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Quantum computation in noiseless subsystems with fast non-Abelian holonomiesJ. Zhang,^{1,2,*} L.-C. Kwek,^{2,3,4,†} Erik Sjöqvist,^{2,5,‡} D. M. Tong,^{1,§} and P. Zanardi^{2,6,||}¹*Department of Physics, Shandong University, Jinan 250100, China*²*Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543*³*National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616*⁴*Institute of Advanced Studies, Nanyang Technological University, 60 Nanyang View, Singapore 639673*⁵*Department of Quantum Chemistry, Uppsala University, Box 518, SE-751 20 Uppsala, Sweden*⁶*Department of Physics and Astronomy and Center for Quantum Information Science & Technology, University of Southern California, Los Angeles, California 90089-0484, USA*

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Quantum-information processing requires a high degree of isolation from the detrimental effects of the environment as well as an extremely precise level of control of the way quantum dynamics unfolds in the information-processing system. In this paper we show how these two goals can be ideally achieved by hybridizing the concepts of noiseless subsystems and of holonomic quantum computation. An all-geometric universal computation scheme based on nonadiabatic and non-Abelian quantum holonomies embedded in a four-qubit noiseless subsystem for general collective decoherence is proposed. The implementation details of this synergistic scheme along with the analysis of its stability against symmetry-breaking imperfections are presented.

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I. INTRODUCTION

Implementation of quantum-information processing (QIP) poses daunting challenges. First, for most of the QIP protocols, quantum coherence has to be maintained throughout the whole computational process in spite of the decoherence induced by the unavoidable coupling with environmental degrees of freedom. Second, one has to achieve an unprecedented level of control to enact quantum gates within the required high accuracy.

To the aim of accomplishing these somewhat contradictory tasks, several theoretical schemes have been devised since the early days of QIP. Broadly speaking, all the information-stabilizing strategies developed to date fall into three categories: active techniques such as quantum-error-correcting codes [1], symmetry-aided passive techniques such as decoherence-free subspaces and subsystems [2–4], and geometrical [5–7] and topological techniques [8].

Geometric QIP exploits different types of quantum holonomies, e.g., Berry phases, to implement quantum gates. Following the first non-Abelian [5] and Abelian [6] adiabatic proposals, many others have been considered; see, e.g., Refs. [7,9–12]. The motivating idea is that the geometric nature of the proposed quantum gates endows them with some degree of inherent robustness against control imprecisions as well as against environment-induced decoherence [13]. One of the drawbacks of the original holonomic quantum computation (HQC) [5] is its relative slowness due to the adiabaticity constraint. This potential limitation can be circumvented by resorting to nonadiabatic Abelian [14,15] and non-Abelian [16,17] quantum holonomies.

The idea of noiseless subsystem (NS) was first introduced in Ref. [3] and experimentally demonstrated in Ref. [18]. Noiseless subsystems are a natural generalization of the concept of noiseless quantum code or decoherence-free subspace (DFS) [2] and are effective when the decohering interactions possess some nontrivial algebra of symmetries. On general theoretical grounds, NSs have been argued to provide the unified algebraic structure underlying all the known quantum-information protection schemes [4] including topological quantum computation [19].

The goal of this paper is to merge synergistically ideas from geometric QIP and NSs in order to take advantage of the appealing features of both. More specifically, we will hybridize nonadiabatic HQC [16], with the powerful theory of NSs [3,4]. The possibility of achieving robust quantum control of NSs by non-Abelian quantum holonomies was first envisioned in Refs. [20,21]; universal HQC schemes embedded in DFSs and NSs were proposed in Refs. [9,17] and (for a strongly dissipative case) in Ref. [22].

In this paper we extend significantly the results of Ref. [17] by showing how a universal non-Abelian and nonadiabatic holonomic processor can be embedded within a NS for general collective decoherence. In this way a universal computational scheme protected against general collective decoherence while featuring at the same time that the robustness of HQC against imprecisions in gate control can be ideally achieved.

II. NOISELESS SUBSYSTEM

We start by briefly recalling the basic notions concerning NSs. Let \mathcal{H} be the Hilbert space of a quantum system S coupled to its environment through a set of error operators $\{E_\alpha\}_\alpha \subset \mathcal{B}(\mathcal{H})$ [23]. The key object is provided by the interaction algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, i.e., the C^* algebra [24] generated by the error operators. The state space of the system decomposes into the different d_J -dimensional irreducible representations of \mathcal{A} (labeled by J and with multiplicity n_J) as $\mathcal{H} \cong \bigoplus_J \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}$.

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The corresponding orthogonal decomposition of \mathcal{A} is given by

$$\mathcal{A} \cong \bigoplus_J \mathbf{1}_{n_J} \otimes M_{d_J}, \quad (1)$$

where M_{d_J} denotes the full matrix algebra of $d_J \times d_J$ complex matrices [24]. When $d_J = 1$, one recovers the concept of DFS [2]. The interaction algebra \mathcal{A} acts irreducibly on the \mathbb{C}^{d_J} factors of \mathcal{H} , whereas Eq. (1) shows that the error algebra elements, responsible for decoherence, have a *trivial* action on the \mathbb{C}^{n_J} factors. It follows that quantum information can be protected by encoding in these virtual subsystems [20] that are then termed noiseless subsystems [3]. In order to perform manipulations of the NS-encoded information, one has to resort to a nontrivial set of operations that belong to the commutant algebra $\mathcal{A}' := \{X \in \mathcal{B}(\mathcal{H})/[X, A] = 0 \forall A \in \mathcal{A}\}$. This crucial fact follows from the dual irreducible representation decomposition $\mathcal{A}' \cong \bigoplus_J M_{n_J} \otimes \mathbf{1}_{d_J}$ for which one sees that \mathcal{A}' has an irreducible action on the NSs and a trivial one on the \mathbb{C}^{d_J} factors, now playing the role of multiplicity spaces. These constructions are useful if at least one of the n_J is larger than or equal to 2, which in turn gives a lower bound on the dimension of the commutant \mathcal{A}' . In other words, the existence of a NS encoding relies on the existence of a sufficiently large number of symmetries of the interaction algebra, i.e., of the noise.

The prototypical symmetric noise is provided by collective decoherence [2] whose experimental relevance has been demonstrated in Refs. [18,25,26]. This is also the model that will be considered in this paper. In this collective case the interaction algebra \mathcal{A} is given by the algebra of totally symmetric operators on the state space of N qubits, i.e., $\mathcal{H} \cong (\mathbb{C}^2)^{\otimes N}$ and its commutant \mathcal{A}' is the C^* algebra generated by permutations $\sigma \in \mathcal{S}_N$ acting on \mathcal{H} according to the natural representation, i.e., $\sigma : \bigotimes_{p=1}^N |\alpha_p\rangle \rightarrow \bigotimes_{p=1}^N |\alpha_{\sigma(p)}\rangle$. In the following, we show how to enact a universal set of operators in \mathcal{A}' using nonadiabatic quantum holonomies only.

III. NONADIABATIC HOLONOMIC QUANTUM COMPUTATION

Nonsadiabatic HQC, proposed in Ref. [16] and experimentally implemented in Ref. [27], is based on the concept of nonadiabatic non-Abelian geometric phases [28]. The key idea is to implement a suitably designed Hamiltonian that induces cyclic evolution of a quantum computational system encoded in a subspace $\mathcal{M}(0)$ in such a way that all dynamical phases vanish. The primitive structure is of Λ type, where an excited state $|e\rangle$ is coupled by a pair of simultaneous laser pulses to ground-state levels $|g_0\rangle, |g_1\rangle$ according to ($\hbar = 1$ from now on)

$$H(t) = \Omega(t)(\omega_0 |e\rangle \langle g_0| + \omega_1 |e\rangle \langle g_1| + \text{H.c.}). \quad (2)$$

Here $\Omega(t)$ is the Rabi frequency and ω_0, ω_1 are complex-valued time-independent driving frequencies satisfying $|\omega_0|^2 + |\omega_1|^2 = 1$. The generic Hamiltonian $H(t)$ describes transitions between energy levels induced by oscillating laser fields in the rotating-wave approximation and can be implemented in a wide range of different physical systems.

The subspace spanned by

$$|\psi_j(t)\rangle = \exp\left(-i \int_0^t H(t') dt'\right) |g_j\rangle = U(t,0) |g_j\rangle, \quad j=0,1,$$

undergoes a cyclic evolution if the Rabi frequency satisfies $\int_0^\tau \Omega(t') dt' = \pi$. The resulting time evolution operator $U(\tau,0)$, projected onto the computational subspace $\mathcal{M}(0) = \text{Span}\{|g_0\rangle, |g_1\rangle\}$, defines the traceless Hermitian gate $U(C_n) = \mathbf{n} \cdot \boldsymbol{\sigma}$, where $\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ with $\omega_0/\omega_1 = -\tan(\theta/2)e^{i\phi}$ and $\boldsymbol{\sigma}$ are the standard Pauli operators acting on $\mathcal{M}(0)$. An arbitrary $SU(2)$ can be realized by sequentially applying two such gates with different \mathbf{n} . The evolution is purely geometric since $\langle \psi_j(t) | H(t) | \psi_k(t) \rangle, j, k = 0,1$, vanish for $t \in [0, \tau]$. Thus, $U(C_n)$ is fully determined by the path C_n of $\mathcal{M}(t)$ in the space of all two-dimensional subspaces of the three-dimensional Hilbert space, i.e., in the complex-valued Grassmannian $G(3;2)$. Together with an entangling holonomic two-qubit gate, $U(C_n)$ constitutes a universal all-geometric set of quantum gates [29].

IV. QUANTUM HOLONOMY IN NOISELESS SUBSYSTEMS

The collective decoherence on a quantum system S consisting of N physical qubits is characterized by the spin- $\frac{1}{2}$ error operators $E_\alpha = \sum_{p=1}^N \sigma_p^\alpha, \alpha = \pm, z$. For a fixed total spin J , the dimension of the noiseless part is $d_J = 2J + 1$. By using angular momentum addition rules, one can prove that

$$n_J = \frac{(2J+1)N!}{(N/2+1+J)!(N/2-J)!}. \quad (3)$$

This n_J , which is the dimension of the NS part, provides the possibility of performing HQC.

Quantum holonomy appears when the subspace $\mathcal{M}(t)$ returns to the original one after a nontrivial cyclic transformation. The NS spans the total space, which is the total Hilbert space of a HQC. In general, the NS should be larger than the computational space in order to admit nontrivial holonomies. A subspace of NS emerges as $\mathcal{M}(0)$ and the effective Hamiltonian of NS acts as the Hamiltonian that generates the nontrivial loop relating to the unitary transformation. Since NS theory guarantees that the states in $\mathcal{M}(0)$ are never evolved out of the NS, the subspace of NS can return to the original one and that ensures that HQC can be conducted.

A. One-qubit gate

A nonadiabatic one-qubit holonomic gate can be implemented in a NS provided there exists a J for which $n_J \geq 3$. Four physical qubits, which contain a $\mathbb{C}^3 \otimes \mathbb{C}^3$ subspace, provides the smallest possible realization of such a gate. Here we demonstrate how noiseless holonomic one-qubit gates can be implemented in this $\mathbb{C}^3 \otimes \mathbb{C}^3$ subspace of the four-qubit code.

First, note that the E_α act on the $\mathbb{C}^3 \otimes \mathbb{C}^3$ subspace in the following way:

$$E_\alpha = I_{\text{NS}} \otimes S_\alpha, \quad \alpha = \pm, z, \quad (4)$$

where S_α is the spin-1 representation of the angular momentum operators. An important observation here is that the inherent symmetry in the action of the decoherence operators E_α on the basis states $|i\rangle |j\rangle_4$ affects only the second part of the basis and leaves the first part unchanged (see the Appendix for more details on the four-qubit code). Moreover, the basis changed by the error operators stays within $|i\rangle |j\rangle_4$. Thus, the information

being stored in this subspace depends only on the first index; it is therefore not spoiled by the interaction between the system and the environment.

To perform holonomic one-qubit gates with the four-qubit code, a set of operators is needed to achieve the appropriate transitions so that the computation stays within the subspace. To this end, the operators that we seek should commute with the E_α . Let us consider the permutation operator $P_{pq} = \frac{1}{2}(I_{pq} + \sigma_p \cdot \sigma_q)$ of qubits p and q such that $P_{pq} |x\rangle_p |y\rangle_q = |y\rangle_p |x\rangle_q$ for $x, y \in \{0, 1\}$. Here I_{pq} is the identity and σ_p, σ_q are the Pauli operators acting on this qubit pair. It is straightforward to check that $[P_{pq}, E_\alpha] = 0$. Three- and four-body permutation operators emerge as a product of two-body ones. Thus, if the Hamiltonian is constructed using a combination of the permutation operators, it will not destroy the subspace. Explicitly, we may take the Hamiltonian that generates the holonomic one-qubit gates to be

$$\begin{aligned} H^{(1)}(t) &= \Omega(t) \left[\frac{J_1}{\sqrt{3}}(P_{23} - P_{13}) + i \frac{J_2}{\sqrt{3}}(P_{23}P_{13} - P_{13}P_{23}) \right. \\ &\quad \left. + \frac{J_4}{2\sqrt{6}}(P_{13} - P_{23} - 3P_{14} + 3P_{24}) \right] \\ &= \Omega(t)[(J_1 - iJ_2)|3\rangle\langle 1| + J_4|3\rangle\langle 2| + \text{H.c.}] \otimes I_{\text{nf}}, \end{aligned} \quad (5)$$

where the first tensor factor corresponding to the NS is identical to the Hamiltonian in Eq. (2) by identifying $|1\rangle = |g_0\rangle$, $|2\rangle = |g_1\rangle$, and $|3\rangle = |e\rangle$; I_{nf} is the identity operator acting on the noiseless subsystem. The Hamiltonian vanishes on the noiseless qubit subspace $\mathcal{M}^{(1)}(0) = \text{Span}\{|1\rangle, |2\rangle\}$, which guarantees the geometric nature of the evolution. By setting $(J_1 - iJ_2)/J_4 = -\tan(\theta/2)e^{i\phi}$ and defining the unit vector $\mathbf{n} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$, a traceless one-qubit holonomic gate

$$U^{(1)}(C) = \mathbf{n} \cdot \boldsymbol{\sigma} \otimes I_{\text{nf}} \quad (6)$$

acting nontrivially on the two-dimensional subspace of the NS can thus be implemented by choosing $\int_0^\tau \Omega(t)dt = \pi$. By combining two such gates, an arbitrary $\text{SU}(2)$ operation acting on the noiseless qubit subspace $\mathcal{M}^{(1)}(0)$ can be realized.

B. Two-qubit gate

It is well known that universal quantum computation can be achieved as long as all one-qubit gates and a single nontrivial two-qubit (entangling) gate is possible [29]. Since all single-qubit gates are possible, it remains to demonstrate that we could construct a nontrivial two-qubit gate.

To guarantee the holonomic scheme to be scalable, we encode each qubit in a two-dimensional subspace of a three-level NS by using four physical qubits. In this scheme, a two-qubit gate requires an eight-qubit code where two noiseless qubits are represented by two sets L, L' of four physical qubits. By choosing an appropriate Hamiltonian for the eight physical qubits, we demonstrate a holonomic controlled-NOT (CNOT) gate that can entangle these noiseless qubits.

Consider the eight-qubit Hamiltonian expressed in terms of permutation operators as

$$\begin{aligned} H^{(2)}(t) &= \frac{\Omega(t)}{12}(P_{13} - P_{23} - 3P_{14} + 3P_{24})_L \\ &\quad \otimes \left[P_{23} - P_{13} - \frac{1}{2\sqrt{2}}(P_{13} - P_{23} - 3P_{14} + 3P_{24}) \right]_{L'}. \end{aligned} \quad (7)$$

By reexpressing the two factors in terms of the NS-noiseless basis for each four-qubit set, we obtain

$$H^{(2)}(t) = \Omega(t)(H_0 + H_1) \otimes I_{\text{nf}}, \quad (8)$$

where

$$\begin{aligned} H_0 &= \frac{1}{\sqrt{2}}|33\rangle\langle 21| - \frac{1}{\sqrt{2}}|33\rangle\langle 22| + \text{H.c.}, \\ H_1 &= \frac{1}{\sqrt{2}}|31\rangle\langle 23| - \frac{1}{\sqrt{2}}|32\rangle\langle 23| + \text{H.c.}, \end{aligned} \quad (9)$$

and I_{nf} is now the identity on the nine-dimensional noiseless subsystem of the eight qubits. The two time-independent operators H_0 and H_1 vanish on the computational two-qubit subspace $\mathcal{M}^{(2)}(0) = \text{Span}\{|11\rangle, |12\rangle, |21\rangle, |22\rangle\}$ of the NS, which ensures the geometric nature of the evolution. Furthermore, H_0 and H_1 commute, which implies that

$$\exp\left(-i \int_0^\tau H^{(2)}(t')dt'\right) = e^{-i\pi H_0} e^{-i\pi H_1} \otimes I_{\text{nf}}, \quad (10)$$

by choosing $\int_0^\tau \Omega(t')dt' = \pi$. The second factor $e^{-i\pi H_1}$ acts trivially on $\mathcal{M}^{(2)}(0)$ and can therefore be ignored. The holonomic gate $U^{(2)}(C)$ is the projection of the first factor $e^{-i\pi H_0}$ onto $\mathcal{M}^{(2)}(0)$ and reads

$$\begin{aligned} U^{(2)}(C) &= (|11\rangle\langle 11| + |12\rangle\langle 12| \\ &\quad + |21\rangle\langle 22| + |22\rangle\langle 21|) \otimes I_{\text{nf}}. \end{aligned} \quad (11)$$

We see that $U^{(2)}(C)$ is a CNOT gate acting on $\mathcal{M}^{(2)}(0)$, which completes the universal set of nonadiabatic holonomic gates in NSs.

V. ROBUSTNESS OF GATES

Our NS encoding allows for perfect protection in the ideal collective decoherence case where the system-bath interactions are fully invariant under arbitrary permutations of the physical qubits. However, in realistic situations, symmetry-breaking interactions will be unavoidably present and spoil the ideal behavior. In order to investigate the robustness of our scheme against such unwanted interactions we introduce a simple decoherence model with a single parameter g that controls the degree of symmetry breaking. The noise Lindblad operators are given by $E'_\alpha = \sum_{p=1}^4 e^{-ps} \sigma_\alpha^p$; clearly when $g = 0$ one recovers the permutational invariant collective decoherence. Within the usual Born-Markov approximation the system evolution is dictated by the Lindblad master

equation

$$\begin{aligned} \dot{\rho}(t) = & -i[H_s(t), \rho] + \Gamma(E'_z \rho E'_z - \frac{1}{2}\{E'_z E'_z, \rho\}) \\ & + \gamma(\bar{n} + 1)(E'_- \rho E'_+ - \frac{1}{2}\{E'_+ E'_-, \rho\}) \\ & + \gamma\bar{n}(E'_+ \rho E'_- - \frac{1}{2}\{E'_- E'_+, \rho\}), \end{aligned} \quad (12)$$

where \bar{n} is the temperature-dependent average number of quanta in the environment ($\bar{n} = 0$ at zero temperature) and Γ and γ are the dephasing rate and dissipation rate, respectively. Here $H_s(t)$ is the qubit system Hamiltonian, which is generated by linear combination of the permutation operators as described above. A square pulse with magnitude Ω and duration π/Ω is used.

As a figure of merit to quantify the robustness of our logical gates, we adopt the gate fidelity F defined as the Bures-Uhlmann fidelity $F(\rho_{\text{id}}, \rho_f) := \text{Tr}\sqrt{\rho_f^{1/2}\rho_{\text{id}}\rho_f^{1/2}}$ averaged over initial conditions. Here ρ_{id} is the NS state obtained by the ideal holonomic gate, i.e., the actual one in the presence of collective decoherence only, and ρ_f is the corresponding faulty one obtained by solving Eq. (12) and tracing over the noiseful degrees of freedom.

We solved numerically Eq. (12) and examined the gate fidelity as a function of g for a one-qubit holonomic gate. In Fig. 1 it is shown that gate fidelity, both at zero temperature (solid line, $\bar{n} = 0$) and nonzero temperature (dashed line, $\bar{n} = 1$), decreases with increasing g as expected. However, the inset in Fig. 1 also shows that in the physically relevant regime of small g , the gate fidelity behaves as $F \approx 1 - g^a$, where $a \approx 2$. This demonstrates that holonomic manipulations of the NS have some degree of resilience against slight violations of the collective symmetry assumption. We would like to stress that Eq. (12) is just a way to describe the system-bath

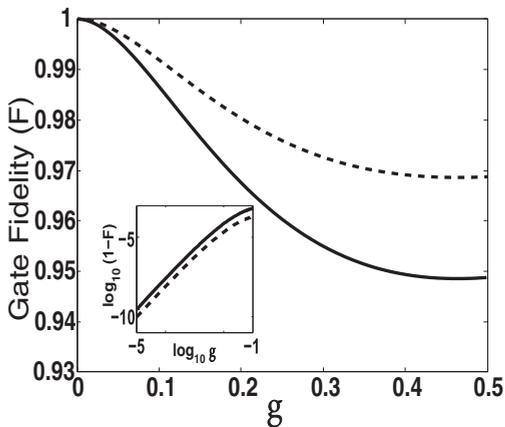


FIG. 1. Gate fidelity in the presence of a noncollective environment with g controlling the degree of symmetry breaking. A square pulse with magnitude Ω and duration π/Ω is used. The dephasing rate Γ and dissipation rate γ are chosen to satisfy $\Gamma = \gamma = 0.1\Omega$. The logical unitary gate is the standard Pauli Z operator and the fidelity is averaged over the six axial pure states on the Bloch sphere as input states. The dashed line and the solid line show the gate fidelity in the NS for mean number of environmental quanta $\bar{n} = 0$ and 1, respectively. The inset shows the plot of $\log_{10}(1 - F)$ as a function of $\log_{10} g$; the dashed line and the solid line are for $\bar{n} = 0$ and 1, respectively.

coupling that allows one to interpolate, in a simple phenomenological fashion, between the fully permutational symmetric ($g = 0$) and increasingly nonsymmetric and unprotected regimes (large g). However, we expect the conclusions drawn from our simulations to be generic for mild violations of permutational symmetry. Namely, gate robustness should be independent of the details of the specific decoherence model, e.g., Markovianity.

VI. CONCLUSION

In this paper we have shown how to implement a universal set of one- and two-qubit gates by nonadiabatic and non-Abelian quantum holonomies acting entirely within a noiseless subsystem for general collective decoherence. Each noiseless qubit can be encoded using four physical qubits and geometrically manipulated by Heisenberg-like two- and four-body interactions. The requested ability to enact four-body interactions certainly presents a major challenge to the realization of our scheme with current experimental techniques. In order to overcome this limitation one may think of resorting to geometric techniques to simulate many-body interactions in terms of simpler interactions [30] or to the so-called perturbation gadgets [31]. In both cases ancillary degrees of freedom are needed. Finally, by numerical simulations, we have provided evidence of the robustness of the proposed hybrid scheme against symmetry-breaking interactions with the environment.

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APPENDIX

The Hilbert space of a four-qubit system can be decomposed as

$$(\mathbb{C}^2)^{\otimes 4} = \mathbb{C}^2 \otimes \mathbb{C} \oplus \mathbb{C}^3 \otimes \mathbb{C}^3 \oplus \mathbb{C} \otimes \mathbb{C}^5.$$

By using the notation

$$|0\rangle = |1/2, 1/2\rangle, \quad |1\rangle = |1/2, -1/2\rangle,$$

we find the $\mathbb{C}^3 \otimes \mathbb{C}^3$ basis states

$$|1\rangle |1\rangle_4 = |1, 1\rangle = \frac{1}{\sqrt{6}}(2|0010\rangle - |0100\rangle - |1000\rangle),$$

$$\begin{aligned} |1\rangle |2\rangle_4 = |1, 0\rangle = & \frac{1}{2\sqrt{3}}(2|0011\rangle - |0101\rangle - |1001\rangle \\ & + |0110\rangle + |1010\rangle - 2|1100\rangle), \end{aligned}$$

$$|1\rangle |3\rangle_4 = |1, -1\rangle = \frac{1}{\sqrt{6}}(|0111\rangle + |1011\rangle - 2|1101\rangle);$$

$$|2\rangle |1\rangle_4 = |1,1\rangle = \frac{1}{2\sqrt{3}}(3|0001\rangle - |0010\rangle - |0100\rangle - |1000\rangle),$$

$$|2\rangle |2\rangle_4 = |1,0\rangle = \frac{1}{\sqrt{6}}(|0011\rangle + |0101\rangle + |1001\rangle - |0110\rangle - |1010\rangle - |1100\rangle),$$

$$|2\rangle |3\rangle_4 = |1,-1\rangle = \frac{1}{2\sqrt{3}}(|0111\rangle + |1011\rangle + |1101\rangle - 3|1110\rangle);$$

$$|3\rangle |1\rangle_4 = |1,1\rangle = \frac{1}{\sqrt{2}}(|0100\rangle - |1000\rangle),$$

$$|3\rangle |2\rangle_4 = |1,0\rangle = \frac{1}{2}(|0101\rangle - |1001\rangle + |0110\rangle - |1010\rangle),$$

$$|3\rangle |3\rangle_4 = |1,-1\rangle = \frac{1}{\sqrt{2}}(|0111\rangle - |1011\rangle).$$

The NS holonomies are realized in the first tensor factor of these $|i\rangle |j\rangle_4$ states.

The Gell-Mann matrices $\lambda_1, \dots, \lambda_8$ on the NS in the $\mathbb{C}^3 \otimes \mathbb{C}^3$ subspace can be expressed in terms of qubit permutation operators as

$$\begin{aligned} \lambda_1 \otimes I_{\text{nf}} &= (|3\rangle \langle 1| + \text{H.c.}) \otimes I_{\text{nf}} \\ &= \frac{1}{\sqrt{3}}(P_{23} - P_{13}), \end{aligned}$$

$$\begin{aligned} \lambda_2 \otimes I_{\text{nf}} &= (-i|3\rangle \langle 1| + \text{H.c.}) \otimes I_{\text{nf}} \\ &= i\frac{1}{\sqrt{3}}(P_{23}P_{13} - P_{13}P_{23}), \end{aligned}$$

$$\begin{aligned} \lambda_3 \otimes I_{\text{nf}} &= (|3\rangle \langle 3| - |1\rangle \langle 1|) \otimes I_{\text{nf}} \\ &= \frac{1}{3}(P_{13} + P_{23} - 2P_{12}), \end{aligned}$$

$$\begin{aligned} \lambda_4 \otimes I_{\text{nf}} &= (|3\rangle \langle 2| + \text{H.c.}) \otimes I_{\text{nf}} \\ &= \frac{1}{2\sqrt{6}}(P_{13} - P_{23} - 3P_{14} + 3P_{24}), \end{aligned}$$

$$\begin{aligned} \lambda_5 \otimes I_{\text{nf}} &= (-i|3\rangle \langle 2| + \text{H.c.}) \otimes I_{\text{nf}} \\ &= i\frac{1}{2\sqrt{6}}(2P_{321} - 2P_{231} + P_{342} \\ &\quad - P_{432} - P_{341} + P_{431} \\ &\quad + 4P_{241} - 4P_{421}), \end{aligned}$$

$$\begin{aligned} \lambda_6 \otimes I_{\text{nf}} &= (|1\rangle \langle 2| + \text{H.c.}) \otimes I_{\text{nf}} \\ &= -\frac{1}{6\sqrt{2}}(2P_{13} + 2P_{23} - 4P_{12} \\ &\quad + 3P_{2341} + 3P_{3421} + 3P_{4321} \\ &\quad + 3P_{2431} - 6P_{3241} - 6P_{4231}), \end{aligned}$$

$$\begin{aligned} \lambda_7 \otimes I_{\text{nf}} &= (-i|1\rangle \langle 2| + \text{H.c.}) \otimes I_{\text{nf}} \\ &= i\frac{1}{2\sqrt{2}}(P_{341} + P_{342} - P_{432} - P_{431}), \end{aligned}$$

$$\begin{aligned} \lambda_8 \otimes I_{\text{nf}} &= \frac{1}{\sqrt{3}}(|3\rangle \langle 3| + |1\rangle \langle 1| - 2|2\rangle \langle 2|) \otimes I_{\text{nf}} \\ &= \frac{1}{\sqrt{3}}(I - P_{12} - P_{13} - P_{23}). \end{aligned}$$

Note that the realization of some of the Gell-Mann matrices ($\lambda_2, \lambda_5, \lambda_6, \lambda_7$) requires higher than two-body interaction.

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