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<td><strong>Author(s)</strong></td>
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Estimating Mean First Passage Time of Biased Random Walks with Short Relaxation Time on Complex Networks

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Abstract

Biased random walk has been studied extensively over the past decade especially in the transport and communication networks communities. The mean first passage time (MFPT) of a biased random walk is an important performance indicator in those domains. While the fundamental matrix approach gives precise solution to MFPT, the computation is expensive and the solution lacks interpretability. Other approaches based on the Mean Field Theory relate MFPT to the node degree alone. However, nodes with the same degree may have very different local weight distribution, which may result in vastly different MFPT. We derive an approximate bound to the MFPT of biased random walk with short relaxation time on complex network where the biases are controlled by arbitrarily assigned node weights. We show that the MFPT of a node in this general case is closely related to not only its node degree, but also its local weight distribution. The MFPTs obtained from computer simulations also agree with the new theoretical analysis. Our result enables fast estimation of MFPT, which is useful especially to differentiate between nodes that have very different local node weight distribution even though they share the same node degrees.


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Introduction

Scale-free node degree distribution, small network diameter, large clustering coefficients – these are common properties found to be present in complex networks arising from seemingly disparate fields such as biology, computer science, cosmology, etc. [1,2]. It is widely believed that there should be common underlying principles behind the formation of these networks that resulted in the observed properties. As such, complex networks have received much research attention during the past decade.

One of the studies pertaining to complex networks is the network efficiency and capacity analysis [3–7]. In these studies, answers to the questions such as ‘how fast can a message be delivered to a given destination’ and ‘how many packets may be generated in the system before congestion arises’ are essential in understanding the performance of a network [5,7,8]. While there are existing results on routing and flow balancing in networks with certain topologies [9], they usually assume the underlying topology or the knowledge of link formation mechanism. There are cases where the assumptions do not hold, for example, the animal foraging strategy [10] and the web searching process as depicted by the PageRank algorithm [11]. Approaches based on random walk can be applied when the detailed information of network formation mechanism is absent. Quantities such as stationary distribution and mean first passage time (MFPT) are important as they can be used to answer the questions about the performance of networks as mentioned above.

The concept of random walk has also been applied in social networks. Even-Dar et al. [12] studied the process of spreading influences in social networks by means of Voter Model and showed that the pathways in which the influences propagate are equivalent to series of random walks. Thus, the MFPT to a node A, yields the expected time for the other nodes in the network to be influenced by node A. Selecting a node with low MFPT for spreading the news could result in fast information propagation.

In fact, MFPT can be calculated by using the fundamental matrix method [13]. However, the computation involves multiple matrix multiplications. When the method is applied on large scale networks with millions of nodes, the computation becomes practically infeasible. Moreover, the solutions obtained from the fundamental matrix approach are too generic and hard to interpret. For instance, it is unclear which factors, be it node degree, eigenvalue, local connectivity, or others, govern the MFPT degree by just looking at the solution expression. Further research is needed to better characterize MFPT and to reduce the computational cost.

In [10], Condamin et al. showed a mean first passage time analysis using the pseudo Green function. They related MFPT to the network size and diameter. The general applicability of their result to non-fractal networks has been discussed in [14]. Fromczak et. al [8] applied the mean field theory to study the MFPT based on the Erdos-Renyi (ER) random graphs and networks generated by using the Barabasi-Albert (BA) preferential attachment model. Lau et. al [14] showed asymptotic analysis of the first passage time of unbiased random walk for a class of networks with short relaxation time by using the mean field theory. However, the solution relates MFPT to the node degree alone. There are cases where the nodes share the same node degree while having vastly different local topology such as those depicted in Figure 1.
Random walks with short relaxation time are also known as random walks with non-compact exploration [15–17]. In [15], Bénichou et al. presented the conditions for which a random walk on fractal network falls into the compact or non-compact exploration regime. In [16], Hwang et al. presented a fundamental result on the relationship between the node degree and the MFPT of heterogeneous networks, i.e., networks with power-law degree distribution. They focused on uniform random walks where it is densely connected as depicted in Figure 1. We will show a new expression allows us to differentiate between the case where the nodes have arbitrary weights. In addition, we analyse the first passage time at an improved level of precision by incorporating the weights assigned to the nodes. Then, we apply a flow-based heuristics to estimate the quasi-stationary distribution of the random walk when a sink node is introduced. Next, we show that the FPT distribution follows an exponential decay, and further show that the decay exponent is related to the quasi-stationary distribution of the neighbours of the target node. Finally, we obtain the MFPT by approximating the integration over the FPT distribution.

Stationary and Quasi-stationary Distribution

A complex network is modelled by a connected and undirected network $G=(V,E,W)$, where $V$ denotes the set of nodes with $|V|=N$, $E$ denotes the set of edges $(i,j)$, and $W$ denotes the weights assigned to the nodes. An edge $(i,j)$ represents the existence of relationship between nodes $i$ and $j$. For the sake of simplifying expression, we assume without loss of generality that self-loops are present for all nodes, i.e., $(i,i) \in E$ for $i \in [1,N]$. We define a biased random walk on $G$ with arbitrary positive weights assigned on nodes. A transition from node $i$ to a neighbouring node $j$ is based on the following transition rule:

$$Pr(i \rightarrow j) = \frac{w_j}{\sum_{h \in \Omega(i)} w_h}$$  \hspace{1cm} (1)

where $w_i$ denotes the weight of node $i$, and $\Omega(i)=\{h|(i,h)\in E\}$ denotes the neighbourhood of node $i$. $Pr(i \rightarrow j)$ denotes the probability of a random walker at node $i$ moving to node $j$ at the next time step. When self-loop is present, the staying probability of a node is not a constant but is dependent on the local weight distribution. Let the transition matrix be denoted by $P=[Pr(i \rightarrow j)]$, and $P(t)$ denote the probability of the random walker appearing at node $i$ exactly at timestep $t$. The master equation is given by:

$$P(t+1) = \sum_{j \in \Omega(i)} P_j(t) Pr(j \rightarrow i)$$  \hspace{1cm} (2)

Recall that a Markov Chain (MC) is said to be regular if $\exists q \in \mathbb{Z}^+$, $\forall (i,j) P^q(i \rightarrow j) > 0$ where $P^q(i \rightarrow j)$ denotes the $(i,j)$ entry of the $q$-th order transition matrix $P^q$. The probability distribution of a regular MC will converge to a unique stationary distribution $\pi$ regardless of the starting position as $t \rightarrow \infty$. The relaxation time $\tau$ of a Markov Chain is the time for the state probability distribution to be close to the stationary distribution, i.e., the standard deviation of $\mathbf{P}^t$ is bounded by $1/e$ where $e$ denotes the Euler Number (see

**Analysis**

Outline

In this section, we will show the detailed analysis for approximating the FPT decay rate and MFPT of random walks with short relaxation time. First, we will show the stationary distribution for random walk with arbitrary node weight assignment scheme. Then, we apply a flow-based heuristics to estimate the quasi-stationary distribution of the random walk when

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**Figure 1. Nodes with same degree may have very different local connectivity.** The figure shows two examples where a node with degree 3 may be part of a sparsely-connected star network or a densely-connected clique.

doi:10.1371/journal.pone.0093348.g001
Chapter 12). The relaxation time of a Markov Chain is given by \( \frac{1}{\lambda_2} \), where \( \lambda_2 \) is the second largest eigenvalue \([18]\).

In the networks that we consider, since every node has a self-loop and the graph is strongly connected, the corresponding Markov Chain is regular. In the following, we show the exact form of stationary distribution for the biased random walk as defined in Eq.(1).

Let \( w_i \) be the weight of node \( i \), \( w_i' \) be the neighbourhood weight of node \( i \) given by \( w_i' = w_i \sum_{j \in \text{neighbours}(i)} w_j \), and \( \langle w' \rangle = \frac{1}{n} \sum_i w_i' \). The stationary distribution of the biased random walk is given by

\[
\pi_i = \frac{w_i}{\sum_{j} w_j}
\]

Figure 2. Plots of empirical first passage time distribution against theoretical prediction according to the approximate bound given by Ineq.(13) for different networks and weighting factors. Each row corresponds to a network in the following order: Actor, BA, ER, and arXiv. The columns, from left to right, correspond to \( \alpha = -1, 0, 1 \) respectively. For most cases, the tail of the first passage time distribution can be predicted fairly accurately except for Figure 2(j), which is due to the high relaxation time as shown in Table 2. doi:10.1371/journal.pone.0093348.g002
denote the probability of visiting node and \( w \) Next, \( 0.9269 \) Correlation Relaxation time \( \tau \) and hence making \( = \) \( \Delta t \) as the starting node. Then by definition, \( \) edges CC Dm Source \( a \) \( 555.56 \) \( 555.56 \) \( 1666.7 \). Finally, we approximate \( 0.9231 \) \( 0.9851 \) \( 0.9241 \) \( 0.9812 \) \( 0.9940 \), \( \) \( 16.502 \) \( 3.7327 \) \( 2.8417 \). where \( \phi_{\text{total}}(t) = \sum_{n \in V} \phi_n(t) \) denotes the total survival probability at time step \( t \).

We approximate the quasi-stationary distribution of \( G \) around the sink node with a flow-based heuristic [14] described in the following. We begin with the stationary distribution \( \pi \) in \( G \). Under the stationary distribution, based on Eq. (3), the probability of the random walker traversing an edge \( (u,v) \) is \( P_{\Delta t}(u,v) = \frac{w_n(u,v)}{w_n(u,v)} \). Next, we treat an undirected edge as a combination of in-link and out-link. We remove the out-links from \( v \) and hence making \( v \) a sink node. Thus, the equilibrium will be broken and nodes \( m(V(v)) \) will have ‘flow’ constantly drawn by \( v \). Finally, we approximate the quasi-stationary distribution of such nodes by discounting the probability of utilizing the edge \( (u,v) \):

\[
\Psi_u \approx (1 - \frac{P_{\Delta t}(u,v)}{\pi_u}) \pi_u = (1 - Pr(u \rightarrow v)) \pi_u
\]  

### Asymptotic First Passage Time Analysis

To obtain FPT, we re-designate the destination node \( v \) as a sink node such that the random walk process terminates once the random walker moves into the sink node. The time taken for it to be absorbed into the sink node is then the same as the FPT. Let \( F(v_t, t|v_s) \) denote the probability of visiting node \( v_s \) at timestep \( t \) for the first time with \( v_s \) as the starting node. Then by definition, we have

\[
F(v_t, t|v_s) = \phi_{\text{total}}(t-1) - \phi_{\text{total}}(t)
\]

i.e. the first passage probability at time \( t \) is given by the difference in the total survival probability between time \( t \) and \( t-1 \).

On the other hand, we can also obtain the first passage probability by using the transition rule.

### Table 1. Summary of networks studied.

<table>
<thead>
<tr>
<th>Network</th>
<th># nodes</th>
<th># edges</th>
<th>CC</th>
<th>Dm</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>arXiv</td>
<td>4158</td>
<td>15501</td>
<td>0.5569</td>
<td>17</td>
<td>[21]</td>
</tr>
<tr>
<td>BA</td>
<td>4158</td>
<td>31136</td>
<td>0.0174</td>
<td>5</td>
<td>[1]</td>
</tr>
<tr>
<td>ER</td>
<td>4158</td>
<td>21020</td>
<td>0.0022</td>
<td>7</td>
<td>[24]</td>
</tr>
<tr>
<td>Actor</td>
<td>968</td>
<td>13324</td>
<td>0.6751</td>
<td>9</td>
<td>[1]</td>
</tr>
</tbody>
</table>

C denotes the average clustering coefficients and Dm denotes the network diameter.

doi:10.1371/journal.pone.0093348.t001

\[
\pi_t = \lim_{t \to \infty} P(t) = \frac{w_j \sum_{j \in \text{out}(i)} w_j}{\sum_i \sum_{j \in \text{out}(i)} w_j} = \frac{w_i'}{N \cdot \langle w_i' \rangle}
\]  

The expression can be verified by applying the equilibrium condition on the master equation as given by Eq. (2).

Let \( v_d \) denote the sink node and let \( G_s \) denote the resultant graph from designating node \( v_d \) as sink node. Let \( \Phi(t) \) denote the corresponding random walk probability distribution and \( \phi_d(t) \) represent the probability for a random walker to be present at node \( v \) at time \( t \). The following set of rules describes the new random walk on \( G_s \):

\[
\begin{align*}
\phi_s(0) &= \delta_{v_s} \\
\phi_d(t) &= 0 \\
\phi_i(t) &= \sum_{u \in \text{in}(i)} \phi_u(t-1) \sum_{v \in \text{out}(v)} \frac{w_v}{w_i} \\
\end{align*}
\]  

(initial condition) (sink node) (transition rule)

where \( v_s \) denotes the source node and \( \delta_{v_i} = 1 \) for \( v_s \), and 0 otherwise. For connected network where \( v_d \) is reachable by every node, \( \phi_d(t) \) tends to zero when \( t \) tends to infinity. However, for random walks with short relaxation time [14,19], the conditional probability distribution \( P_{\Delta t}(T > t) \) (i.e. conditioned on the survival of the random walker) will converge as \( t \to \infty \), where \( T \) denotes the time to reach the sink node. As such, the converged conditional probability distribution is called the quasi-stationary distribution denoted as \( \Psi_i \). For \( t \to \infty \), we have the following approximation:

\[
\Psi_s \approx \frac{\phi_s(t)}{\phi_{\text{total}}(t)}
\]

### Table 2. Pearson’s correlation coefficient for MFPT.

<table>
<thead>
<tr>
<th>Network</th>
<th>Correlation</th>
<th>Relaxation time ( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \beta = -1 )</td>
<td>0</td>
</tr>
<tr>
<td>arXiv</td>
<td>0.2619</td>
<td>0.7435</td>
</tr>
<tr>
<td>BA</td>
<td>0.9241</td>
<td>0.9812</td>
</tr>
<tr>
<td>ER</td>
<td>0.4860</td>
<td>0.9846</td>
</tr>
<tr>
<td>Actor</td>
<td>0.0064</td>
<td>0.9231</td>
</tr>
</tbody>
</table>

Overall, the correlations are high whenever the relaxation time is low. For the BA network, the relaxation time is consistently low and thus the correlation is extremely good.

doi:10.1371/journal.pone.0093348.t002
Figure 3. Comparison of MFPTs obtained from simulation against that obtained from using the bound shown in Ineq. (16). The column on the left compares the empirical MFPT to the results obtained by (i) using the bound in Ineq. (16); (ii) using the result presented in [8]. The column on the right shows the scatter plot and correlation between the empirical results and our proposed theoretical results. The rows correspond to the Actor, arXiv, and BA network respectively. While the result presented by Fronczak et al. [8] gives the general trend of MFPTs with respect to node degree, we find that the MFPTs for a given node degree are distributed across a wide range and cannot be fitted with a function of the node degree alone. The scatter plots show a strong correspondence between the empirical results and our proposed theoretical results. This is further supported by the high Pearson correlation coefficients which are shown on top of the scatter plots.

doi:10.1371/journal.pone.0093348.g003
### Table 3. Spectral dimension $d_s$ w.r.t. weighting factor $\alpha$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Actor</th>
<th>arXiv</th>
<th>BA</th>
<th>ER</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-50$</td>
<td>0.042</td>
<td>0.016</td>
<td>0.026</td>
<td>0.110</td>
</tr>
<tr>
<td>$-25$</td>
<td>0.068</td>
<td>0.028</td>
<td>0.142</td>
<td>0.173</td>
</tr>
<tr>
<td>$-5$</td>
<td>0.360</td>
<td>0.300</td>
<td>3.760</td>
<td>0.990</td>
</tr>
<tr>
<td>$-4$</td>
<td>0.476</td>
<td>0.360</td>
<td>4.500</td>
<td>1.537</td>
</tr>
<tr>
<td>$-3$</td>
<td>0.590</td>
<td>1.735</td>
<td>5.066</td>
<td>2.220</td>
</tr>
<tr>
<td>$-2$</td>
<td>2.000</td>
<td>2.561</td>
<td>$\infty$</td>
<td>3.526</td>
</tr>
<tr>
<td>$-1$</td>
<td>1.932</td>
<td>3.440</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$0$</td>
<td>1.500</td>
<td>2.505</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$1$</td>
<td>1.456</td>
<td>1.060</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$2$</td>
<td>1.400</td>
<td>0.667</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$3$</td>
<td>1.025</td>
<td>0.420</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$4$</td>
<td>0.920</td>
<td>0.229</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$5$</td>
<td>0.880</td>
<td>0.220</td>
<td>$\infty$</td>
<td>4.378</td>
</tr>
<tr>
<td>$25$</td>
<td>0.180</td>
<td>0.295</td>
<td>$\infty$</td>
<td>0.536</td>
</tr>
<tr>
<td>$-50$</td>
<td>0.107</td>
<td>0.090</td>
<td>$\infty$</td>
<td>0.220</td>
</tr>
</tbody>
</table>

The data is obtained by conducting simulation and fitting the exponent of RTO probability according to the definition given in Eq.(17). We mainly consider the range $[-5, 5]$ as they are mostly considered in the literature. The entries $[-50, -25, 25, 50]$ are used to examine the effect on $d_s$ for large $\alpha$. The entries with $d_s \sim \infty$ are obtained by the observation that the RTO probability stabilized fairly quickly in less than 10 time steps. The data in this table is plotted in Figure 4.

doi:10.1371/journal.pone.0093348.t003

**Figure 4. Relationship between spectral dimension $d_s$ and the weighting factor $\alpha$.** The data used for drawing this figure is tabulated in Table 3. The lines corresponding to BA and ER network appear disconnected as they have $d_s = \infty$ for certain values of $\alpha$ and cannot be adequately shown in the figure. The spectral dimension generally peaks in the interval $[-1, 1]$ and drops significantly for $\alpha$ of greater magnitude. This is especially true for the BA and ER networks (from infinity to a finite value). In the extreme case, by setting $\alpha = \infty$, the ‘random walk’ is no longer random as the node with largest degree will always be chosen at every step. Similar reasoning also applies for $\alpha = -\infty$. Therefore, towards both extremes, we would expect the random walk to become more localized and hence falls into the compact exploration regime.

doi:10.1371/journal.pone.0093348.g004
For $t < \tau$, the first passage probability is dependent on the source-sink distance. Here, we focus on the asymptotic behaviour of the random walker for $t \gg \tau$. As such, we can apply Approx.(5) and after simplifying, we get:

$$ F(v_d, t|v_s) \approx \phi_{total}(t-1) \sum_{\text{out}(u)} \Psi_u \sum_{\text{out}(u)} w_{ud} $$

Let $\beta_d = \sum_{\text{out}(u)} \Psi_u \sum_{\text{out}(u)} w_{ud}$. By combining Eq.(7) and Approx.(9), after simplification, we obtain the following recursive approximation:

$$ \phi_{total}(t) \approx (1 - \beta_d) \phi_{total}(t-1) $$

Substituting the result back to Approx.(9) yields:

$$ F(v_d, t|v_s) \approx \beta_d e^{-\beta_d(t-1)} $$

Thus, the first passage probability is approximately bounded below by an exponential function for $t \gg \tau$ and the decay rate is given by $\beta_d$. To calculate the decay rate, we apply Approx.(6):

$$ \beta_d = \sum_{\text{out}(u)} \Psi_u \sum_{\text{out}(u)} w_{ud} $$

$$ \approx \frac{w_{vd}}{N(w_d)} \sum_{\text{out}(u)} \left( 1 - \frac{w_{ud}}{\sum_{\text{out}(u)} w_u} \right) w_u $$

$$ = \frac{w_{vd}}{N(w_d)} \left( \sum_{\text{out}(u)} w_{ud} - \sum_{\text{out}(u)} \sum_{\text{out}(u)} w_{ud} \right) $$

where the last step is just a simplification of the expression by using the definition as shown in Eq.(1) as we assumed earlier that every node in the network has a self-loop. Nevertheless, for the cases where self-loop is absent, the expression $Pr(u \rightarrow u)$ can be
substituted for \( \sum_{n \neq 0} r_n \), as self-loop played no role in our
derivation except for the regularity of the random walk.

The result in Approx.(14) suggests that the decay rate of the first
passage time distribution depends mainly on the sink node’s
stationary distribution and the transition probabilities around the
sink node.

Mean First Passage Time

After obtaining the first passage time distribution, we can estimate
the mean first passage time by using the approximation:

\[
\langle T \rangle = \sum_{1}^{\infty} tF(v_d, t|v_s)
\]

\[
\approx \int_{1}^{\infty} tF(v_d, t|v_s)dt
\]

\[
= \int_{1}^{\infty} tF(v_d, t|v_s)dt + \int_{1}^{\infty} tF(v_d, t|v_s)dt
\]

for some \( c \gg r \). In Approx.(15), the integral is separated into two
parts – from 1 to \( c \) and from \( c \) to \( \infty \) for short-term and long-term
behaviour respectively. While the first passage time analysis is
based on the assumption \( t \gg r \), our result is an upper bound on the
first passage probability for small \( t \). This is justified by the
observation that the first passage time probability starts from zero
initially, increases to a peak, then decreases exponentially (see Figure 2
and [13] for non-compact case) with the decay rate as
presented previously. Thus, Ineq.(13) is an overestimation of the
first passage probability for small \( t \). Hence, we provide a lower
bound of mean first passage time as follows:

\[
\langle T \rangle = \int_{1}^{\infty} tJ_d e^{-\beta_d(t-1)} dt
\]

\[
= \int_{1}^{\infty} (1-J_d) e^{-\beta_d(t-1)} dt + \int_{1}^{\infty} e^{-\beta_d(t-1)} dt
\]

(16)

Although Ineq.(16) is a lower bound of the mean first passage time,
results from simulations show that it is strongly correlated to the
actual MFPT.

In order to calculate the MFPT, an edge will be followed exactly
twice to obtain the weight of the connected neighbour. Thus,
assuming the network is represented in adjacency list format, the
time complexity of the approximation is given by \( O(N + M) \). The
time complexity to convert a network from adjacency matrix
representation into adjacency list representation is bounded by
\( O(N^2) \). Hence, the overall time complexity of the approach is
bounded by \( O(N^2 + M) \) for adjacency matrix representation.
Nevertheless, in terms of computational efficiency, the approach is
a great improvement from the fundamental matrix method [13]
which has time complexity of \( O(N^3) \) or \( O(N^{2.376}) \) depending on
the actual implementation of matrix operations [20]. Thus, our
approach can be used to estimate MFPTs of the network quickly
especially for large scale networks.

Results

We verify the theoretical results on two real world networks
and two artificially generated networks: arXiv, Actor collaboration
network, obtained from the
Quantum Cosmology collaboration network, obtained from the
Stanford SNAP website [21], (ii) Barabási-Albert (BA) preferential
attachment network [1,22], (iii) Erdős-Rényi (ER) random graph,
and (iv) Actor collaboration network from the Barabási lab [1].
Because the analysis only applies to connected graphs, we used the
largest component of the arXiv network. The two generated
networks are chosen to test the theoretical results on networks with
different structure and edge density. The ER network is generated
with \( p=0.0011 \), while the BA network is constructed from the first
1000 records of the database, where each record consists of the
actors who collaborated in the same movie. Table 1 summarizes
the networks that we have examined.

We apply the node weight assignment scheme given by \( w_i = k_i^2 \),
where \( k_i \) is degree of node \( i \) and \( x \) is an integer. This node weight
assignment scheme is mainly studied in network traffic
community such as [5,7]. For our experiments, we mostly consider
the range of \( x \) in \([-1,1]\) except for the Actor network as other
values with greater magnitude will result in random walks with
exceedingly long walk lengths.

The experiments are conducted as described below. Firstly, for
a given network, we choose 20 nodes randomly as the source
nodes. The weighting factor \( x \) is then fixed and the node weights
are computed accordingly. For each source node, 250 times of full
random walk simulations are conducted independently. A full
random walk simulation starts with the random walker at the
source node and terminates when every other node has been
visited at least once. The first passage time to each node is
recorded. The procedure is then repeated for other values of \( x \).

To obtain the first passage time distribution, we applied
Gaussian Kernel Density Estimation on the first passage time
statistics collected from the simulations. We randomly selected
different source-sink pairs for each network and different values of
\( x \), and the results are plotted in Figure 2. As shown in the figure,
for most cases, we can predict the tail of the first passage time
distribution fairly accurately except for the case when \( x = -1 \) in
arXiv network. With reference to Table 2, we find that for
\( x = -1 \), the relaxation time for the corresponding random walk is
very high, and thus for certain sink nodes with high absorption
rates, our theoretical result may not be applicable.

In Figure 3, we compare the empirical MFPT to that predicted by
Ineq. (16). Since we relaxed several expressions during the
derivation, the values predicted by Ineq. (16) may not be of the
same scale as the empirical result. Therefore, we renormalized the
predicted values (P-set) with respect to the empirical result (E-set).
The renormalization scheme is described as follows. First, we sort
both the P-set and E-set in non-increasing order. Then we rescule
the middle 90% of the P-set with respect to that of the E-set, i.e.
obtain the rescale parameters (shearing and scaling) by ignoring
both the upper and lower 5 percentile of both sets. Finally, the
renormalization is applied to the whole P-set.

As shown in Figure 3, we can observe that while the result by
Fronczak et al. predicted the general trend of MFPT with respect
to node degree, our result further refined the predicted values by
examining local weight distribution. By zooming into a greater
level of detail, our approach has revealed a useful relationship
between local connectivity and the MFPT especially as highlighted
in Figure 3(a),(c), and (e). The weighting factor \( x \) controls whether
high degree nodes should receive greater attention or vice-versa.
Thus, for small \( x \), high degree nodes should be reached less often
and hence greater MFPT. Surprisingly, even for \( x = -2 \), there are
cases where high degree nodes can be reached fairly quickly as
depicted in Figure 3(a). By examining the structure of the Actor
network, we find that many high degree nodes are connected with
bridge nodes of low degree. On the other hand, the MFPT of the BA network does not show this kind of pattern. We believe that this is due to the fact that the BA network possesses a much simpler structure than the Actor network.

To investigate the overall performance of the bound given in Incq. (16), we also calculated Pearson’s correlation coefficient, and the results are summarized in Table 2. Overall, the correlations are high whenever the relaxation time is low. For the BA network, the relaxation time is consistently low and thus the correlation is extremely good.

Discussions

In [15,16], it was found that the random walk exploration can be divided into compact and non-compact regimes based on their spectral dimension. For the compact case, the random walker spends longer time travelling around certain neighbourhood and hence the relaxation time is long, while the non-compact case is the other way round. In our study, we focus on the non-compact case, i.e., random walk with short relaxation time. While the structure plays a part in deciding the dynamics of random walk, the weighting factor \( \alpha \) also affects the mixing rates. To examine how much the assumption of short relaxation time holds when the weighting factor \( \alpha \) is changed, we also investigated the relationship between \( \alpha \) and the spectral dimension of the corresponding random walk \( d_s \). We numerically estimate \( d_s \) by using the following relation [17]:

\[
P_s(t) \sim t^{-d_s/2}
\]

where \( P_s(t) \) denotes the Return-to-origin (RTO) probability at time \( t \) [17]. To obtain the RTO probabilities, we conducted simulations with 100,000 random walkers which are placed randomly at \( t=0 \). Each random walker walks independently for 100,000 time steps, and the fraction of the random walker returning to their respective starting node is recorded at each time step. Table 3 and Figure 4 summarizes the result.

As shown in Figure 4, the spectral dimension generally peaks in the interval \([-1, 1]\), which explains the applicability of our result. While for the case of uniform random walk, it was shown that \( d_s \) of both BA networks and random graphs are infinity [23], we found finite \( d_s \) for \( \alpha \neq 0 \). In the extreme case, by setting \( \alpha = -\infty \), the ‘random walk’ is no longer random as the node with smallest degree will always be chosen at every step. For this scenario, the network will be broken into cycles where leaf nodes form the smallest cycle. Similar phenomenon also applies for \( \alpha = \infty \). Therefore, towards both extremes, we would expect the random walk to become more localized and hence falls into the compact exploration regime where our result may not be applicable.

We also observe several disparities between our results and that of [16]. For instance, as shown in Figure 5 and Table 3, when \( \alpha = 0 \), we found that the spectral dimension of the Actor network is 1.560. However, the MFPT is found to be following a power-law relationship with respect to node degree instead of being independent of node degree. Similar results have been obtained for the arXiv network for \( \alpha = 1 \), for which \( d_s \approx 1.060 \), as shown in Figure 3(c). We believe that the disparities arise from the following facts: (i) we considered random walks with self-loops where the staying probabilities are proportional to the node weights; and (ii) the random walks are biased by weight assignments. The self-loop changes the RTO probability distribution and thus also affects the estimated \( d_s \). The nodes are weighted differently, therefore the cross-over threshold for \( d_s \) may not be the same as that of [16]. Further research is needed to better understand the effects of self-loops and node weights on the spectral dimension of a random walk.

In summary, we have shown the exact form of stationary distribution for a class of biased random walks on networks where the nodes are assigned arbitrary weights. By using this result, we have presented a new method that gives improved estimation of MFPT for random walks with short relaxation time. We have verified that the decay rate of the first passage time distribution can be estimated fairly accurately and the MFPT’s are found to be better revealed by local weight distributions. Given its low computational cost, our method enables quick inspection of the MFPT for large scale networks. This is especially true for cases where the ranking rather than the actual values of MFPT of nodes is more important. For instance, to contain virus outbreaks, the new method can be used to quickly rank the nodes based on estimated MFPT and judiciously apply security measures on the nodes that are ranked highly. Our result can also be readily extended to the case of cyclic search [9] where the random walker will scan the direct neighbours of current node for the target as opposed to blindly following the transition rule.

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Author Contributions

Conceived and designed the experiments: ZQL WJH ML. Performed the experiments: ZQL WJH. Analyzed the data: ZQL. Contributed reagents/materials/analysis tools: ZQL. Wrote the paper: ZQL WJH.