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<td>Author(s)</td>
<td>Bose, Sumanta; Ramaraj, M.; Raghavan, S.; Kumar, Swadhyaya</td>
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Mathematical Modeling, Equivalent Circuit Analysis and Genetic Algorithm Optimization of an N-sided Regular Polygon Split Ring Resonator (NRPSRR)

Sumanta Bose\textsuperscript{a}*, Swadhyaya Kumar\textsuperscript{a}, M. Ramaraj\textsuperscript{a}, Dr. S. Raghavan\textsuperscript{a}

\textsuperscript{a}Department of Electronics and Communication Engineering, National Institute of Technology, Trichy-620015, India.
Email: sumantabose@gmail.com, swadhyayak@gmail.com, mu_ramaraj@yahoo.co.in, raghavan@nitt.edu

Abstract

In this paper, a mathematical model is proposed to estimate the resonant frequency of a N-sided Regular Polygon Split Ring Resonator (NRPSRR), which is extended to the Circular Split Ring Resonator (CSRR) for N tending to infinity. The model also predicts the variation in the resonant frequency with the angle of rotation (skew) between the inner and the outer polygon. It is mathematically shown that the minimum resonant frequency is obtained at zero skew. Genetic Algorithm Optimization is used over the independent parameters deciding the geometry of the NRPSRR to optimize them for a particular desired resonant frequency by minimizing the relative error between the calculated and desired frequency. NRPSRR metamaterial loaded microstrip antenna could be used to obtain highly directional beam pattern and increased gain for miniaturized (electrically small) antenna systems in the microwave regime.

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Keywords: Genetic Algorithm; Metamaterials; Mathematical Analysis; N-sided Regular Polygon Split Ring Resonator

* Corresponding author. Tel.: +91-431-2503342; fax: +91-431-2500133.
E-mail address: sumantabose@gmail.com
1. Introduction and Background

Metamaterials are defined as artificial materials which have the ability to exhibit electromagnetic characteristics not readily found in naturally occurring material, such as, negative refractive index and artificial magnetism [1,2]. Split Ring Resonator (SRR) are common in the metamaterial design as they exhibit negative permittivity and permeability for frequencies close to their resonant frequency. It is a major engineering concern to estimate the resonant frequency due to the existence of negative value of permeability over a very narrow band of frequency. Generally, SRRs vary in their shape and structure such as square SRR [3] and circular SRR [4]. Rarer structures include Triangular SRR [5] and Elliptical SRR [6,13]. In this work, a new structure and associated mathematical analysis and derivation of a N-sided Regular Polygon SRR (NRPSRR) is introduced. This work is novel in itself and no previous reference or discussion about a generalized NRPSRR has been discussed in literature to the best of the author's knowledge.

2. Mathematical Analysis

2.1. NRPSRR Geometry

Fig. 1(a) shows the schematic geometry of an NRPSRR (with \( N = 6 \), for clarity) and other dimensions indicated in the figure. The structure of NRPSRR is newly designed with strip width 'c' and spacing 'd' between the polygons. The NRPSRR has gaps of the splits \('g_1'\) and \('g_2'\). Fig. 1(b) shows the equivalent circuit model of the NRPSRR, forming an L-C network. The inductance is due to the gap between the rings and the capacitance is due to the rings and the gaps in the rings itself.

![NRPSRR Geometry and Equivalent Circuit](image)

**Nomenclature**

- **a**: Effective Radius of the polygon of the NRPSRR
- **g**: Split Gap in the conductor rings of the NRPSRR
- **c**: Width of the conductor ring in the NRPSRR
- **d**: Spacing between the conductor rings in the NRPSRR
- **h**: Depth of substrate on which the NRPSRR is embedded
2.2. Resonant Frequency Computation

With a magnetic field applied along the z-axis (refer fig. 1(a)), an electromotive force appears around the NRPSRR which makes the structure behaves like an L-C network [7] having resonant frequency, \( f_0 \), expressed as:

\[
f_0 = \frac{1}{2\pi \sqrt{2a_{eq} L_{net} C_{net}}}\]

(1)

where \( a_{eq} \) is the effective radius, \( L_{net} \) is the net inductance and \( C_{net} \) is the net effective capacitance of the equivalent L-C network of the NRPSRR. Applying general trigonometry to Fig. 1(a) and assuming \( g_1 = g_2 = g \), we estimate the effective \( a (a_{eq}) \) as:

\[
a_{eq} = 2a \cdot \sin \left( \frac{\pi}{N} \right) - \frac{g}{N}\]

(2)

By classical electromagnetism definition [7], \( L \) can be analytically expressed as:

\[
L = \frac{\mu_0 \pi^2}{12} \int_0^{\infty} [\Im(k)]^2 k^2 dk
\]

(3)

where \( \Im(k) \) is the Fourier-Bessel Transformation of \( I(k) \), the current function of the ring. For computational convenience, Bashenoff deduced a simpler expression to compute the self-inductance of any closed loop conductor using semi-empirical methods [8]. The self-inductance of any closed loop conductor is given as:

\[
L_{net} = 0.002 \cdot l \left[ \log e \frac{4l}{\rho} - \left(2 \log e \frac{l}{\sqrt{S}} + \phi \right) + \frac{\mu}{4} \right]
\]

(4)

where \( l \) is the perimeter of the NRPSRR (Appendix A); \( S \) is the area of the NRPSRR (Appendix B); \( \rho \) is the width of the cross-section of the conductor (\( = c \)); and \( \mu \) is the permeability of the conductor, taken in order to incorporate the internal linkage of the flux in the cross section of the conductor. The parameter \( \phi \) is a constant which depends on the geometry of the closed loop, which varies with the number of sides of the NRPSRR, i.e. \( N \). Values of \( \phi \) for different \( N \) are tabulated in [9]. Simplified further by Terman [10], the net inductance, \( L_{net} \) can be analytically expressed as:

\[
L_{net} = 0.00508 \cdot l \left(2.303 \log_{10} \frac{4l}{\rho} - 2.636 \right)
\]

(5)

Considering the rotation of inner ring by angle \( \theta \) in counter-clockwise direction as shown in fig. 1(a), we express \( \theta \) as:

\[
\theta = m \cdot \frac{2\pi}{N} + \psi
\]

(6)

where \( m \) is some integer and \( \psi < \frac{2\pi}{N} \). For this counter-clockwise rotation of angle \( \theta \), the perimeter of the upper half-ring decreases and that of lower half-ring increases by an amount \( \Delta \), computed using general trigonometry as:

\[
\Delta = a \sin \left( \frac{\pi}{N} \right) \cdot (2m + 1) - a \cos \left( \frac{\pi}{N} \right) \cdot \tan \left( \frac{\pi}{N} - \psi \right)
\]

(7)

Now the capacitance of the upper half-ring (\( C_u \)) and lower half-ring (\( C_l \)) can be easily computed from \( \Delta \) & \( C_{pul} \) as:

\[
C_u = \left[ N \cdot a \cdot \sin \left( \frac{\pi}{N} \right) - \Delta \right] \cdot C_{pul} ; \quad C_l = \left[ N \cdot a \cdot \sin \left( \frac{\pi}{N} \right) + \Delta \right] \cdot C_{pul}
\]

(8 - 9)
The capacitance due to the gaps (splits) in the NRPSRR, \( C_g \), can be estimated using parallel plate approximation method, while assuming \( g = g_1 = g_2 \), as:

\[
C_g = \frac{\varepsilon_r \cdot \varepsilon_r \cdot c \cdot h}{g}
\]  

(10)

Therefore, the net equivalent capacitance of the L-C network, \( C_{net} \), stands out to be:

\[
\frac{1}{C_{net}} = \frac{1}{(C_u + C_g)} + \frac{1}{(C_l + C_g)}
\]

(11)

which can be simplified using general algebra and expressed as:

\[
C_{net} = \left( \frac{N \cdot \sin \left( \frac{\pi}{N} \right) + \beta}{2} \right)^2 \times a \cdot C_{pul}
\]

(12)

where \( \beta = \frac{C_g}{a \cdot C_{pul}} \); and \( C_{pul} \) is the capacitance per unit length of the NRPSRR (Appendix C).

Thus by using equation (2), (5) and (12) in equation (1), we can determine the resonant frequency of the NRPSRR. Solving \( \frac{\partial f_\circ}{\partial \theta} = 0 \) for minimization of \( f_\circ \), gives us minimum \( f_\circ \) at \( \theta = 0^\circ \).

2.3. Extension to CSRR for \( N \to \infty \)

The derived formula for NRPSRR is used to plot the variation of resonant frequency \( f_\circ \) with varying skew angle \( \theta \). In this consideration, we use Calculus to limit \( N \to \infty \), thereby approaching towards oblique circular split ring resonator (CSRR) [4]. Fig. 2 gives a plot of \( f_\circ \) vs. \( \theta \), \( f_\circ \) being least at \( \theta = 0^\circ \).

![Fig. 2. Resonant frequency \( f_\circ \) vs. skew angle \( \theta \) of a NRPSRR, with \( N \to \infty \), \( a = 2.6 \text{ mm} \), \( \varepsilon_r = 2.43 \), \( c = 0.5 \text{ mm} \), \( d = 0.2 \text{ mm} \), \( h = 0.49 \text{ mm} \), \( g = g_1 = g_2 = 0.4 \text{ mm} \).](image)

2.4. Application in Wireless Communication

Metamaterial loaded antennas [11] are extremely useful for performance enhancement of miniaturized (electrically small) antenna systems. Incorporated with multiple miniscule NRPSRR metamaterial structure, they can produce unusual physical properties. Conventional microstrip antennas suffer from signal reflection due to comparable size with wavelength. The NRPSRR loaded microstrip antenna, on the other hand makes it behave as if it were much larger than it actually is. It stores and re-radiates energy because of the intrinsic...
physical property of the structure. Fig. 3 shows the structure of the NRPSRR metamaterial loaded microstrip antenna.

![Microstrip Antenna](image)

**Fig. 3.** NRPSRR metamaterial loaded microstrip antenna.

This novel NRPSRR metamaterial loaded antenna is useful for wireless systems encompassing wireless communication hardware like RF, microwave, THz & optical frequencies transmission and devices used for non-invasive detection or monitoring in medicine and biology such as detection of breast cancer. With the usage of metamaterial loaded antennas they have continued to decrease in electrical size while delivering better antenna directivity, gain and power transmission.

3. Genetic Algorithm Optimization

3.1. Introduction to Genetic Algorithm

The Genetic Algorithm (GA) [12] is an optimization technique inspired from natural combination of genes in living organism. It is regarded as one of the efficient way to optimize various parameters related by constrains. The main advantage of GA is that, we can modify the algorithm so as to get better results. In this process, the parameters to be optimized is encoded as a fixed length bits and all these sequence are concatenated to form a ‘chromosome’ (inspired from the real chromosomes in organism). The parameters are encoded such that its minimum and maximum value vary from all bits 0 to all bits 1. By this conversion, each parameter would have different step size and would assume all possible combination of the sequence. The values of the parameter are randomly selected and each sequence becomes a part of the chromosomes. By this we get a group of chromosomes called as ‘population’. The chromosomes of the population participate in ‘cross-breeding’ to give raise to a new generation. This is iteratively done for a number of generations. The percentage of chromosomes which take part in cross-breeding is fixed, but the candidates are randomly chosen in each generation. During cross-breeding, a single or multiple positions on each of the chromosomes are selected and their respective blocks of bits are exchanged, leading to newer combination of parameters. The chromosomes are ranked according to their fitness, the fitness in our assignment is the error between desired frequency and derived frequency. The values of the parameter is plugged into the frequency formula to get the derived frequency. The best few (the least error) chromosomes of each generation are stored to get a collection of best fit values. This algorithm is adapted so that any best fit chromosomes from each generation is not neglected. Along with cross-breeding, ‘mutation’ of bits is also performed to get a diverse value. In mutation, a random bit of a selected chromosome is toggled. The cross breeding operation helps to obtain best possible solutions from the given population and mutation process helps to explore for new solutions and prevents the solution from settling in narrow range for optimum solution. After allowing breeding over various generations, we get a collection of best values of parameters which have the least error in frequency. This collection is also ranked so as to get ‘Best of Best’ values, thus obtaining the optimized results.
The steps of Genetic Algorithm Optimization can be summarized as:

**Step 1:** Create an initial ‘population’ of all the parameters forming a binary sequence of ‘chromosomes’.

**Step 2:** Rank the ‘chromosomes’ according to their fitness using fitness function and select the best few.

**Step 3:** Perform ‘cross-breeding’ and ‘mutation’ to get the next generation offspring with new characteristics.

**Step 4:** Repeat Step 2 and Step 3 for a number of generations to get the overall best candidate chromosomes.

**Step 5:** Extract the optimal valued chromosomes and plot them to get the least error and best candidates.

It should be noted that the initial population must be large enough so that there is no domination of genes. This ensures satisfactory convergence to obtain the optimal values.

3.2. Implementation of GA for NRPSRR Optimization

The implementation of GA for the optimization of physical parameters of an NRPSRR ($\theta = 0^\circ$; $N = 6$) for minimization of relative error between desired & calculated frequency is carried out using MATLAB simulation (achieving appreciable chromosome convergence) with the numerical data tabulated in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Range</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius</td>
<td>$3\text{mm} &lt; a &lt; 8\text{mm}$</td>
<td>$\mu$</td>
<td>1 (FR4)</td>
</tr>
<tr>
<td>Split Gap</td>
<td>$0.2\text{mm} &lt; g &lt; 0.6\text{mm}$</td>
<td>$\varepsilon_r$</td>
<td>4 (FR4)</td>
</tr>
<tr>
<td>Ring Width</td>
<td>$0.2\text{mm} &lt; c &lt; 0.8\text{mm}$</td>
<td>Desired Frequency</td>
<td>22 Ghz</td>
</tr>
<tr>
<td>Ring Gap</td>
<td>$0.2\text{mm} &lt; d &lt; 0.8\text{mm}$</td>
<td>% Crossover</td>
<td>45%</td>
</tr>
<tr>
<td>Depth</td>
<td>$0.2\text{mm} &lt; h &lt; 0.8\text{mm}$</td>
<td>% Mutation</td>
<td>4%</td>
</tr>
</tbody>
</table>

3.3. Optimized Graphical Results

The optimized population swarm of the 5 physical parameter viz. ‘a’, ‘g’, ‘c’, ‘d’ & ‘h’ for best performance of an NRPSRR ($\theta = 0^\circ$; $N = 6$) using MATLAB based GA optimization of the NRPSRR are tabulated in Table 2.

<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>Parameter</th>
<th>GA Optimized Value</th>
<th>Best When</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Radius ‘a’</td>
<td>3mm</td>
<td>Minimum</td>
</tr>
<tr>
<td>2</td>
<td>Split Gap ‘g’</td>
<td>0.6mm</td>
<td>Maximum</td>
</tr>
<tr>
<td>3</td>
<td>Ring Width ‘c’</td>
<td>0.2mm</td>
<td>Minimum</td>
</tr>
<tr>
<td>4</td>
<td>Ring Gap ‘d’</td>
<td>0.8mm</td>
<td>Maximum</td>
</tr>
<tr>
<td>5</td>
<td>Depth ‘h’</td>
<td>0.2mm</td>
<td>Minimum</td>
</tr>
</tbody>
</table>
Fig. 4. Plot of Relative error in frequency v/s Range of 'a'

Fig. 5. Plot of Relative error in frequency v/s Range of 'g'

Fig. 6. Plot of Relative error in frequency v/s Range of 'c'

Fig. 7. Plot of Relative error in frequency v/s Range of 'd'

Fig. 8. Plot of Relative error in frequency v/s Range of 'h'
4. Results and Conclusions

A new design of the N-sided Regular Polygon Split Ring Resonator with angular rotation (skew) between inner and outer ring is demonstrated with associated mathematical analysis and derivation of a closed form expression of its primary resonant frequency. The expression is successfully extended to Circular Split Ring Resonator (CSRR) for N → ∞ with graphical simulation exhibiting minimum resonant frequency at zero skew which is in agreement with the mathematical analysis. MATLAB based Genetic Algorithm is used to optimize the physical parameters of the NRPSRR for a desired resonant frequency value. This can be effectively used by a designer to estimate the specifications of an NRPSRR to incorporate it with a microstrip antenna to get highly directional beam patterns because of the enhanced Negative Refractive Index properties of NRPSRR.

References


Appendix

**Appendix A**: The perimeter of an NRPSRR (refer fig. 1(a)), \(l\) is estimated as: \(l = 2 \cdot a \cdot N \cdot \sin \left( \frac{\pi}{N} \right) \).

**Appendix B**: The area of an NRPSRR (refer fig. 1(a)), \(S\) is estimated as: \(S = \left( \frac{a^2 \cdot N}{2} \right) \cdot \sin \left( \frac{2 \pi}{N} \right) \).

**Appendix C**: The Capacitance per unit length of the NRPSRR, \(C_{pul}\) is estimated as:

\[
C_{pul} = \varepsilon_0 \cdot \left( \frac{\sigma}{\frac{d+2c}{2}} \right) \cdot \frac{\varepsilon \left( \sqrt{1 - \sigma^2} \right)}{\varepsilon \left( \sigma \right)}
\]

where \(\sigma = \frac{d}{d+2c}\) and \(\varepsilon (\cdot)\) is the complete elliptical integral of the second kind, defined as: \(\varepsilon (k) = \int_0^{\pi/2} \sqrt{1 - (k \sin \theta)^2} \, d\theta\)