = (2j) w_2 \). Comparing the power of \( x_{j+1} \), we have \( n_2 = n_j \). So \( n_2 = \ldots = n_p \).

Let \( V \subseteq \mathbb{P}^{n-1}(k) \) be a projective variety of dimension \( m \). The intersection of \( V \) and a generic linear subspace \( W \) of \( \mathbb{P}^{n-1} \) of dimension \( n - m - 1 \) is a union of \( s \) points for some fixed positive integer \( s \). This positive integer is the degree of the projective variety \( V \). In case \( \dim W < n - m - 1 \), then \( V \cap W = \emptyset \).

**Corollary 4.8.** For \( p \geq 3 \), the degree of the projectivized variety \( V^L_{\mathbb{E}}(S^{(mp)}) \) is non-zero and divisible by \( (p-1)^2 \).
Corollary 5.2. Let $W \subseteq k^p$ be a generic linear subspace of dimension 2. For any component $V_i(S^{(p^q)})$ such that $0 \leq i \leq p - 2$, we have $W_i(S^{(p^q)}) \cap W = \{0\}$. So

$$V_2^*(S^{(p^q)}) \cap W = W_{p+1}(S^{(p^q)}) \cap W = s$$

where $s$ is the degree of the homogeneous variety $W_{p+1}(S^{(p^q)})$, i.e., the degree of a homogeneous polynomial $f$ defining $W_{p+1}(S^{(p^q)})$ such that $\sqrt{f} = \langle f \rangle$. So $s$ is non-zero and divisible by $(p-1)^2$.

5. The Variety for the Specht Module $S^\mu$ where $\mu$ is $p$-regular

5.1. The permutation modules. Let $m, n$ be two non-negative integers and $m = \ldots + m_1p + m_0$, $n = \ldots + n_1p + n_0$ be their $p$-adic expansions. The number $m$ is $p$-contained in $n$ if $m_i \leq n_i$ for all $i \geq 0$. In this case we write $m \subseteq_p n$. Note that $m \subseteq_p n$ if and only if $n^m \neq 0 \pmod{p}$. Corollaries 5.2 and 5.3 rely on Theorem 3.3 of \cite{S}.

Theorem 5.1 (3.3 of \cite{S}). The Young module $Y^{(r-s,s)}$ is a direct summand of the permutation module $M^{(r-m,m)}$ if and only if $m - s \subseteq_p r - 2s$.

Corollary 5.2. For any integer $p < m \leq p^2/2$, we have a decomposition

$$M^{(p^2-m,m)} \cong M^{(p^2-m+p,m-p)} \oplus Q$$

where $Q$ has a filtration with Specht factors

$$S^{(p^2-m+p-1,m-p+1)}, S^{(p^2-m+p-2,m-p+2)}, \ldots, S^{(p^2-m,m)}$$

Proof. It suffices to show that if $Y^{(p^2-s,s)}$ is a direct summand of $M^{(p^2-m+p,m-p)}$, then it is also a direct summand of $M^{(p^2-m,m)}$. Note that the trivial module $k \cong Y^{(p^2)}$ is not a direct summand of $M^{(p^2-w,w)}$ for any $0 < w \leq p^2/2$. Let $0 < s \leq m - p$ and $Y^{(p^2-s,s)}$ be a direct summand of $M^{(p^2-m+p,m-p)}$, i.e.,

$$a_1p + a_0 = m - p - s \subseteq_p p^2 - 2s = b_1p + b_0$$

where $0 \leq a_0, a_1, b_0, b_1 \leq p - 1$. Note that $a_0 + (a_1 + 1)p$ is the $p$-adic expansion of $m - s$. If $a_1 + 1 > b_1$, then $a_1 = b_1$. So $(p^2 - 2s) - (m - p - s) = b_0 - a_0 < p$. On the other hand, since $s \leq m \leq p^2/2$, we have $(p^2 - 2s) - (m - p - s) = p^2 + p - (m + s) \geq p$, which is a contradiction. This shows that $m - s \subseteq_p p^2 - 2s$.

Corollary 5.3. Let $p$ be an odd prime.

(i) If $n \not\equiv 1, 2 \pmod{p}$, then $M^{(np-2p,2p)} \cong M^{(np-p,p)} \oplus Q$;

(ii) If $n \equiv 1 \pmod{p}$, then $M^{(np-p,p)} \cong k \oplus N$ for some $k \Sigma_{np}$-module $N$ and $M^{(np-2p,2p)} \cong N \oplus R$.

In both cases, $Q, R$ have filtrations with Specht factors

$$S^{(np-n-1,p+1)}, S^{(np-n-2,p+2)}, \ldots, S^{(np-2p,2p)}$$

Proof. Suppose that $0 < s \leq p$ and let $Y^{(np-s,s)}$ be a direct summand of $M^{(np-p,p)}$, i.e.,

$$a_0 = p - s \subseteq_p np - 2s = \ldots + b_2p^2 + b_1p + b_0$$

...
where $0 \leq a_i, b_i < p$ for all $i \geq 0$. Note that $b_0 + 2s \equiv 0 \pmod{p}$ and $a_0 + p$ is the $p$-adic expansion of $2p - s$. On the other hand, we have

$$p + 1 \leq p + s = a_0 + 2s \leq b_0 + 2s \leq (p - 1) + 2p = 3p - 1$$

So $b_0 + 2s = 2p$. Substituting into the $p$-adic expansion of $np - 2s$, we see that $n \equiv 2 \pmod{p}$ unless $b_1 \neq 0$. This shows that $2p - s \subseteq np - 2s$ in both (i) and (ii).

Suppose that $n \not\equiv 1 \pmod{p}$, the equation

$$\binom{np}{2p} = \binom{np}{np-2p+1}(np-2p+2)\ldots(np-p)\over (p+1)(p+2)\ldots2p$$

implies that the trivial module $k$ is a direct summand of $M^{(np-p, p)}$ if and only if it is a direct summand of $M^{(np-2p, 2p)}$. So (i) is proved. If $n \equiv 1 \pmod{p}$, then $k$ is a direct summand of $M^{(np-p, p)}$ but not of $M^{(np-2p, 2p)}$. □

**Lemma 5.4.** Let $\mu = (n_1 p, \ldots, n_s p)$ be a partition of $np$ such that $n_1 \geq n_2 \geq \ldots \geq n_s > 0$. Then the generic Jordan type of the module $M^\mu|_{E_n}$ is $(p^a, 1^b)$ where

$$a = \frac{\dim_k M^\mu - b}{p}, \quad b = \frac{n!}{n_1!n_2!\ldots n_s!}$$

**Proof.** The permutation module $M^\mu$ is isomorphic to the trivial module induced from the Young subgroup $\mathfrak{S}_\mu$ to $\mathfrak{S}_{np}$. Apply the Mackey decomposition formula

$$M^\mu|_{E_n} \cong k\mathfrak{S}_\mu \uparrow \mathfrak{S}_{np}|_{E_n} \cong \bigoplus_{E_n^{\mu} \mathfrak{S}_\mu} gk\uparrow|_{E_n^{\mu} \mathfrak{S}_\mu}$$

Double coset representatives of $E_n, \mathfrak{S}_\mu$ in $\mathfrak{S}_{np}$ correspond to the orbits $O_{E_n}(g_i)$ of the action of $E_n$ on the left coset representatives $g_1, \ldots, g_m$ of $\mathfrak{S}_\mu$ in $\mathfrak{S}_{np}$. The stabilizer $\text{Stab}_{E_n}(g_i)$ of $g_i$ consists precisely of the elements $e \in E_n$ such that $eg_i \in g_i\mathfrak{S}_\mu$.

So $\text{Stab}_{E_n}(g_i) = E_n$ if and only if $O_{E_n}(g_i) = \{g_i\}$ if and only if the $\mu$-tabloid corresponding to $g_i$ is fixed by $E_n$. For each $\mu$-tabloid $\{t\}$, denote by $R_i(t)$ the set consisting of integers in the $i$th row of $\{t\}$. Take $\{t_0\}$ as the $\mu$-tabloid such that

$$R_i(t_0) = \left\{1 + \sum_{j=0}^{i-1} \mu_j, 2 + \sum_{j=0}^{i-1} \mu_j, \ldots, i + \sum_{j=0}^{i-1} \mu_j \right\}$$

So the $\mu$-tabloid corresponding to $g_i$ is $\{g_it_0\}$. Note that the $\mu$-tabloids $\{t\}$ fixed by $E_n$ are precisely those satisfying the property that for each $1 \leq i \leq n$ there exists some $1 \leq u \leq s$ such that $\{1 + (i-1)p, \ldots, ip\} \subseteq R_u(t)$. So

$$k\mathfrak{S}_\mu \uparrow \mathfrak{S}_{np}|_{E_n} \cong \left( \bigoplus_{E_n^{\mu} \mathfrak{S}_\mu} kE_n^{\mu} \uparrow \mathfrak{S}_\mu \uparrow E_n \right) \oplus \left( \bigoplus_{E_n^{\mu} \mathfrak{S}_\mu \supset E_n} k \right)$$

The summand $kE_n^{\mu} \mathfrak{S}_\mu \uparrow \mathfrak{S}_\mu \uparrow E_n$ is generically free if $E_n \cap g\mathfrak{S}_\mu < E_n$; otherwise, it has generic Jordan type $(1)$, and there are precisely $\frac{(n_1 + \ldots + n_s)!}{n_1! \ldots n_s!}$ of them. □
5.2. The map $\Phi$. Every partition can be associated to a set of $\beta$-numbers and represented by an abacus (§2.7 of [1]). For the rest of our discussion, whenever we speak of an abacus of $\mu$, we mean the abacus associated to the choice of $\beta$-numbers given by the first column hook lengths of $\mu$. So the abacus of the partition $\mu = (\mu_1, \ldots, \mu_s)$ has beads $\mu_i + (s - i)$ where $1 \leq i \leq s$. In the case $s \leq p$, the $p$-core $\tilde{\mu}$ of $\mu$ is empty if and only if for each $1 \leq i \leq s$ there is a unique $0 \leq j_i \leq s - 1$ such that $\mu_i + (s - i) \equiv j_i (\text{mod } p)$. If $\mu_i \not\equiv 0 (\text{mod } p)$ for some $1 \leq i \leq s$, let $1 \leq b \leq s$ be the number such that $\mu_b \not\equiv 0 (\text{mod } p)$ and $\mu_i \equiv 0 (\text{mod } p)$ for all $b + 1 \leq i \leq s$. If we insert the beads $\mu_s, \mu_{s-1} + 1, \ldots, \mu_b + (s - b)$ successively into the abacus and ignore the position of the bead in each runner, we see the following

$$
\bullet \quad \cdots \quad \bullet \quad \circ \quad \cdots \quad \circ \quad \bullet_b$$

where there are $s - b$ beads $\bullet$ on the left-hand side, more than one $\circ$ in the middle and the last bead $\bullet_b$ corresponds to $\mu_b$. Since $\tilde{\mu} = \emptyset$, there is a unique number $1 \leq a < b$ such that $\mu_a + (s - a) \equiv \mu_b + (s - b) - 1 (\text{mod } p)$, i.e., the bead immediately to the left of $\bullet_b$.

**Hypothesis 5.5.** Suppose that $\mu = (\mu_1, \ldots, \mu_s)$ is a partition with $s \leq p$, $\tilde{\mu} = \emptyset$ and there are unique numbers $1 \leq a < b \leq s$ with the properties:

(i) $\mu_i \equiv 0 (\text{mod } p)$ for all $b + 1 \leq i \leq s$ and $\mu_b \not\equiv 0 (\text{mod } p)$,

(ii) $\mu_a - a \equiv \mu_b - 1 - b (\text{mod } p)$.

Let $\mu$ be a partition satisfying Hypothesis 5.5, $\eta = (\mu_1, \ldots, \mu_{b-1}, \mu_b-1, \mu_{b+1}, \ldots, \mu_s)$ and $\Omega$ be the set consisting of all (proper) partitions $\mu(j)$ where $1 \leq j \leq s + 1$ such that $\mu(j)$ is the partition obtained from $\eta$ by adding a node to the end of the $j$th row (assuming $\mu_{s+1} = 0$). We claim that $\mu(a), \mu(b) \in \Omega$ and there is no $\mu(j) \in \Omega$ with empty $p$-core other than $\mu = \mu(b)$ and $\mu(a)$. Suppose that $\mu_{a-1} = \mu_a$, we have $\mu_{a-1} + (s - (a - 1)) \equiv \mu_b - (s - b) (\text{mod } p)$. This implies that the $p$-core of $\mu$ is not empty. This contradiction shows that $\mu_{a-1} > \mu_a$ and so $\mu(a) \in \Omega$. In the case $j = a$, $\mu(a)_i + (s - i) = \mu_i + (s - i)$ for all $i \neq a, b$ and

$$
\mu(a)_a + (s - a) \equiv \mu_b + (s - b) (\text{mod } p)
$$

$$
\mu(a)_b + (s - b) \equiv \mu_a + (s - a) (\text{mod } p)
$$

So $\mu(a)$ has empty $p$-core because $\mu$ has empty $p$-core. It is clear that $\mu = \mu(b) \in \Omega$.

For $1 \leq j \leq s \leq p$ such that $j \neq a, b$, we have

$$
\mu(j)_b + (s - b) \equiv \mu(j)_a + (s - a) (\text{mod } p)
$$

So the $p$-core of $\mu(j)$ is not empty. In the case $j = s + 1$, there are at most $p + 1$ beads in the abacus of $\mu(s + 1)$. It is clear that

$$
\mu(s + 1)_a + ((s + 1) - a) \equiv \mu_b + (s - b) \equiv \mu(s + 1)_b + ((s + 1) - b) (\text{mod } p)
$$

So $\mu(s + 1)$ has non-empty $p$-core unless $s = p$ and $\mu_b + (s - b) \equiv 0 (\text{mod } p)$, i.e., $\mu_b \equiv b (\text{mod } p)$. In this case, we must have $\mu_p \equiv 0 (\text{mod } p)$: if not, then $b = p$ and $\mu_b \equiv 0 (\text{mod } p)$, contradicts to the hypothesis of $\mu$. However there are two beads corresponding to $\mu(p + 1)_{p+1}$ and $\mu(p + 1)_p$ lying in the first runner of the abacus, so $\mu(p + 1)$ has non-empty $p$-core.
Fix a positive integer \( n \). Let \( \Lambda \) be the set consisting of all partitions \( \mu = (\mu_1, \ldots, \mu_s) \) of \( np \) with \( s \leq p \) and \( \tilde{\mu} = \emptyset \). We define a map \( \phi : \Lambda \to \Lambda \) as follows.

\[
\phi(\mu) = \begin{cases} 
\mu(a) & \text{if } \mu \text{ satisfies Hypothesis 5.5} \\
\mu & \text{otherwise}
\end{cases}
\]

**Example 5.6.** Let \( p \) be an odd prime, \( \mu = (u, v, 2^m) \) with \( 0 \leq m \leq p - 2 \) and \( \tilde{\mu} = \emptyset \); for instance, taking \( u \equiv p - m - 1 \pmod{p} \) and \( v \equiv p - m + 1 \pmod{p} \). If \( m > 0 \), then \( b = m + 2 \) and \( a = 2 \). So \( \phi(\mu) = (u, v + 1, 2^{m-1}, 1) \). Now we follow the procedure for \( \phi(\mu) \), we get \( b = m + 2 \) and \( a = 1 \). So \( \phi^2(\mu) = (u + 1, v + 1, 2^{m-1}) \). By induction on the integer \( m \), we see that \( \phi^{2m}(\mu) = (U, V) \) with \( U \equiv p - 1 \pmod{p} \), \( V \equiv 1 \pmod{p} \) and \( U + V = |\mu| \). So \( \phi^{2m+1}(\mu) = (U + 1, V - 1) \), and indeed \( \phi^t(\mu) = (U + 1, V - 1) \) for all \( t \geq 2m + 1 \) by our definition.

**Definition 5.7.** Let \( \mu \) be a partition not more than \( p \) parts with empty \( p \)-core. There is a positive integer \( t(\mu) \) such that for all integers \( t \geq t(\mu) \) we have \( \phi^t(\mu) = \phi^{t(\mu)}(\mu) \). We define \( \Phi(\mu) = \phi^{t(\mu)}(\mu) \).

Recall from Section 2 the definition of generic Jordan type and stable generic Jordan type of a finitely generated \( kE \)-module \( M \) where \( E \) is an elementary abelian \( p \)-group. We have \( V_{E_n}^\sharp(M) = V_{E_n}^\sharp(k) \) if and only if \( M \) is not generically free. For each \( 1 \leq i \leq p \), we define \( n_M(i) \) as the number of Jordan blocks of size \( i \) in the generic Jordan type of \( M \). In the case \( M = S^\mu \downarrow_{E_n} \) where \( \mu \) is a partition of \( np \), we write \( n_\mu(i) \) for \( n_{S^\mu \downarrow_{E_n}}(i) \). Let \( N \) be another finitely generated \( kE \)-module. If \( n_M(i) = n_N(p - i) \) for all \( 1 \leq i \leq p - 1 \), we say that \( M, N \) is a pair of modules of complementary stable Jordan type.

**Lemma 5.8.** Suppose that \( \mu = (\mu_1, \ldots, \mu_s) \) is a partition of \( np \) with \( 1 \leq s \leq p \) and \( \tilde{\mu} = \emptyset \). Let \( 1 \leq a < b \leq s \) be the unique numbers such that

\begin{enumerate}
\item[(i)] \( \mu_i \equiv 0 \pmod{p} \) for all \( b + 1 \leq i \leq s \) and \( \mu_b \not\equiv 0 \pmod{p} \),
\item[(ii)] \( \mu_a \equiv \mu_b - 1 \bmod{p} \).
\end{enumerate}

Then \( \phi(\mu) \) has empty \( p \)-core and \( S^\mu \downarrow_{E_n}, S^{\phi(\mu)} \downarrow_{E_n} \) is a pair of modules of complementary stable Jordan type. In particular, \( V_{E_n}^\sharp(S^\mu) = V_{E_n}^\sharp(k) \) if and only if \( V_{E_n}^\sharp(S^{\phi(\mu)}) = V_{E_n}^\sharp(k) \).

**Proof.** Let \( \eta \) and \( \Omega \) as before. Since there are precisely two partitions \( \mu, \phi(\mu) \) in \( \Omega \) with empty \( p \)-core, using Proposition 2.2 (i), (ii) and (iv), we get

\[
S^n \uparrow_{S^{np}} \downarrow_{E_n} \cong Q \oplus \text{generically free direct summand}
\]

where \( Q \) is a direct summand with factors \( S^\mu \downarrow_{E_n} \) and \( S^{\phi(\mu)} \downarrow_{E_n} \), and Specht modules not in the principal block contribute to the generically free summand. Since

\[
\dim V_{E_n}(S^n \uparrow_{S^{np}}) \leq \dim V_{S^{np}}^*(S^n \uparrow_{S^{np}}) = \dim \text{res}_{S^{np}, S^{np-1}}^* V_{S^{np-1}}^*(S^n) \leq n - 1
\]

the module \( S^n \uparrow_{S^{np}} \downarrow_{E_n} \) is generically free. Hence \( S^\mu \downarrow_{E_n}, S^{\phi(\mu)} \downarrow_{E_n} \) is a pair of modules of complementary stable Jordan type. Since \( S^{\phi(\mu)} \downarrow_{E_n} \) has stable generic Jordan type either the same or complementary to that of \( S^\mu \downarrow_{E_n} \), we get the second assertion.

**Lemma 5.9.** Let \( n \geq 2 \) be a positive integer and \( \mu = (np - p, p) \). If \( p \) is an odd prime, then \( n - 2 \leq n_\mu(1) \leq n + 1 \). If \( p = 2 \), then \( n_\mu(1) = n - 2 \).
Proof. Let $p$ be an odd prime. Note that $S^{(np)} \downarrow E_n \cong k_{E_n}$ has generic Jordan type (1) and $S^{(np-1,1)} \downarrow E_n$ has stable generic Jordan type $(p-1)$, which is complementary to (1). The permutation module $M^{(np-p,p)}$ decomposes into $Q \oplus S$ where $S \downarrow E_n$ is a generically free $k_{E_n}$-module and $Q$ has three Specht factors $S^{(np)}$, $S^{(np-1,1)}$ and $S^{(np-p,p)}$ reading from the top. The stable generic Jordan type for $M^{(np-p,p)} \downarrow E_n$ is given in Lemma 5.4 as $(1^n)$. Suppose that $A$ is the quotient $Q/S^{(np-p,p)}$. We want to figure out the possible stable generic Jordan types of $A \downarrow E_n$. Considering the short exact sequence $0 \to S^{(np-1,1)} \downarrow E_n \to A \downarrow E_n \to k \to 0$ and using Proposition 2.2 (iii), we see that $A \downarrow E_n$ is generically free or has stable generic Jordan type $(p-1,1)$. Now consider the short exact sequence $0 \to S^{(np-p,p)} \downarrow E_n \to Q \downarrow E_n \to A \downarrow E_n \to 0$. In the case $A \downarrow E_n$ is generically free, $S^{(np-p,p)} \downarrow E_n$ has stable generic Jordan type the same as $M^{(np-p,p)} \downarrow E_n$, which is $(1^n)$. Suppose that the stable generic Jordan type of $A \downarrow E_n$ is $(p-1,1)$. Using Proposition 2.2 (iii), one can figure out that the possible stable generic Jordan types of $S^{(np-p,p)} \downarrow E_n$ are one of the listed below:

(a) $(p-1,1^{n+1})$,
(b) $(p-1,2,1^{n-1})$,
(c) $(1^n)$,
(d) $(2,1^{n-2})$.

In any case, $n - 2 \leq n_{\mu}(1) \leq n + 1$.

Suppose that $p = 2$. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$ be a generic point and $u_{\alpha} = 1 + \sum_{i=1}^{n} \alpha_i((2i-1,2i)-1)$. Consider the map

$$M^{\mu} \downarrow_{k(u_{\alpha})} = M_1 \oplus P_1 \xrightarrow{f} S^{(2n-1,1)} \downarrow_{k(u_{\alpha})} \oplus S^{(2n)} \downarrow_{k(u_{\alpha})} = M_2 \oplus P_2$$

given by $f(t_{i,j}) = (t_i + t_j, 1)$ where $t_{i,j}$ is the $\mu$-tabloid with $i,j$ in the second row and $t_i$ is the $(2n-1,1)$-tabloid with $i$ in the second row, also $M_1, M_2$ are the non-free parts of $M^{\mu} \downarrow_{k(u_{\alpha})}, S^{(2n-1,1)} \downarrow_{k(u_{\alpha})} \oplus S^{(2n)} \downarrow_{k(u_{\alpha})}$ respectively. Note that the Jordan types of $M_1, M_2$ are $(1^n), (1^2)$ respectively, and $P_1, P_2$ are projective modules or equivalently free modules. The kernel of the map $f$ is precisely $S^{\mu} \downarrow_{k(u_{\alpha})}$. The module $M_1$ is $k$-spanned by $t_{2i-1,2i}$ for $1 \leq i \leq n$; meanwhile, $P_1$ is $k$-spanned by those $t_{i,j}$’s not listed before. The module $M_2$ is $k$-spanned by both the tabloid in $S^{(2n)}$ and $\sum t_i \in S^{(2n-1,1)} \subseteq M^{(2n-1,1)}$. Consider the map $f$ in the stable module category, the induced map $\tilde{f} : M_1 \to M_2$ splits and it has kernel with Jordan type $(1^{n-2})$. So ker $f \cong \ker \tilde{f} \oplus P_3$ for some projective module $P_3$. This shows that the stable generic Jordan type of $S^{\mu} \downarrow E_n$ is $(1^{n-2})$, i.e., $n_{\mu}(1) = n - 2$.

5.3. Proof of Theorem 3.3. Suppose that $\mu$ is a partition satisfying Hypothesis 3.2. In order to prove Theorem 3.3 by Lemma 5.8, it suffices to show that $V_{E_n}^{\ell}(S^{(\Phi(\mu))}) = V_{E_n}^{\ell}(k)$. The idea of the proof is to show that $n_{\Phi(\mu)}(r) > 0$ for some $1 \leq r < p - 1$.

Proof of Theorem 3.3. The result is obvious for Type (H3); indeed, $n_{\Phi(\mu)}(1) = 1$.

Type (H1): For any $1 \leq m < p/2$, let $\eta(m) = (p^2 - mp, mp)$ and $\lambda(m) = (p^2 - mp - 1, mp + 1)$. Note that $\Phi(\lambda(m)) = \eta(m)$. Taking $n = p$ in Lemma 5.9 we have $n_{\eta(1)}(1) \geq p - 2$. We claim that for all $1 \leq m < p/2$

$$n_{\eta(m)}(1) + n_{\eta(m)}(2) + \ldots + n_{\eta(m)}(m) \geq p - 2 \quad (\ast)$$
By induction on \( m \), suppose that we have the inequality as in (*). By Corollary 5.2, we have a decomposition

\[ \mathcal{M}^{\eta(m+1)} \cong \mathcal{M}^{\eta(m)} \oplus S \oplus F \]

where \( F \mid_{E_p} \) is a generically free \( kE_p \)-module and \( S \) has a filtration with Specht factors \( S^{\lambda_i(m)}, S^{\eta(m+1)} \) reading from the top. By Lemma 5.4, \( S \mid_{E_p} \) has stable generic Jordan type 1 \((\mu \geq 3, \lambda)\). Note that \( n^{\eta(m+1)} \mid_{E_p}, S^{\lambda(m)} \mid_{E_p} \) is a pair of modules of complementary stable Jordan type, so \( S^{\lambda(m)} \mid_{E_p} \) has stable generic Jordan type satisfying the inequality

\[ n^{\lambda(m)}(p - m) + n^{\lambda(m)}(p - m + 1) + \ldots + n^{\lambda(m)}(p - 1) \geq p - 2 \]

The short exact sequence \( 0 \rightarrow S^{\eta(m+1)} \mid_{E_p} \rightarrow S^{\lambda(m)} \mid_{E_p} \rightarrow 0 \) leads to the short exact sequences \( 0 \rightarrow J_i \rightarrow J_a \oplus J_b \rightarrow J_j \rightarrow 0 \) with \( a, b \in \{0, 1, p\} \). We focus on the cases where \( p - m \leq j \leq p - 1 \). Note that we must have either \( J_p \oplus J_1 \) or \( J_p \) in the middle. Using Proposition 2.2 (iii), in the first case, \( i + j = 1 + p \) and hence \( 2 \leq i \leq m + 1 \); in the latter case, \( i + j = p \) and hence \( 1 \leq i \leq m \). So we conclude that

\[ n^{\eta(m+1)}(1) + n^{\eta(m+1)}(2) + \ldots + n^{\eta(m+1)}(m + 1) \geq p - 2 \]

Since \( n^{\eta(m)}(i) \) is non-zero for some \( 1 \leq i \leq m \), \( S^{\eta(m)} \mid_{E_p} \) is not generically free. So the rank variety \( V^2_{E_p}(S^{\Phi(\mu)}) = V^2_{E_p}(k) \) in this case.

Type (H2): Note that \( n > 2 \). The proof is akin to the proof of Type (H1), except that we use Corollary 5.3 instead of Corollary 5.2. So \( n^{\Phi(\mu)}(1) + n^{\Phi(\mu)}(2) \geq n - 2 > 0 \).

Type (H4): Consider the case where \( \Phi(\mu) = (2n - 2, 2) \neq (2, 2) \). Note that \( n > 2 \). Now we use Lemma 5.9 for \( p = 2 \) to deduce that \( n^{\Phi(\mu)}(1) = n - 2 > 0 \). Now take \( \Phi(\mu) = (2n - 4, 4) \neq (4, 4) \) where \( n > 4 \). The permutation module \( M^{\Phi(\mu)} \) has a filtration with Specht factors \( S^{\lambda_i(i)} \) one for each \( 0 \leq i \leq 4 \) where \( \lambda(i) = (2n - i, i) \).

Over \( p = 2 \), any short exact sequence of \( kE \)-modules \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) satisfies the inequality \( n_B(1) \leq n_A(1) + n_C(1) \). So

\[ \left( \begin{array}{c} n \\ 2 \end{array} \right) = n_{M^{\Phi(\mu)} \mid_{E_n}}(1) \leq \sum_{i=0}^{4} n^{\lambda(i)}(1) \]

Note that \( n^{\lambda(0)}(1) = n^{\lambda(1)}(1) = 1 \) and \( n^{\lambda(3)}(1) = n^{\lambda(2)}(1) = n - 2 \). Suppose that \( S^{\Phi(\mu)} \mid_{E_n} \) is generically free, i.e., \( n^{\Phi(\mu)}(1) = 0 \), we deduce that \( n^2 - 5n + 4 \leq 0 \), i.e., \( 1 \leq n \leq 4 \), a contradiction. So \( S^{\Phi(\mu)} \mid_{E_n} \) is not generically free, i.e., \( V^2_{E_n}(S^{\Phi(\mu)}) = V^2_{E_n}(k) \).

**Example 5.10.** Let \( p \) be an odd prime and \( \mu \) be the partition as in Example 5.6. Then \( V^2_{E_p}(S^{\mu}) = V^2_{E_p}(k) \) and the complexity of \( S^{\mu} \) is \( p \).

**Example 5.11.** Let \( p = 3 \) and \( \mu \) be a partition of 9. Suppose that \( \mu \neq (3^3) \). In the case \( \bar{\mu} \neq \varnothing \), we use Proposition 2.1 (iv) and the hook formula to calculate the rank variety \( V^2_{E_3}(S^{\mu}) \); otherwise, we use Proposition 2.2 (v) and Theorem 3.3. In the case \( \mu = (3^3) \), the software MAGMA was used by Jon Carlson to determine the
rank variety $V_{E_3}^2(S^{(3^3)})$ which is precisely the zero set of the radical ideal

$$\langle x_1^2x_2^2 + x_2^2x_3^2 + x_1^2x_3^2 \rangle$$

Note that it fits into Theorem 3.1 (ii), where $\tilde{f} = 0$ and $n = 1$.

The Variety $V_{E_3}(S^\mu)$ for the case $p = 3$ and $|\mu| = 9.$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$V_{E_3}(S^\mu)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(9), (1^9)</td>
<td>$V_3$</td>
</tr>
<tr>
<td>(81), (21^7)</td>
<td>$V_3$</td>
</tr>
<tr>
<td>(72), (2^21^5)</td>
<td>$V_1$</td>
</tr>
<tr>
<td>(63), (2^31^3)</td>
<td>$V_3$</td>
</tr>
<tr>
<td>(54), (2^41)</td>
<td>$V_3$</td>
</tr>
<tr>
<td>(71^2), (31^6)</td>
<td>$V_3$</td>
</tr>
<tr>
<td>(621), (321^4)</td>
<td>$V_3$</td>
</tr>
<tr>
<td>(531), (32^21^2)</td>
<td>${0}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$V_{E_3}(S^\mu)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4^21), (32^3)</td>
<td>$V_3$</td>
</tr>
<tr>
<td>(52^2), (3^21^3)</td>
<td>$V_3$</td>
</tr>
<tr>
<td>(432), (3^221)</td>
<td>$V_3$</td>
</tr>
<tr>
<td>(521^2), (421^3)</td>
<td>$V_1$</td>
</tr>
<tr>
<td>(431^2), (42^21)</td>
<td>$V_1$</td>
</tr>
<tr>
<td>(61^3), (41^5)</td>
<td>$V_3$</td>
</tr>
<tr>
<td>(51^4)</td>
<td>$V_3$</td>
</tr>
</tbody>
</table>

In Table 1, $V_1 = \text{res}_{E_1}^{E_3} V_{E_1}(k)$, $V_2 = V(x_1^2x_2^2 + x_2^2x_3^2 + x_1^2x_3^2)$ and $V_3 = V_{E_3}(k)$ where $E_1 = \langle (1, 2, 3) \rangle \subseteq S_9$. The subscript $i$ of $V_i$ gives the dimension, which is the complexity of $S^\mu$.

Acknowledgement. I thank my supervisor David Benson for his guidance and suggestions in this paper.

References